1. Introduction

Though still a relatively young subject within discrete mathematics, the study of pattern classes of permutations is one of the fastest growing and is useful in theoretical computer science as well as various areas of mathematics. There are two commonly used forms of notation for permutations: $a_1a_2\ldots a_n$ denotes the permutation which sends $i$ to $a_i$ for $i = 1, 2, \ldots, n$, where each $a_i$ represents one of $\{1, 2, \ldots, n\}$. We call this the image notation. In cyclic notation, on the other hand, $(a_1a_2\ldots a_n)$ denotes the permutation which sends $a_1$ to $a_2$, $a_2$ to $a_3$, $a_n$ to $a_1$, etc. The length of a permutation $\alpha = a_1a_2\ldots a_n$ is denoted by $|\alpha| = n$.

An important idea in dealing with permutations is order isomorphism, where two permutations $\gamma = c_1c_2c_3\ldots$ and $\delta = d_1d_2d_3\ldots$ are said to be order isomorphic if, for all $i, j$, we have $c_i < c_j$ if and only if $d_i < d_j$. We write $\gamma \cong \delta$.

Also essential for the manipulation of permutations is the concept of involvement. A permutation $\alpha$ involves a permutation $\beta$ if there exists a subsequence of $\alpha$ which is order isomorphic to $\beta$ and we write $\alpha \triangleright \beta$. We say that a permutation $\alpha$ avoids a permutation $\beta$ if $\beta$ is not involved in $\alpha$ and write $\beta \not\triangleright \alpha$. The avoidance class of a set of permutations $Y$ is the set of all permutations that avoid the elements of $Y$ and is denoted by $A(Y)$.

This paper is concerned with the study of closed classes. A closed class $X$ is a set of permutations which is closed under involvement in the sense that $\alpha \in X$ and $\beta \triangleright \alpha$ implies that $\beta \in X$. The basis of a set $X$ of permutations is defined as the set of permutations, minimal with respect to involvement, which are avoided by every element of $X$. Any closed class $X$ can be defined as the avoidance class of its basis elements and since all closed class have bases they are often referred to as pattern avoidance classes, or simply pattern classes.

In this paper we will examine some specific examples from topics within permutations theory which may be of interest to researchers of the subject. For a more comprehensive introduction to these topics see [1] and [4].

2. Group Closed Sets

As defined in [2] and [3], a group closed set $X$ is a closed set in which the permutations of each length form a group. More precisely, let $X$ be a closed set and let $X(n) = X \cap S_n$ be all permutations of the form $a_1a_2\ldots a_n$ in $X$ where $S_n$ denotes the symmetric group of order $n$. Then $X$ is a group closed set if, for all $n$, $X(n)$ forms a group.

Date: July 2004.
Consider the following examples of group closed sets $X$ and $Y$: Fix some $k$ and $l \in \mathbb{N}$, let $X(n) = S_n$ if $n < k + l$ and $X(n) = S_k \times S_l$, if $n \geq k + l$, where $S_k \times S_l$ acts as $S_k$ on $\{1, \ldots, k\}$, fixes each point in $\{k+1, \ldots, n-1\}$ and acts as $S_l$ on $\{n-l+1, \ldots, n\}$. Then

$$X = \bigcup_{n \in \mathbb{N}} X(n)$$

is a group closed set.

**Definition 2.1.** The sum of two permutations $\alpha$ and $\beta$, denoted by $\alpha \circ \beta$, is the permutation $\alpha \gamma$ of length $|\alpha| + |\beta|$ where $\gamma$ permutes $|\alpha| + 1$, $|\alpha| + 2, \ldots, |\alpha| + |\beta|$. Similarly, the difference $\alpha \oplus \beta$ of $\alpha$ and $\beta$ is the permutation $\delta \beta$ where $\delta \equiv \alpha$ and $\delta$ permutes $|\beta| + 1, |\beta| + 2, \ldots, |\beta| + |\alpha|$. To define $Y$, fix some $k \in \mathbb{N}$, let $Y(n) = S_n$ if $n < 2k$ and $Y(n) = (S_k \times S_k) \cdot \rho_n$ if $n \geq 2k$, where $S_k \times S_k$ is defined as before (now with $l = k$), $\rho_n = n(n-1)\ldots 21$ is the ‘reversal’ permutation of length $n$ and $Z_2$ is the cyclic group of order 2. Then

$$Y = \bigcup_{n \in \mathbb{N}} Y(n)$$

is a group closed set.

In order to understand the structure of $Y(n)$ more clearly, we see that $\rho_n$ composed with an element of $(S_k \times S_k)$ has the effect of transforming a permutation of the form $\alpha \oplus \beta$ into one of the form $\gamma \ominus \delta$. Where $n > 2k$ the symbols $\{k+1, \ldots, n-1\}$ which were previously fixed now occur in reversed order, with $k+1$ mapping to $n-k-1$, $k+2$ mapping to $n-k-2$, etc. More precisely, for an element $\alpha = a_1 a_2 \ldots a_n$ of $Y$, we have $\alpha \circ \rho_n = (n+1-a_1)(n+1-a_2)\ldots(n+1-a_n)$ which is the complement of $\alpha$. Also, $\rho_n \circ \alpha = a_n a_{n-1} \ldots a_2 a_1$ is the reverse of $\alpha$. Hence the elements of $Y(n)$ take either the form $\gamma \ominus \rho_n \ominus \delta$, or the form $\alpha \oplus \iota_{n-2k} \ominus \delta$, where $\iota_{n-2k}$ is the identity permutation of length $n-2k$. It is not possible for elements of $Y$ to take any other form, since $\rho_n$ is the identity map.

