Some cyclically presented groups

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Declaration

I, Elizabeth Kimber, hereby certify that this dissertation has been composed by myself, that it is a record of my own work, and that it has not been accepted in any previous application for any degree.

Signed:

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Abstract

In 1999, D.L. Johnson, A.C. Kim and E.A. O’Brien proved that the corresponding groups in two pairs of infinite families of cyclically presented groups are isomorphic. The details of their proofs are given here. Upper and lower bounds for the size of a minimum generating set for the groups in each family were proved in 2001 by M.F. Newman, with an indication that smaller upper bounds could be found. Proofs of Newman’s bounds are given, and the proof of a smaller upper bound, which is a new result, completes this dissertation.
Introduction

The main results in this dissertation concern two infinite families of cyclically presented groups, studied by D.L. Johnson, A.C. Kim, and E.A. O’Brien in [4], and by M.F. Newman in [6]. In [4], two pairs of infinite families of cyclically presented groups are considered and it is shown that corresponding groups in each pair are isomorphic (Theorem 1.1 in [4]). In two subsequent papers, further results about the two infinite families were found: in 2001 George Havas, Derek F. Holt and M.F. Newman showed that almost all of the groups in each of these families are infinite, see [1], and in another 2001 paper, [6], M.F. Newman provided upper and lower bounds for the size of a minimum generating set for each of the groups in these families; see Theorem 1 and Theorem 8 in [6].

As the methods used in the proofs in [4] and [6] are similar for each of the families, the full details of the proofs of the results for both families are not given in these papers. The details of the proof of Theorem 1.1 in [4] are only given for the second pair of families of groups. In [6], only the proof
of Theorem 1 is given, with a statement of the results for the second family (Theorem 8). Summaries of the proofs in these papers and full proofs of the other results are given here.

The proof of a further result completes this dissertation. The mathematics computer package ACE, see [9] was used to find generating sets of the appropriate size in order to help produce a proof of Theorem 8 in [6]. In doing this, smaller generating sets were found for some of the groups in the second family. The pattern suggested by these experiments with ACE was generalized and proved to hold for all the groups in the family. This provided a smaller upper bound for the size of a minimum generating set for each of the groups in the second family, and the proof of this new result is given.
Chapter 1

Preliminary ideas

1.1 Some theory

1.1.1 Tietze transformations

In the proofs of the first results in this dissertation, Tietze transformations are used to produce new presentations for given groups. Let \( \langle X \mid R \rangle \) be a presentation for a group \( G \). Let \( F \) be the free group on \( X \) and \( \bar{R} \) be the normal closure of \( R \) in \( F \). Tietze transformations are defined below, with the resulting generating set \( X' \) and the resulting set of relators \( R' \) given for each of the transformations; see [3], p.35.

If \( r \) is contained in \( \bar{R} \), but \( r \) is not in \( R \), then relator \( r \) can be added and \( X' = X, R' = R \cup \{r\} \).

If a \( r \) is in \( \bar{R} \setminus \{r\} \), then \( r \) can be removed and \( X' = X, R' = R \setminus \{r\} \).
If \( y \notin X \), but \( y = w \in F \), then a generator \( y \) can be added and \( X' = X \cup \{y\}, R' = R \cup \{y^{-1}w\} \).

If \( y^{-1}w \) is the only element of \( R \) that involves \( y \) and \( w \) is in the subgroup generated by \( X \setminus \{y\} \), then generator \( y \) can be removed. This gives \( X' = X \setminus \{y\} \) and \( R' = R \setminus \{y^{-1}w\} \).

Given two groups, \( G_1 \) and \( G_2 \), with presentations \( P_1 \) and \( P_2 \) respectively, if \( P_2 \) can be obtained from \( P_1 \) by a finite sequence of Tietze transformations, then \( G_1 \cong G_2 \).

1.1.2 Group automorphisms

A group automorphism is an isomorphism of a group onto itself. Let \( G \) be a group and let \( a \) be an element of \( G \). Define a map \( \alpha : G \to G \) to be conjugation by \( a \), i.e. \( g\alpha = a^{-1}ga \). Then \( \alpha \) is an automorphism; see [2] p.68.

Group automorphisms that are conjugation are called inner automorphisms. In particular, if \( c \) and \( a \) are in the same conjugacy class of \( G \), then there is an inner automorphism of \( G \) mapping \( c \) to \( a \).

