Representation Theory In and With GAP

Lecture 3

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- Rob Wilson's Atlas of Group Representations
- The GAP Share Package `AtlasRep`
- The Modular Atlas

Throughout my lectures, G denotes a finite group and K a field.

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the representation of M of degree $196\,882$ over $\mathbb{F}_2$ by Linton, Parker, Walsh and Wilson.

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Most of this information is also available in GAP through the `AtlasRep` share package.

The AtlasRep Share Package

```
gap> RequirePackage( "atlasrep" );
```

The AtlasRep Share Package

```
gap> RequirePackage( "atlasrep" );
true
```

The AtlasRep Share Package

```
gap> RequirePackage( "atlasrep" );
true
gap> ??AtlasRep
```

The `AtlasRep` Share Package

```
gap> RequirePackage( "atlasrep" );
true
gap> ??AtlasRep
# Output deleted
```

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```
gap> RequirePackage( "atlasrep" );
true
gap> ??AtlasRep
# Output deleted
gap> DisplayAtlasInfo( );
```

The `AtlasRep` Share Package

```
gap> RequirePackage( "atlasrep" );
true
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```

The AtlasRep Share Package

```
gap> DisplayAtlasInfo( "Co3", IsPermGroup );
```

The AtlasRep Share Package

```
gap> DisplayAtlasInfo( "Co3", IsPermGroup );
Representations for G = Co3:  (all refer to
std.  generators 1)
----------------------------
 1:   G <= Sym(276)
 2:   G <= Sym(552)
 3:   G <= Sym(11178)
 4:   G <= Sym(37950)
 5:   G <= Sym(48600)
 6:   G <= Sym(128800)
```

The AtlasRep Share Package

```
gap> p552 := AtlasGenerators( "Co3", 2 );;
```

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gap> p552 := AtlasGenerators( "Co3", 2 );;

# The generators of Co3 in the above
permutation representation of degree 552 are
now contained in the list ‘p552.generators’
and you can access them as follows:
```

The AtlasRep Share Package

```
gap> p552 := AtlasGenerators( "Co3", 2 );;

# The generators of Co3 in the above
permutation representation of degree 552 are
now contained in the list ‘p552.generators’
and you can access them as follows:

gap> p1 := p552.generators[ 1 ];;
gap> p2 := p552.generators[ 2 ];;
```

The AtlasRep Share Package

```
gap> DisplayAtlasInfo( "M11",
Characteristic, 2 );
```

The AtlasRep Share Package

```
gap> DisplayAtlasInfo( "M11",
Characteristic, 2 );
Representations for G = M11:  (all refer to
std.  generators 1)
----------------------------
 6:  G <= GL(10,2)
 7:  G <= GL(32,2)
 8:  G <= GL(44,2)
16:  G <= GL(16b,4)
17:  G <= GL(16a,4)
```

The `AtlasRep` Share Package

```
gap> r10 := AtlasGenerators( "M11", 6 );;
gap> r32 := AtlasGenerators( "M11", 7 );;
```

The AtlasRep Share Package

```
gap> r10 := AtlasGenerators( "M11", 6 );;
gap> r32 := AtlasGenerators( "M11", 7 );;
gap> r10.generators;
[ <an immutable 10x10 matrix over GF2>, <an
immutable 10x10 matrix over GF2> ]
```

The AtlasRep Share Package

```
gap> A1 := r10.generators[ 1 ];;
gap> A2 := r10.generators[ 2 ];;
```

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```
gap> A1 := r10.generators[ 1 ];;
gap> A2 := r10.generators[ 2 ];;
gap> B1 := r32.generators[ 1 ];;
gap> B2 := r32.generators[ 2 ];;
```

The AtlasRep Share Package

```
gap> A1 := r10.generators[ 1 ];;
gap> A2 := r10.generators[ 2 ];;
gap> B1 := r32.generators[ 1 ];;
gap> B2 := r32.generators[ 2 ];;
gap> C1 := KroneckerProduct( A1, B1 );;
gap> C2 := KroneckerProduct( A2, B2 );;
```

The AtlasRep Share Package

```
gap> A1 := r10.generators[ 1 ];;
gap> A2 := r10.generators[ 2 ];;
gap> B1 := r32.generators[ 1 ];;
gap> B2 := r32.generators[ 2 ];;
gap> C1 := KroneckerProduct( A1, B1 );;
gap> C2 := KroneckerProduct( A2, B2 );;

gap> V:=GModuleByMats( [ C1, C2 ], GF(2) );;
```