These examples of group closed sets $X$ and $Y$ were introduced in [3] where it was proven that these are the only intransitive group closed sets. We will now determine the bases of $X$ and $Y$.

**Theorem 2.2.** The basis $B(X)$ of $X$ is given by the set $B$ of all permutations of length $k+l$ that are not of the form $\alpha \oplus \beta$ for some $\alpha \in S_k$, $\beta \in S_l$.

**Proof.** First we will show that $B \subseteq B(X)$. All permutations $\gamma \in X$ of length $k+l$ are in $S_k \times S_l$ and so are of the form $\alpha \oplus \beta$ for some $\alpha \in S_k$, $\beta \in S_l$. Hence, by definition, $B \cap X = 0$. Also, if $\gamma \in B$ then any proper subsequence $\gamma^\ominus$ of $\gamma$ will be of length less than $k+l$ and, since $X(n) = S_n$ for $n < k+l$, we have that $\gamma^\ominus \in X$. So no elements of $B$ are in $X$ but all their proper subsequences are and so they must be basis elements; i.e. $B \subseteq B(X)$.

Next we will show that $A(B) \subseteq X$. Let $\delta \in A(B)$ be arbitrary. If $|\delta| < k+l$ then $\delta \in X$ since $X$ contains all $S_n$ for $n < k+l$. If $|\delta| = k+l$ then $\delta$ is of the form $\alpha \oplus \beta$ for some $\alpha \in S_k$ and $\beta \in S_l$ and hence $\delta \in S_k \times S_l \subseteq X$.

So let $|\delta| = n = k+l+m$, where $m > 0$ and $\delta = d_1 d_2 \ldots d_{k+l+m}$. Suppose there exists $d_i \in \{d_1, \ldots, d_k\}$ such that $d_i > k$. Then choose a subsequence of $\delta$ including $d_i$, all of $1, \ldots, k$ and $l-1$ other symbols to obtain a subsequence $\delta^\ominus$ of
length $k + l$. $d_i$ will still be in the first $k$ symbols of $\delta^\circ$ and will be greater than at least $k$ other symbols in this subsequence. Hence $\delta^\circ$ is not of the form $\alpha \oplus \beta$ for any $\alpha \in S_k, \beta \in S_l$. So $\delta^\circ$ defines a permutation in $B$, but since $\delta^\circ$ is involved in $\delta$ this is a contradiction. So $d_1, \ldots, d_k$ each take a value in $\{1, \ldots, k\}$.

Now consider $d_{k+1}$ and suppose $d_{k+1} > k+1$. Then choose a subsequence $\delta^{\bullet}$ of $\delta$ consisting of $d_1, \ldots, d_{k+1}$ and $l$ of the symbols occurring after $d_{k+1}$, including $k+1$. Then $\delta^{\bullet}$ defines a permutation of length $k + l$ whose $k$th symbol has value greater than $k$ other symbols. But then $\delta^{\bullet} \in B$ and $\delta^{\bullet} \leq \delta$ contradicting the fact that $\delta$ avoids $B$. Also $d_{k+1}$ cannot be less than $k + 1$ since all symbols less than $k + 1$ are in the first $k$ positions and $\delta$ is a bijection. So $d_{k+1} = k + 1$. Now either $m = 1$, or else we can continue this argument to show that the symbols $d_{k+2}, d_{k+3}, \ldots, d_{k+m}$ must take precisely the values of their respective indices.

However, this argument only works for $d_i$ when $k < i \leq n - l$ since it relies on the fact that we can choose $l$ symbols occurring after $d_i$. Therefore, considering $d_{n-l+1}$, we only know that it must take a value greater than $n - l$ since the values $1, \ldots, n - l$ have already been taken. The same applies to $d_{n-l+2}, \ldots, d_n$ which then means that $d_{n-l+1}, \ldots, d_n$ each take a value in $\{n - l + 1, \ldots, n\}$. But then $\delta \in X$.

So we have shown that $\delta \in \mathcal{A}(B)$ implies $\delta \in X$ and hence $B(X) \subseteq B$ which completes the proof.

Next we consider the group closed set $Y$ as defined above which contains the previous pattern class $X$ with $k = l$ together with the ‘reversal’ permutation $\rho_n = n \ (n - 1) \ldots 2 1$.

**Theorem 2.3.** The basis $B(Y)$ of $Y$ is given by the set $C$ of all permutations of length $2k$ that are not of the forms $\alpha \oplus \beta$ or $\alpha \ominus \beta$ for any $\alpha, \beta \in S_k$.

**Proof.** Again we will begin by showing that $C \subseteq B(Y)$. All permutations $\gamma \in Y$ of length $2k$ are in $(S_k \times S_k) \cdot \langle \rho_n \rangle$ and so are of the form $\alpha \oplus \beta$ or $\alpha \ominus \beta$ for some $\alpha, \beta \in S_k$. Hence, by definition, $C \cap Y = \emptyset$. Also, if $\gamma \in C$ then any proper subsequence $\gamma^{\circ}$ of $\gamma$ will be of length less than $2k$ and since $Y(n) = S_n$ for $n < 2k$, we have that $\gamma^{\circ} \in Y$. So no elements of $C$ are in $Y$ but all their proper subsequences are and so they must be basis elements; i.e. $C \subseteq B(Y)$.