Lemmas counting the number of elements in \( A_5 \) that satisfy certain conditions are used in the proofs of the lower bounds for the size of a minimum generating set for the groups in the two families. These lemmas are proved using GAP; see [8]. The conjugacy classes of \( A_5 \) are used to restrict the number of computations required in the proofs. For example, in Lemma 5 of
[6], for any element $a$ of order three in $A_5$, it is required to find the number of pairs of elements $(c, d)$ of order three that satisfy the conditions

$$(c^{-1}d)^2 = 1, \text{ and } (a^{-1}c)^2 = (d^{-1}a)^2 = 1.$$  \hfill (1.1)

As the 3-cycles are all in one conjugacy class in $A_5$, it is sufficient to consider just one choice for $a$, for if $b$ is any other 3-cycle and $c_b, d_b$ satisfy (1.1), then there is an element $h$ in $A_5$ such that $h^{-1}bh = a$, and $h^{-1}ch$ and $h^{-1}dh$ satisfy (1.1).

### 1.2 Some cyclically presented groups

We define a cyclically presented group, following the notation used in [6], and then give some examples.

Let $F$ be the free group on a countable set of generators $\{x_1, x_2, \ldots\}$. Let $w = w(x_1, x_2, \ldots, x_r)$ be a word in the generators $x_1, x_2, \ldots, x_r$ and for any $n$ with $n \geq r$, let $G_w(n)$ be the group generated by $x_1, x_2, \ldots, x_n$ with defining relators $w, w\theta_n, w\theta_n^2, \ldots, w\theta_n^{n-1}$ where $\theta_n$ is the automorphism of $F$ that maps $x_i$ to $x_{i+1}$ for $i = 1, 2, \ldots, n-1$, maps $x_n$ to $x_1$, and maps all other generators to themselves. For convenience, if $G$ is the group defined by the presentation $\langle X \mid R \rangle$, then we write $G = \langle X \mid R \rangle$, so

$G_w(n) = \langle x_1, x_2, \ldots, x_n \mid w(x_1, x_2, \ldots, x_r), w(x_1, x_2, \ldots, x_r)\theta_n,$

$$w(x_1, x_2, \ldots, x_r)\theta_n^2, \ldots, w(x_1, x_2, \ldots, x_r)\theta_n^{n-1} \rangle.$$  \hfill (1.2)

A group $G$ is said to be cyclically presented if $G \cong G_w(n)$ for some $w, n$. 

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For example the Fibonacci groups $F(2,n)$, which are defined by the presentation
\[
\langle x_1, \ldots, x_n \mid x_1x_2x_3^{-1}, x_2x_3x_4^{-1}, \ldots, x_{n-2}x_{n-1}x_n^{-1}, x_{n-1}x_nx_1^{-1}, x_nx_1x_2^{-1} \rangle
\]
are cyclically presented. In particular, $C_{11}$, the cyclic group of order eleven, is cyclically presented because $C_{11} \cong F(2,5)$, and the quaternion group of order eight $Q_8$ is cyclically presented because $Q_8 \cong F(2,3)$. Other examples of cyclically presented groups are contained in larger sets of groups. For example, the Mennicke groups, $M(a,b,c)$, are the groups defined by the presentation
\[
\langle x_1, x_2, x_3 \mid x_2^{-1}x_1x_2x_1^{-a}, x_3^{-1}x_2x_3x_2^{-b}, x_1^{-1}x_3x_1x_3^{-c} \rangle.
\]
The groups $G_w(3)$, where $w = x_1^{-1}x_3x_1x_3^{-a}$, which are defined by this presentation with $a = b = c$, are a subset of the Mennicke groups; see [3].

There is a polynomial $f_{G_w(n)}(t)$ associated with $G_w(n)$. The coefficient of $t^{i-1}$ is the exponent sum of $x_i$ in $w$. For instance, the polynomial associated with the Fibonacci groups is $-t^2 + t + 1$, and the polynomial associated with the cyclically presented Mennicke groups is $(1-a)t^2$. 
Chapter 2

Two pairs of infinite families of cyclically presented groups

This chapter contains the details of the proofs given in [4] that the corresponding groups in each of two pairs of families of groups are isomorphic. With a change of notation in order to be consistent with our definition of a cyclically presented group, the groups in the first pair of families are

\[ G_{u_1}(n) = \langle x_1, \ldots, x_n \mid u_1, u_1 \theta_n, \ldots, u_1 \theta_n^{n-1} \rangle, \]

(2.1)

\[ G_{u_2}(n) = \langle y_1, \ldots, y_n \mid u_2, u_2 \theta_n, \ldots, u_2 \theta_n^{n-1} \rangle, \]