The AtlasRep Share Package

```
gap> ccfV := MTX.CollectedFactors( V );;
```

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```
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gap> time;
6610
```

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gap> time;
6610
gap> Length( ccfV );
4
```

The AtlasRep Share Package

```
gap> ccfV := MTX.CollectedFactors( V );;
gap> time;
6610
gap> Length( ccfV );
4
gap> List(ccfV, x -> MTX.Dimension( x[1] ));
[ 1, 10, 32, 44 ]
```

The AtlasRep Share Package

```
gap> ccfV := MTX.CollectedFactors( V );;
gap> time;
6610
gap> Length( ccfV );
4
gap> List(ccfV, x -> MTX.Dimension( x[1] ));
[ 1, 10, 32, 44 ]
gap> List( ccfV, x -> x[ 2 ] );
[ 4, 2, 1, 6 ]
```

Brauer Characters

Assume from now on that K is algebraically closed and has prime characteristic p.

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$\chi_{\mathfrak{X}}(1)$ only gives the degree of $\mathfrak{X}$ modulo p.

Instead one considers its Brauer character $\varphi_{\mathfrak{X}}$.

Brauer Characters

This is obtained from consistently lifting the eigenvalues of the matrices $\mathfrak{X}(g)$ for $g \in G_{p'}$ to $\mathbb{C}$.

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More precisely: Let R denote the ring of algebraic integers in the field $\mathbb{Q}(\exp(2\pi\, i/|G|))$.

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Choose a ring homomorphism $\alpha : R \longrightarrow K$.

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Choose a ring homomorphism $\alpha : R \to K$.

(A reduction modulo p of the $|G|$-ths roots of unity.)

Brauer Characters

The Brauer character (with respect to α)

$$\varphi_{\mathfrak{X}} : G_{p'} \longrightarrow R$$

satisfies

$$\alpha(\varphi_{\mathfrak{X}}(g)) = \chi_{\mathfrak{X}}(g) \quad \text{for all} \quad g \in G_{p'}.$$

The Brauer Character Table

Put $\text{IBr}_p(G) :=$ set of irreducible Brauer characters of G (all with respect to the same α), $\text{IBr}_p(G) = \{\varphi_1, \ldots, \varphi_l\}$.

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The square matrix

$$[\varphi_i(g_j)]_{1 \leq i,j \leq l}$$

is called the Brauer character table of G.

An Example in GAP

```
gap> ct := CharacterTable( "S7" )
mod 2;
```

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```
gap> ct := CharacterTable( "S7" )
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BrauerTable( "A7.2", 2 )
```

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gap> ct := CharacterTable( "S7" )
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gap> Display( ct, options );
```

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gap> ct := CharacterTable( "S7" )
mod 2;

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gap> Display( ct, options );

# The GAP-record "options" contains
display options defined previously.
```

An Example in GAP

A7.2mod2

	1a	3a	3b	5a	7a
X.1	1	1	1	1	1
X.2	8	-4	2	-2	1
X.3	6	3	.	1	-1
X.4	14	2	-1	-1	.
X.5	20	-4	-1	.	-1

The Decomposition Numbers

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These integers are called the decomposition numbers of G modulo p.

The matrix $[d_{ij}]$ is the decomposition matrix of G.

Decomposition Matrices in GAP

```
gap> c2 := CharacterTable( "A6" ) mod 2;;

gap> Display( DecompositionMatrix( c2 ) );

[ [  1,  0,  0,  0,  0 ],
  [  1,  1,  0,  0,  0 ],
  [  1,  0,  1,  0,  0 ],
  [  0,  0,  0,  1,  0 ],
  [  0,  0,  0,  0,  1 ],
  [  1,  1,  1,  0,  0 ],
  [  2,  1,  1,  0,  0 ] ]
```

Remarks

$\mathrm{IBr}_p(G)$ is linearly independent and so the decomposition numbers are uniquely determined.

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Knowing $\mathsf{Irr}(G)$ and the decomposition matrix is equivalent to knowing $\mathsf{Irr}(G)$ and $\mathsf{IBr}_p(G)$.

Tasks and Results

Describe all ordinary character tables and Brauer character tables of all finite simple groups.

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Almost finished for ordinary character tables.
For sporadic groups and other "small" groups:
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Wide open for Brauer character tables.
For Atlas groups up to McL:

An Atlas of Brauer Characters (Jansen et al.)

The Modular Atlas Project

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Methods: GAP, MOC, Meat-Axe, Condensation

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- Felix Noeske

Thank you for your attention!