Next we will show that $\mathcal{A}(C) \subseteq Y$. Let $\delta \in \mathcal{A}(C)$ be arbitrary. If $|\delta| < 2k$ then $\delta \in Y$ since $Y$ contains all permutations of length $n$ for $n < 2k$. If $|\delta| = 2k$ then $\delta$ is either of the form $\alpha \oplus \beta$ or of the form $\alpha \ominus \beta$ for some $\alpha, \beta \in S_k$ and hence $\delta \in S_k \times S_k \subseteq Y$.

So let $|\delta| = n = 2k + m$, where $m > 0$ and $\delta = d_1d_2\ldots d_{2k+m}$. Suppose there exists $d_i, d_j \in \{d_1, \ldots, d_k\}$ such that $d_i > k$ and $d_j \leq k$. Then choose a subsequence of $\delta$ including $d_i$, all of $1, \ldots, k$ (including $d_j$) and $k - 1$ other symbols to obtain a subsequence $\delta^{\circ}$ of length $2k$. $d_i$ will still be in the first $k$ symbols of $\delta^{\circ}$ and will be greater than at least $k$ other symbols in this subsequence, hence $\delta^{\circ}$ is not of the form $\alpha \oplus \beta$ for any $\alpha, \beta \in S_k$. Also, $d_j$ will still be in the first $k$ symbols of $\delta^{\circ}$ and will be smaller than at least $k$ other symbols in the sequence. Therefore $\delta^{\circ}$ is not of the form $\alpha \ominus \beta$ for any $\alpha, \beta \in S_k$ either. $C$ therefore contains a permutation which is order isomorphic to $\delta^{\circ}$, contradicting the fact that $\delta$ avoids $C$. So either $d_1, \ldots, d_k$ each take a value in $\{1, \ldots, k\}$ or each of them takes a value in $\{k+1, k+2, \ldots, n\}$. 

Suppose then that $d_1, \ldots, d_k$ each take a value in $\{k+1, k+2, \ldots, n\}$ and suppose there exists $d_i, d_j \in \{d_1, \ldots, d_k\}$ such that $d_i < n - k + 1$ and $d_j \geq n - k + 1$. We choose a subsequence $\delta^\bullet$ of $\delta$ consisting of $d_i$, all of the symbols $n - k + 1, \ldots, n$ (including $d_j$) and $k - 1$ other symbols to obtain a subsequence $\delta^\bullet$ of length $2k$. $d_i$ will still be in the first $k$ symbols of $\delta^\bullet$ and will be smaller than at least $k$ other symbols in this subsequence, hence $\delta^\bullet$ is not of the form $\alpha \oplus \beta$ for any $\alpha, \beta \in S_k$. Also, $d_j$ is still in the first $k$ symbols of $\delta^\bullet$ and $d_j$ is one of the $k$ highest symbols of $\delta^\bullet$. Hence $\delta^\bullet$ is not of the form $\alpha \ominus \beta$ for any $\alpha, \beta \in S_k$ either. Then $\delta^\bullet \in C$ and $\delta^\bullet \leq \delta$ which contradicts the fact that $\delta$ avoids $C$. Hence if $d_1, \ldots, d_k$ do not each take a value in $\{1, \ldots, k\}$, then either they each take a value in $\{k+1, \ldots, n-k\}$ or they each take a value in $\{n-k+1, \ldots, n\}$. We now consider the three possible cases in turn.

Suppose firstly that $d_1, \ldots, d_k$ each take a value in $\{k+1, \ldots, n-k\}$ and consider where the lowest $k$ symbols $\{1, \ldots, k\}$ and the highest $k$ symbols $\{n-k+1, \ldots, n\}$ occur in $\delta$. There are three possibilities: Either the two sets form an interleaving subsequence of $\delta$, or they are entirely disjoint in which case we can either have all of $\{1, \ldots, k\}$ occurring before all of $\{n-k+1, \ldots, n\}$ or vice versa. If the highest $k$ symbols and the lowest $k$ symbols of $\delta$ are interleaved in $\delta$, we can take a subsequence $\delta_1$ of $\delta$ consisting of $\{1, \ldots, k\}$ and $\{n-k+1, \ldots, n\}$. Since the two sets of symbols are interleaved in $\delta_1$ and $\delta_1$ is of length $2k$, $\delta_1$ is not of the form $\alpha \ominus \beta$ or $\alpha \oplus \beta$ for any $\alpha, \beta \in S_k$ and so is order isomorphic to a permutation in $C$. Again, this contradicts the fact that $\delta$ avoids $C$.

So suppose now that $\{1, \ldots, k\}$ and $\{n-k+1, \ldots, n\}$ occur disjointly in $\delta$, with all of $\{1, \ldots, k\}$ occurring before all of $\{n-k+1, \ldots, n\}$. Then one may take a subsequence $\delta_2$ of $\delta$ containing $d_1$, all of $\{1, \ldots, k\}$ and $\{n-k+2, \ldots, n\}$. We know $k + 1 \leq d_1 \leq n - k$ and so $\delta_2$, a subsequence of length $2k$, has $k - 1$ of its $k$ highest symbols in the last $k - 1$ positions and the remaining one of them in its first position. Hence $\delta_2$ is order isomorphic to a permutation contained in $C$ which contradicts the fact that $\delta$ avoids the elements of $C$. Lastly, we may have $\{1, \ldots, k\}$ and $\{n-k+1, \ldots, n\}$ occurring disjointly in $\delta$, with all of $\{n-k+1, \ldots, n\}$ occurring before all of $\{1, \ldots, k\}$. But this case is very similar to the last one: We may then take a subsequence $\delta_3$ of $\delta$ containing $d_1$, all of $\{n-k+1, \ldots, n\}$, and $\{1, \ldots, k-1\}$. Then by a similar argument, $C$ contains a permutation order isomorphic to $\delta_3$ which causes the same contradiction as before. Hence the case where each of $d_1, \ldots, d_k$ take a value in $\{k+1, \ldots, n-k\}$ leads to a contradiction and so we must have the first $k$ symbols of $\delta$ either each taking one of the $k$ highest values or each taking one of the $k$ lowest values of $\delta$. We will now consider these two remaining possible cases in turn.