(2.2)

where

\[ u_1 = x_2^{-1} x_3 x_2^{-1} x_3 x_1 x_2^{-1} x_1 \]

and

\[ u_2 = y_2^{-1} y_3 y_1 y_2^{-1} y_3 y_2^{-1} y_1. \]
and the groups in the second pair are

\[ H_{v_1}(n) = \langle x_1, \ldots, x_n \mid v_1, v_1\theta_n, \ldots, v_1\theta_n^{n-1} \rangle, \]

\[ H_{v_2}(n) = \langle y_1, \ldots, y_n \mid v_2, v_2\theta_n, \ldots, v_2\theta_n^{n-1} \rangle, \]

where

\[ v_1 = x_2^{-1}x_3x_2^{-1}x_3x_2^{-2}x_1x_2^{-1}x_1 \]

and

\[ v_2 = y_2^{-1}y_3y_2^{-1}y_1y_2^{-1}y_3y_2^{-2}y_1. \]

The families are the sets of these groups as \( n \) ranges over \( \mathbb{N} \).

The groups defined by the presentations \( G_{u_1}(n) \) and \( G_{u_2}(n) \) are listed in the table on page 219 of \([5]\). This table contains cyclically presented groups that are the fundamental groups of manifolds that are cyclic branched coverings of knots or links. The polynomials associated with these presentations are equivalent to the Alexander polynomials of the appropriate knots; see Theorem 5.1 and Corollary 5.1 in \([5]\). As this dissertation is about cyclically presented groups, the details of how these groups arise are not included. Some of these details can be found in \([5]\) and \([7]\). In 1999, D.L. Johnson, A.C. Kim, and E.A. O’Brien considered the two pairs of families of cyclically presented groups given above because they provided a partial answer to the question of whether all cyclically presented groups with the same Alexander
polynomial are isomorphic. In [4] (Theorem 1.1) it is proved that, for all 

\[ n \geq 1, \; G_1(n) \cong G_2(n) \text{ and } H_1(n) \cong H_2(n). \]

The polynomials associated with these presentations are

\[ f_{G_{u_1}(n)}(t) = 2t^2 - 3t + 2 = f_{G_{u_2}(n)}(t) \]

and

\[ f_{H_{v_1}(n)}(t) = 2t^2 - 5t + 2 = f_{H_{v_2}(n)}(t). \]

Using the notation above, Theorem 1.1 in [4] is:

**Theorem 2.0.1** For all \( n \geq 1, \; G_{u_1}(n) \cong G_{u_2}(n) \text{ and } H_{v_1}(n) \cong H_{v_2}(n). \)

In [4] the details of the proof are given for the second family, but some are
omitted for the first family. A more detailed proof for each family is given here.

We begin with the proof that \( G_{u_1}(n) \cong G_{u_2}(n). \) Tietze transformations are
used to obtain new presentations for \( G_{u_1}(n) \) and \( G_{u_2}(n). \) A map between the
generators of these new presentations is extended to a map between \( G_{u_2}(n) \)
and \( G_{u_1}(n), \) and this map is shown to be an isomorphism.

The first step is to find a new presentation for \( G_{u_2}(n). \) We begin by re-
indexing the generators \( y_1, \ldots, y_n \) of \( G_{u_2}(n). \) We do this by writing \(-i\) for
\( i, \) and then replacing \( y_i \) by \( y_{-i}. \) With respect to this new set of generators,
$G_{u_2}(n)$ has presentation

$$\langle y_1, y_2, \ldots, y_n \mid \hat{u}_2, \hat{u}_2\theta_n, \ldots, \hat{u}_2\theta_n^{n-1} \rangle$$

where $\hat{u}_2 = y_2^{-1}y_1y_3y_2^{-1}y_1y_2^{-1}y_3$.

Relator $\hat{u}_2\theta_{n-1}^i$ in this presentation is $y_{2+i-1}^{-1}y_{1+i-1}y_{3+i-1}y_{2+i-1}^{-1}y_{1+i-1}y_{2+i-1}^{-1}y_{3+i-1}$, which is

$$y_{i+1}^{-1}y_{i+2}y_{i+1}^{-1}y_{i+1}y_{i+2}^{-1}.$$ 

Therefore, from the relation $1 = \hat{u}_2\theta_{n-1}^i$ we have

$$y_i = y_{i+1}y_{i+2}y_{i+1}^{-1}y_{i+1}y_{i+2}^{-1}.$$ 

Writing $b_i$ for $y_{i+1}y_i^{-1}$, we have

$$y_i = b_i^{-1}b_i^{-1}. \quad (2.5)$$

Using Tietze transformations, we add new generators $b_1, \ldots, b_n$ to give a new presentation