Suppose that $d_1, \ldots, d_k$ each take a value in $\{1, \ldots, k\}$ and consider $d_{k+1}$. Suppose $d_{k+1} > k+1$. Then choose a subsequence $\delta^\circ$ of $\delta$ consisting of $d_1, \ldots, d_{k-1}, d_{k+1}$ and $k$ of the symbols occurring after $d_{k+1}$, including $k+1$. Then $\delta^\circ$ defines a permutation of length $2k$ whose first $k - 1$ symbols each take one of the $k$ smallest values but the $k$th symbol has value greater than $k$ other symbols. Hence $\delta^\circ \not\alpha \oplus \beta$ and $\delta^\circ \not\alpha \ominus \beta$ for any $\alpha, \beta \in S_k$. So again, $C$ contains a permutation that is order isomorphic to $\delta^\circ$, which contradicts the fact that $\delta$ avoids $B$. Also $d_{k+1}$ cannot be less than $k + 1$ since all symbols less than $k + 1$ are in the first $k$ positions and $\delta$
is a bijection. So $d_{k+1} = k + 1$. Now either $m = 1$, or else we can continue this argument to show that the symbols $d_{k+2}, d_{k+3}, \ldots, d_{k+m}$ must take precisely the values of their respective indices.

However, this argument only applies to $d_i$ when $k < i \leq n - k$ since it relies on the fact that we can choose $k$ symbols occurring after $d_i$. Therefore, considering $d_{n-k+1}$, we only know that it must take a value greater than $n - k$ since the values $1, \ldots, n - k$ have already been taken. The same applies to $d_{n-k+2}, \ldots, d_n$ which then means that $d_{n-k+1}, \ldots, d_n$ each take a value in $\{n - k + 1, \ldots, n\}$. But then $\delta \in Y$.

Finally, suppose that $d_1, \ldots, d_k$ each take a value in $\{n - k + 1, \ldots, n\}$ and consider $d_{k+1}$. Suppose $d_{k+1} < n - k$. Then choose a subsequence $\delta^*$ of $\delta$ consisting of $d_1, \ldots, d_{k-1}, d_{k+1}$ and $k$ of the symbols occurring after $d_{k+1}$ including $n - k$. Then $\delta^*$ defines a permutation of length $2k$ whose first $k - 1$ symbols each take one of the $k$ highest values but the $k$th symbol has value less than $k$ other symbols. Hence $\delta^* \not\sim \alpha \oplus \beta$ and $\delta^* \not\sim \alpha \ominus \beta$ for any $\alpha, \beta \in S_k$. So $C$ contains a permutation that is order isomorphic to $\delta^*$, which again contradicts the fact that $\delta$ avoids $B$. Also $d_{k+1}$ cannot be greater than $n - k$ since all symbols greater than $n - k$ are in the first $k$ symbols; hence $d_{k+1} = n - k$. Now either $m = 1$, or else we can continue this argument to show that $d_i = d_{n-i+1}$ for all $i \in \{k + 1, \ldots, k + m\}$.

However, this argument is only valid for $d_i$ when $k < i \leq n - k$ since it relies on the fact that we can choose $k$ symbols occurring after $d_i$. Therefore, considering $d_{n-k+1}$, we only know that it must take a value less than $k + 1$ since the values $k + 1, \ldots, n$ have already been taken. The same applies to $d_{n-k+2}, \ldots, d_n$ which then means that $d_{n-k+1}, \ldots, d_n$ each take a value in $\{1, \ldots, k\}$. But then $\delta \in Y$.

So, we have shown that $\delta \in \mathcal{A}(C)$ implies $\delta \in Y$ and hence $B(Y) \subseteq C$ which completes the proof.

\[\Box\]

3. Merges of Pattern Classes and their Bases

For any permutations $\alpha$ and $\beta$ their merge, denoted by $\alpha$ merge $\beta$ in this paper, is considered to be any permutation consisting of two not necessarily disjoint subsequences isomorphic to $\alpha$ and $\beta$. An interleaving of $\alpha$ and $\beta$ is an example of a merge; note that $\alpha$ merge $\beta$ is by no means unique. The merge of two sets $A, B$ of permutations is the set of all permutations that are a merge of some $\alpha \in A, \beta \in B$.

For a closed class $A$, if $d$ is the minimum length of any basis element of $A$, then by shortest basis elements we will mean all basis elements of length $d$.

**Lemma 3.1.** Let $A, B$ be closed classes with shortest basis elements $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ and $\{\beta_1, \beta_2, \ldots, \beta_n\}$ respectively. Then the shortest basis element of the merge $M$ of $A$ and $B$ has length at least $|\alpha_1| + |\beta_1| - 1$.