$$\langle y_1, y_2, \ldots, y_n, b_1, b_2, \ldots, b_n \mid y_1 = b_2^{-1}b_1b_2^{-1}, \ldots, y_{n-1} = b_n^{-1}b_{n-1}b_n^{-1}, \quad \text{where} \quad y_n = b_1^{-1}b_nb_1^{-1}, b_1 = y_2y_1^{-1}, \ldots, b_{n-1} = y_ny_{n-1}, b_n = y_1y_n^{-1} \rangle. \quad (2.6)$$

As (2.6) is obtained from (2.2) by a finite sequence of Tietze transformations, (2.6) is also a presentation of $G_{u_2}(n)$; see [3].

We also give a new presentation for $G_{u_1}(n)$. Relator $u_1\theta_{n-1}^i$ is

$$x_i^{-1}x_{i+1}^{-1}x_i^{-1}x_{i+2}^{-1}x_ix_{i+1}^{-1}x_i^{-1}x_i.$$
We rearrange the relation \(1 = u_1 \theta_n^{-1}\) to get

\[
x_i = x_{i+2}^{-1}x_{i+1}^{-1}x_{i+2}^{-1}x_{i+1}^{-1}x_i^{-1}x_{i+1}^{-1}
\]

\[
= (x_{i+2}^{-1}x_{i+1}^{-1})x_{i+1}^{-1}x_i^{-1}x_{i+1}^{-1}
\]

\[
= (x_{i+2}^{-1}x_{i+1}^{-1})(x_i^{-1}x_i^{-1})^{-1}.
\]

Using Tietze transformations we add the generators \(a_1, a_2, \ldots, a_n\), where

\[
a_i = x_{i+1}^{-1}x_i^{-1}.
\]

This gives us a new presentation for \(G_{u_1}(n)\) :

\[
\langle x_1, \ldots, x_n, a_1, \ldots, a_n \mid x_1 = a_2a_1^{-1}, \ldots, x_{n-1} = a_na_{n-1}^{-1}, x_n = a_1a_n^{-1},
\]

\[
a_1 = x_2^{-1}x_1^{-1}x_2^{-1}, \ldots, a_{n-1} = x_n^{-1}x_{n-1}^{-1}x_n^{-1}, a_n = x_1^{-1}x_n^{-1}\rangle.
\]

We define a map between the generating sets in (2.6) and (2.7), and extend it to a map between \(G_{u_2}(n)\) and \(G_{u_1}(n)\), which is then shown to be a homomorphism. Note that \(G_{u_2}(n)\) is generated by the set \(\{y_1, \ldots, y_n\}\) and \(G_{u_1}(n)\) is generated by the set \(\{a_1, \ldots, a_n\}\). Let \(y_i\) be mapped to \(a_i\) for \(i = 1, \ldots, n\).

Extend this map to a map \(\alpha_G\) from (2.6) to (2.7) by

\[
w(y_1, \ldots, y_n)\alpha_G = w(y_1\alpha_G, \ldots, y_n\alpha_G) \quad \text{for any word } w \text{ in the } y_i.
\]

Therefore \(b_i\alpha_G = (y_{i+1}\alpha_G)(y_i\alpha_G)^{-1} = a_{i+1}a_i^{-1} = x_i\).

We check that the relations in (2.6) are preserved by \(\alpha_G\); i.e. we check that
$y_i \alpha_G = (b_{i+1}^{-1} b_i b_{i+1}^{-1}) \alpha_G$, for $i = 1, \ldots, n$.

$$(b_{i+1}^{-1} b_i b_{i+1}^{-1}) \alpha_G = (b_{i+1}^{-1} \alpha_G)^{-1} (b_i \alpha_G) (b_{i+1}^{-1} \alpha_G)^{-1}$$

$$= x_{i+1}^{-1} x_i x_{i+1}^{-1}$$

$$= a_i$$

$$= y_i \alpha_G,$$

so $\alpha_G$ is a homomorphism by the Substitution test, see [3] p.29.