**Proof.** Consider an arbitrary permutation $\pi$ of length $|\alpha_1| + |\beta_1| - 2 - k$ for some $k \geq 0$. It can be subdivided into two subsequences $\gamma$ and $\delta$ where $|\gamma| = |\alpha_1| - 1 - k$ and $|\delta| = |\beta_1| - 1$ such that $\pi = \gamma$ merge $\delta$. Then since all basis elements of $A$ and $B$ have length at least $|\alpha_1|$ or $|\beta_1|$ respectively, we know that $\gamma \in A, \delta \in B$ and so $\gamma$ merge $\delta = \pi \in M$. Hence all permutations of length up to $|\alpha_1| + |\beta_1| - 2$ are in $M$ and the result follows. \[\Box\]
Lemma 3.2. If, for some permutation \( \beta \), there exists a permutation \( \alpha \) with \( |\alpha| < |\beta| \) such that \( \gamma \preceq \beta \) and \( |\gamma| = |\alpha| \) implies \( \gamma \succeq \alpha \), then either \( \alpha = 12 \ldots n \) or \( \alpha = n \ldots 21 \).

**Proof.** Let \( \beta = b_1 b_2 \ldots b_m \) and let \( |\alpha| = n < m \). Consider the subsequences \( \gamma_1 = b_1 b_2 \ldots b_n \) and \( \gamma_2 = b_2 b_3 \ldots b_{n+1} \) of \( \beta \). Since \( |\gamma_1| = |\gamma_2| = n \), we have \( \alpha \succeq \gamma_1 \succeq \gamma_2 \).

So by the definition of order isomorphism, \( b_i < b_{i+1} \) if and only if \( b_{i+1} < b_{i+2} \) for all \( i \in \{1, 2, \ldots, n - 1\} \). Similarly, consider \( \gamma_2 \) and \( \gamma_3 = b_3 b_4 \ldots b_{n+2} \leq \beta \).

Again, \( |\gamma_2| = |\gamma_3| = n \) implies that \( \gamma_3 \succeq \alpha \succeq \gamma_2 \succeq \gamma_1 \) and so \( b_i < b_{i+1} \) if and only if \( b_{i+1} < b_{i+2} \) for all \( i \in \{1, 2, \ldots, n\} \). It is clear that by continuing to consider similar combinations of subsequences \( \gamma_1, \gamma_2, \ldots, \gamma_i \) of \( \beta \) for \( i = 4, 5, \ldots, m - n + 1 \), we can show that \( b_i < b_{i+1} \) if and only if \( b_{i+1} < b_{i+2} \) for all \( i \in \{1, 2, \ldots, n - 2\} \). Hence we must have either \( \beta = 12 \ldots n \) or \( \beta = m \ldots 21 \) which in turn implies \( \alpha = 12 \ldots n \) or \( \alpha = n \ldots 21 \) respectively. \( \square \)

**Theorem 3.3.** Let \( A, B \) be pattern classes where \( A \) has a unique shortest basis element \( \alpha \) of length \( k \) and the shortest basis elements of \( B \) are \( \{\beta_1, \ldots, \beta_n\} \) of length \( l \). Also suppose \( \alpha \neq 12 \ldots k \) and \( \alpha \neq k \ldots 21 \). Then all basis elements of the merge \( M \) of \( A \) and \( B \) have length at least \( k + l \).

**Proof.** By Lemma 3.1, all permutations of length up to \( k + l - 2 \) are in \( M \). So let \( \gamma \) be an arbitrary permutation of length \( k + l - 1 \). Since \( \alpha \neq 12 \ldots k \) and \( \alpha \neq k \ldots 21 \), we know that we can find \( \delta \preceq \gamma \) with \( |\delta| = k \) such that \( \delta \neq \alpha \) by Lemma 3.2. We can then find \( \varepsilon \succeq \gamma \) with \( |\varepsilon| = l - 1 \) such that \( \gamma = \delta \merge \varepsilon \). Since the shortest basis elements of \( B \) are of length \( l \), we know that \( \varepsilon \in B \). Also, \( A \) has unique shortest basis element \( \alpha \), both \( \alpha \) and \( \delta \) are of length \( k \) and \( \delta \neq \alpha \), therefore \( \delta \in A \). Hence \( \delta \merge \varepsilon = \gamma \in M \) and the result follows. \( \square \)

Let \( I \) denote the closed class consisting of the identity permutations of all lengths \( n \in \mathbb{N} \), and let \( R \) denote the closed class consisting of the ‘reversal’ permutations \( \rho_n \) for all \( n \in \mathbb{N} \). Also, let \( \alpha - a_i \) represent the permutation which is obtained by deleting \( a_i \) from \( \alpha = a_1 a_2 \ldots a_n \) and relabelling so that \( \alpha - a_i \) is a permutation of \( \{1, 2, \ldots, n - 1\} \) order isomorphic to \( a_1 a_2 \ldots a_{i-1} a_{i+1} \ldots a_n \). Finally, a pattern class \( X \) is sum complete if \( \alpha \in X, \beta \in X \Rightarrow \alpha \oplus \beta \in X \). Similarly, a pattern class \( Y \) is minus complete if \( \gamma \in Y, \delta \in Y \Rightarrow \gamma \ominus \delta \in Y \).

**Theorem 3.4.** Let \( X \) be a sum complete pattern class with basis \( \mathcal{B}(X) \). Then the basis of the merge \( M \) of \( X \) and \( R \) contains \( \alpha \oplus \beta \) for all \( \alpha, \beta \in \mathcal{B}(X) \). Similarly, if \( Y \) is a minus complete pattern class with basis \( \mathcal{B}(Y) \), then the basis of the merge \( N \) of \( Y \) and \( I \) contains \( \gamma \ominus \delta \) for all \( \gamma, \delta \in \mathcal{B}(Y) \).

**Proof.** Let \( \alpha, \beta \) be arbitrary basis elements of \( X \) and assume \( \alpha \oplus \beta = \rho \merge \sigma \) where \( \rho \in R, \sigma \in \mathcal{S} \). Since \( \rho \) is monotonically decreasing it cannot be embedded in \( \alpha \) and \( \beta \) simultaneously and so we have either \( \rho \preceq \alpha \) or \( \rho \preceq \beta \). Without loss of generality, let \( \rho \preceq \alpha \). Then \( \beta \) must be entirely contained in \( \sigma \). But \( \sigma \in X \) and \( \beta \in \mathcal{B}(X) \) causes a contradiction. Hence \( \alpha \oplus \beta \notin M \).

Now let \( a \) be one of the first \( |\alpha| \) points of \( \alpha \oplus \beta \) and let \( b \) be one of the last \( |\beta| \) points of \( \alpha \oplus \beta \). Since \( \alpha, \beta \in \mathcal{B}(X), \alpha - a \) and \( \beta - b \) are both elements of \( X \) and since \( X \) is sum complete, \( (\alpha - a) \oplus (\beta - b) \in X \). So if we remove one of the first \( |\alpha| \) symbols of \( \alpha \oplus \beta \) to obtain \( (\alpha - a) \oplus \beta \), then this permutation is in \( M \) since \( (\alpha - a) \oplus \beta = (\alpha - a) \oplus (\beta - b) \merge b \) where \( (\alpha - a) \oplus (\beta - b) \in X, b \in R \).
Similarly, $\alpha \oplus (\beta - b) = (\alpha - a) \oplus (\beta - b)$ merge $a$ and so removing one of the last $|\beta|$ symbols of $\alpha \oplus \beta$ also results in a permutation which is in $M$. So $\alpha \oplus \beta$ is minimal and hence the basis of $M$ contains $\alpha \oplus \beta$ for all $\alpha, \beta \in \mathcal{B}(X)$. The second part of the theorem follows by symmetry. \hfill \square

**Remark 3.5.** Note that the basis $\mathcal{B}(M)$ of $M$ also contains permutations which are not of the form $\alpha \oplus \beta$ for some $\alpha, \beta \in \mathcal{B}(X)$ in general, as can be seen from the following counterexample. Consider the closed class $C$ containing all permutations $\alpha$ of the form $\alpha = \alpha_1 \oplus \alpha_2 \oplus \cdots \oplus \alpha_m$, $m \in \mathbb{N}$ where $\alpha_1, \alpha_2, \ldots, \alpha_m \in \{1, 21\}$. We call this the *Sum Completion Class* of $1$ and $21$. It may be shown that the basis of $C$ is $\mathcal{B}(C) = \{231, 312, 321\}$. However, the basis of the merge $M$ of $C$ and $R$ contains $45123$ which is not the sum of any basis elements of $C$.

### 4. Sum Completeness of Pattern Classes

**Theorem 4.1.** Let $X$ be a pattern class and let the elements of its basis $\mathcal{B}(X)$ be of the form $\alpha_i = a_1 a_2 \ldots a_n$. If, for all $\alpha_i \in \mathcal{B}(X)$, $a_1 > a_n$, then $X$ is sum complete. Similarly, if for all $\alpha_i \in \mathcal{B}(X)$, $a_1 < a_n$, then $X$ is minus complete.

**Proof.** Suppose $X$ is not sum complete. Then there exist permutations $\beta, \gamma \in X$ such that $\beta \oplus \gamma \notin X$. Hence there exists $\alpha \in \mathcal{B}(X)$ such that $\alpha \preceq \beta \oplus \gamma$. Let $\alpha^* = a_1^* a_2^* \ldots a_n^*$ be the subsequence of $\beta \oplus \gamma$ that is order isomorphic to $\alpha$. $\alpha^*$ cannot be entirely contained in the first $|\beta|$ or last $|\gamma|$ elements of $\beta \oplus \gamma$ since $\beta$ and $\gamma$ avoid $\alpha$ and hence avoid $\alpha^*$. So $a_1^*$ must be in the first $|\beta|$ elements of $\beta \oplus \gamma$ and $a_n^*$ must be in the last $|\gamma|$ elements of $\beta \oplus \gamma$. So $a_1^* < a_n^*$ and hence $a_1 < a_n$ since $\alpha \cong \alpha^*$. So it has been shown that if $X$ is not sum complete, then there exists $\alpha \in \mathcal{B}(X)$ such that $a_1 < a_n$. Hence if $a_1 > a_n$ for all $\alpha \in \mathcal{B}(X)$, then $X$ is sum complete. The second part of the theorem follows by symmetry. \hfill \square

Perhaps the most useful implication of Theorem 4.1 is Corollary 4.2 which follows directly from it.

**Corollary 4.2.** The avoidance class $\mathcal{A}(\alpha)$ of a single element $\alpha$ is always either sum complete or minus complete.

**Remark 4.3.** Theorem 4.1 cannot be changed to an *if and only if* statement because there are classes which are both sum and minus complete, for instance $\mathcal{A}(2413)$.

The next theorem links the sum completeness of a pattern class to its basis elements in such a way that it may also be applied to show that a given class is not sum or minus complete. Note that a permutation $\sigma$ is *sum decomposable* if there exist permutations $\beta, \gamma$ such that $\alpha = \beta \oplus \gamma$. Similarly, a permutation $\rho$ is *minus decomposable* if there exist permutations $\sigma, \tau$ such that $\rho = \sigma \ominus \tau$.

**Theorem 4.4.** A pattern class $X$ is sum complete if and only if none of its basis elements are sum decomposable. Similarly, a pattern class $Y$ is minus complete if and only if none of its basis elements are minus decomposable.