Similarly, we define a map $\beta_G$ from $G_{u_1}(n)$ to $G_{u_2}(n)$ by mapping the generator $a_i$ to the generator $y_i$ and extending this map to a map from $G_{u_1}(n)$ to $G_{u_2}(n)$. Therefore $x_i \beta_G = (a_{i+1} \beta_G) (a_i \beta_G)^{-1} = y_{i+1} y_i^{-1} = b_i$. Again by the Substitution test, $\beta_G$ is a homomorphism because

$$(x_{i+1}^{-1} x_i x_{i+1}^{-1}) \beta_G = (x_{i+1} \beta_G)^{-1} (x_i \beta_G) (x_{i+1} \beta_G)^{-1}$$

$$= b_{i+1}^{-1} b_i b_{i+1}^{-1}$$

$$= y_i$$

$$= a_i \beta_G.$$

By construction, $\beta_G$ is the inverse of $\alpha_G$, and hence $G_{u_2}(n) \cong G_{u_1}(n)$.

The proof that $H_{v_1}(n) \cong H_{v_2}(n)$ follows in a very similar way. As in the proof of the previous result, we use relations rather than relators.

Relator $v_1 \theta_n^i$ is

$$x_i^{-1} x_{i+1} x_i^{-1} x_{i+1} x_i^{-1} x_i^{-1} x_{i-1} x_i^{-1},$$
so from the relation $v_1 \theta_n^{i-2} = 1$, we have

$$x_i = x_{i-1}^{-1} x_i^{-1} x_{i+1}^{-1} x_i x_{i+1} x_i^{-1}$$

$$= (x_{i-1} x_i^{-1} x_{i+1}^{-1}) (x_i x_{i+1}^{-1})$$

$$= (x_{i-1} x_i^{-1}) (x_i x_{i+1}^{-1} x_i x_{i+1} x_i^{-1})$$

so if $a_i = x_{i-1} x_i^{-1}$, we have

$$x_i = a_i^2 a_{i+1}^{-2}.$$  \hspace{1cm} (2.8)

This gives a new presentation for $H_{v_1}(n)$:

$$\langle x_1, \ldots, x_n, a_1, \ldots, a_n \mid x_1 = a_1^2 a_2^{-2}, \ldots, x_{n-1} = a_{n-1}^2 a_n^{-2}, x_n = a_n^2 a_1^{-2}, a_1 = x_n x_1^{-1}, a_2 = x_1 x_2^{-1}, \ldots, a_n = x_{n-1} x_n^{-1} \rangle.$$ \hspace{1cm} (2.9)

A new presentation for $H_{v_2}(n)$ is obtained in a similar way. Relator $v_2 \theta_n^{i-2}$ is

$$y_i^{-1} y_{i+1}^{-1} y_i^{-1} y_{i-1}^{-1} y_{i+1}^{-1} y_i^{-1} y_{i-1}^{-1} y_i^{-1} y_{i+1}^{-1}.$$ \hspace{1cm}

Therefore, from the relation $v_2 \theta_n^{i-2} = 1$ we have

$$y_i = y_{i-1} y_i^{-1} y_{i+1}^{-1} y_i^{-1} y_{i-1}^{-1} y_{i+1}^{-1} y_i^{-1} y_{i+1}^{-1}$$

$$= (y_{i-1} y_i^{-1} y_{i+1}^{-1} y_i^{-1})^2$$

so if we put $b_i = y_i y_{i+1}^{-1}$, we have

$$y_i = (b_{i-1} b_i^{-1})^2.$$  \hspace{1cm} (2.10)

Therefore we have another presentation for $H_{v_2}(n)$:

$$\langle y_1, \ldots, y_n, b_1, \ldots, b_n \mid y_1 = (b_1 b_1^{-1})^2, y_2 = (b_1 b_2^{-1})^2, \ldots, y_n = (b_{n-1} b_n^{-1})^2,$$

$$b_1 = y_1 y_2^{-1}, \ldots, b_{n-1} = y_{n-1} y_n^{-1}, b_n = y_n y_1^{-1} \rangle.$$ \hspace{1cm} (2.11)
We map $x_i$ to $b_i$ and extend this to a map $\alpha_H$ from $H_{v_1}(n)$ to $H_{v_2}(n)$. Then we have

$$a_i\alpha_H = (x_{i-1}x_i^{-1})\alpha_H = (x_{i-1}\alpha_H)(x_i\alpha_H)^{-1} = b_{i-1}b_i^{-1} \quad (2.12)$$

and hence

$$a_i^2\alpha_H = (a_i\alpha_H)^2$$

$$= (b_{i-1}b_i^{-1})^2 \quad (2.13)$$

$$= y_i.$$

As

$$(a_i^2a_{i+1}^{-2})\alpha_H = (a_i^2\alpha_H)(a_{i+1}\alpha_H)^{-2}$$

$$= y_i(b_ib_{i+1}^{-1})^{-2} \quad \text{(by (2.13) & (2.12))}$$

$$= y_iy_{i+1}^{-1}$$

$$= b_i$$

$$= x_i\alpha_H$$

the relations in (2.9) are preserved, and hence, by the Substitution test, $\alpha_H$ is a homomorphism.