**Proof.** We prove the result about the sum complete class $X$; the statement about the minus complete class $Y$ may be established by a similar proof.
Let $X$ be a sum complete pattern class and consider an arbitrary element $\alpha = a_1 \ldots a_n$ of $B(X)$. Aiming for a contradiction, suppose $\alpha \not\equiv \beta \oplus \gamma$ for some permutations $\gamma, \beta$. Then by the definition of a basis, since $\alpha \in B(X)$, its two proper subsequences $\beta, \gamma$ are both elements of $X$ and since $X$ is sum complete, $\beta \oplus \gamma \in X$. But $\beta \oplus \gamma \not\equiv \alpha \in B(X)$ which is a contradiction. Hence if $X$ is sum complete, then none of its basis elements are sum decomposable.

Conversely, let $X$ be a closed class none of whose basis elements are sum decomposable. Now suppose $X$ is not sum complete. Then there exist $\beta, \gamma$ in $X$ such that $\beta \oplus \gamma /\in X$. As in the proof of Theorem 4.1, this is equivalent to the statement that $\beta \oplus \gamma = b_1b_2\ldots b_n g_1g_2\ldots g_n \succeq \alpha$ for some $\alpha \in B(X)$ where $\alpha \not\preceq \beta$ and $\alpha \not\preceq \gamma$. Therefore we must have $\alpha \equiv b_{i_1}\ldots b_{i_k}g_{j_1}\ldots g_{j_l}$ where $\{i_1, \ldots, i_k\} \in \{1, \ldots, m\}$ and $\{j_1, \ldots, j_l\} \in \{1, \ldots, n\}$. But $b_i < g_j$ for all $i \in \{i_1, \ldots, i_k\}$ and all $j \in \{j_1, \ldots, j_l\}$ which means that $\alpha \equiv \beta^* \oplus \gamma^*$ where $\beta^* \preceq \beta$ and $\gamma^* \preceq \gamma$. However, since $\alpha \in B(X)$, it is not sum decomposable and so we have a contradiction. Hence we have that if none of the basis elements of a closed class are sum decomposable, then the class is sum complete.

In summary, Theorem 4.1 provides a convenient way of verifying that some pattern classes are sum complete and some are minus complete. However, although the condition of Theorem 4.4 is much more tedious to check, it becomes useful when we wish to establish the sum or minus completeness of a class which fail to satisfy the condition of Theorem 4.1.

5. Serial Composition of Pattern Classes

The study of pattern classes arises in areas such as Theoretical Computer Science. Of particular interest are permuting mechanisms, such as a stack or deque, which accept a naturally ordered, finite input sequence $1 2 \ldots n$ and generate a permutation, or sort an existing permutation into ascending order. The sets of output sequences which can be generated (or input sequences which can be sorted) are closed pattern classes because they avoid certain patterns which cannot be generated (or sorted) by the mechanism. Permuting mechanisms and their associated pattern classes are studied in greater depth in [1, 2, 3, 4]. We may wish to investigate the situation where an input sequence is fed first into the mechanism with associated pattern class $X$, then the output generated by $X$ is inputted into the mechanism associated with pattern class $Y$. We call this the serial composition of two closed classes $X, Y$ and denote it by $XY$.

The Coxeter Class is the closed class whose permutations of length $n \in \mathbb{N}$ are the identity permutation and the $n-1$ transpositions of the form $(i \ i + 1)$. In [2], Atkinson and Beals proved the following result:

**Proposition 5.1.** Let $X$ be the Coxeter Class and $Y$ any finitely based closed class. Then $XY$ is finitely based.

The Coxeter Class is contained in the Sum Completion Class of 1 and 21 and we can in fact extend the proof of the above proposition to the following.

**Theorem 5.2.** Let $X$ be the Sum Completion Class of 1 and 21 and $Y$ any finitely based closed class. Then $XY$ is finitely based.
Proof. Let $\pi = p_1p_2\ldots p_n$ be a permutation of length $n$. Then $\pi \notin XY$ if and only if $\pi \notin Y$ and $\tau \pi \notin Y$ where $\tau$ is the product, under composition of mappings, of any number of disjoint transpositions in $S_n$ of the form $(i\ i + 1)$. We chose $\pi$ to be minimal with respect to this property so that $\pi$ is a basis element of $XY$.

Since $\pi \notin Y$, it has a subsequence $\beta = b_1b_2\ldots b_k$ which is order isomorphic to a basis element of $Y$. Because $Y$ is finitely based, $k$ is bounded independently of $n$. Since $\beta$ is a subsequence of $\pi$, $\beta$ has a segment of the form $\beta_1\gamma_1\beta_2\gamma_2\ldots \beta_j\gamma_j\beta_{j+1}\gamma_{j+1}$ where all the $\beta_i, \gamma_i$ are non-empty segments of $\pi$, $\beta_1\beta_2\ldots \beta_j = \beta$ and $j \leq k$. For each $i = 1, 2, \ldots, j - 1$ let $g_i$ be an element of $\gamma_i$.

We now investigate the implications of the fact that $\tau\pi \notin Y$, i.e. we consider the effect of interchanging one or more disjoint pairs of adjacent symbols $p_i, p_{i+1}$ in $\pi$. Let $B(Y)$ denote the basis of $Y$. Interchanging $p_i$ and $p_{i+1}$ when at least one of them is contained in some $\gamma_i$ is trivial: the resulting permutation $\tilde{\pi}$ still involves $\beta$ which ensures that $\tilde{\pi} \notin Y$ and we do not obtain any new information. So we now interchange some $b_i, b_{i+1}$ in $\pi$. Note that we can only do this when $b_i b_{i+1} \in \beta_u$ for some $u$: If $b_i \in \beta_u$ and $b_{i+1} \in \beta_v$ for $u \neq v$, then $b_i, b_{i+1}$ cannot be interchanged because they are not adjacent in $\pi$.

So consider any $b_i b_{i+1}$ contained in some $\beta_u$. The interchange of $b_i$ and $b_{i+1}$ in $\pi$ gives a permutation that is not in $Y$. Thus $\pi$ must contain a subsequence $\lambda_r, b_i b_{i+1} \lambda_r \gamma_2$ such that $\lambda_r, b_i b_{i+1} \lambda_r$ is order isomorphic to another basis element of $Y$, $\delta_v$, say. Since $\delta_v$ is of length bounded independently of $n$, the subsequences $\lambda_r$ and $\lambda_{r+1}$ are also of bounded length.

Now consider the same $b_i b_{i+1} \in \beta_u$ together with some $b_u b_{u+1} \in \beta_u$ where neither $b_u$ nor $b_{u+1}$ are equal to either of $b_i$ and $b_{i+1}$. The interchanges of both $b_i$ and $b_{i+1}$ and of $b_u$ and $b_{u+1}$ in $\pi$ also results in a permutation that is not in $Y$. Without loss of generality, assume that $r + 1 < s$. Then this means that $\pi$ must contain a subsequence $\lambda_{r,s} \beta_u b_i b_{i+1} b_{r+2} \ldots b_{s-1} b_s \lambda_{r,s} \gamma_2$ such that $\lambda_{r,s} \beta_u b_i b_{i+1} b_{r+2} \ldots b_{s-1} b_s \lambda_{r,s} \gamma_2$ is order isomorphic to some $\delta_{r,s} \in B(Y)$. Once again, the fact that $Y$ is finitely based ensures that $\lambda_{r,s}$ are of bounded length.

Next, consider the same $b_i b_{i+1} \in \beta_u$ together with some $b_u b_{u+1} \in \beta_v$ for some $u \neq v$. Swapping both $b_i$ with $b_{i+1}$ and $b_u$ with $b_{u+1}$ again results in a permutation which is not in $Y$. Without loss of generality, suppose $u < v$. Then by the above reasoning, we have $\pi \geq \lambda_{r,s} b_i b_{i+1} \lambda_{r,t} b_u b_{u+1} \lambda_{r,t} \lambda_{r,s}$ such that $\lambda_{r,s} b_i b_{i+1} b_u b_{u+1} \lambda_{r,t} \lambda_{r,s} \equiv \delta_{r,t}$ for some $\delta_{r,t} \in B(Y)$. Again, since $Y$ is finitely based, all $\lambda_{r,s}$ are of bounded length.

Similarly, we could swap all of $b_i, b_u$, and $b_t$ with their respective following symbols to obtain more subsequences $\lambda_{r,s,t}$ of $\pi$. Continuing in this way, there are many choices of pairs $p_i, p_{i+1} \in \pi$ that could be interchanged. Up to $n/2$ pairs can be swapped simultaneously, and we can interchange any combination of them, as long as each $p_i$ is not moved more than once. However, since $\pi$ is of finite length, the number of possible combinations is finite. Furthermore, the fact that $Y$ is finitely based will ensure that the resulting subsequences $\lambda_i$ are all bounded in length, irrespective of which symbols we choose to interchange.

Now define $\omega = w_1 w_2 \ldots w_m, m \leq n$ to be the union of all $\beta_i, g_i$ and $\lambda_i$. We now show that this subsequence of $\pi$ is not order isomorphic to any permutation of $XY$.

Since $\omega \geq \beta, \omega \notin Y$. Next, we wish to show that $\tau \omega \notin Y$, so consider the effect of interchanging any number of adjacent symbols $w_i, w_{i+1}$ of $\omega$ whilst ensuring that
no single symbol is moved to a different position more than once. Then all of the above conclusions concerning $\tau\pi$ apply to $\tau\omega$ as well: If at least one of $w_i, w_{i+1}$ does not form part of $\beta$, then the permutation which results from interchanging them still contains $\beta$ and hence is not in $Y$. So we can ignore this case and consider the case where $\tau$ interchanges $w_i, w_{i+1} = b_i b_{i+1}$ for some $i$. As before, the $g_i$ ensure that $b_i b_{i+1}$ must be in the same $\beta_u$ for some $u$ if they are interchanged. But since $\omega$ involves all $\lambda_i$, the only situations which may occur are those which have been discussed above, i.e. $\tau\omega \geq \delta$ for some $\delta \in B(Y)$. Hence $\tau\omega \notin Y$ and since $\omega \notin Y$ also, this implies that $\omega$ is not order isomorphic to any permutation in $XY$.

But $\pi \geq \omega$ and so the minimality of $\pi$ implies that $\pi = \omega$. Now $\omega$ has been defined as the union of a bounded number of sequences of bounded length all of which depend upon the basis elements of $Y$. Since there is only a finite number of elements of $B(Y)$ and they are all of finite length, there is only a finite number of possibilities of interleaving them into sequences of types $\beta_i, \gamma_i, \lambda_i$ and hence there is only a finite number of possibilities for $\pi$. $\square$

References


Acknowledgements

We would like to acknowledge the support of the Nuffield Foundation, the Carnegie Trust and the School of Mathematics and Statistics of the University of St Andrews. We would also like to thank all staff of the school, in particular the members of the CIRCA group and leaving Head of School Prof. Kenneth Falconer for their kind hospitality - including the tea and biscuits. Finally, particular thanks are extended to Dr. Sophie Huczynska and Dr. Nik Ruskuc for their invaluable help and guidance throughout our research project.