As $x_i = a_i^2a_{i+1}^{-2}$ and $a_i = x_{i-1}x_i^{-1}$, we have

$$a_i = x_{i-1}x_i^{-1}$$

$$= (a_i^2a_i^{-2})(a_i^2a_{i+1}^{-2})^{-1}$$

$$= a_i^2a_i^{-2}a_{i+1}a_i^{-2}$$

and hence $G_{u_2}(n)$ is generated by the set $\{a_1^2, \ldots, a_n^2\}$. Let $y_i$ be mapped to
$a_i^2$. This can be extended to a map $\beta_H$ from (2.11) to (2.9). Then

$$b_i \beta_H = (y_i y_{i+1}^{-1}) \beta_H$$

$$= (y_i \beta_H)(y_{i+1}\beta_H)^{-1}$$

$$= a_i^2 a_{i+1}^{-2}$$

$$= x_i,$$

and hence

$$(b_{i-1} b_i^{-1})^2 \beta_H = ((b_{i-1} \beta_H)(b_i \beta_H)^{-1})^2$$

$$= (x_{i-1} x_i^{-1})^2$$

$$= a_i^2$$

$$= y_i \beta_H$$

so $\beta_H$ is a homomorphism, and as it is the inverse of $\alpha_H$, we have proved Theorem 2.0.1.
Chapter 3

Upper and lower bounds for the generator number of the groups in two families of cyclically presented groups

As we have seen that $G_{u_1}(n) \cong G_{u_2}(n)$ and $H_{v_1}(n) \cong H_{v_2}(n)$, from now on we will just consider $G_{u_1}(n)$ and $H_{v_1}(n)$, and denote these groups by $G(n)$ and $H(n)$ respectively. Relator $w\theta_n^{i-1}$ in the presentation of $G_w(n)$ will be denoted by $r_i$.

The generator number of a group $G$ is defined to be the size of a minimum generating set, and denoted by $d(G)$. In 2001 M.F. Newman published a paper containing upper and lower bounds for $d(G(n))$ and $d(H(n))$; see [6]. It is remarked in [4] that both families of groups may be 2-generator groups. The lower bounds for $d(G(n))$ and $d(H(n))$ given in [6] prove that this only
true for small \( n \). The details of the proofs of the results for \( d(G(n)) \) are given in [6], and it is indicated that the proofs of the results for \( d(H(n)) \) follow a similar method, although no proofs are given. Summaries of the proofs given in [6] for \( d(G(n)) \) are given here, followed by proofs of the results for \( d(H(n)) \).

3.1 Upper bounds for \( d(G(n)) \) and \( d(H(n)) \)

The method of proof is similar for the groups in both families: for each \( n \), a generating set for \( G(n) \) or \( H(n) \) of the required size is produced.

3.1.1 An upper bound for \( d(G(n)) \)

**Theorem 3.1.1** The generator number of the group \( G(n) \) is at most \[
\frac{n + 1}{2}.
\]

We begin with a lemma.

**Lemma 3.1.2** For \( i \in \{1, 2, \ldots, n-3\} \), the group defined by the presentation
\[
\langle x_i, x_{i+1}, x_{i+2}, x_{i+3} \mid x_{i+1}^{-1} x_{i+1} x_{i+2} x_{i+3} x_{i+1}^{-1} x_{i+1} x_{i+2} x_{i+3} x_{i+1}^{-1} x_{i+1} x_{i+2} x_{i+3} x_{i+1}^{-1} x_{i+1} \rangle
\]
can be generated by the set \( \{x_i, x_{i+2}, x_{i+3}\} \).
Proof. Taking the product of the two relators in the presentation, we have
\[
1 = (x_{i+2}^{-1}x_{i+3}^{-1}x_{i+2}^{-1}x_{i+3}x_{i+1}^{-1}x_{i+2}x_i^{-1}x_{i+1}^{-1}x_i)(x_{i+1}^{-1}x_{i+2}x_{i+1}^{-1}x_{i+2}x_i^{-1}x_{i+1}^{-1}x_i)
\]
\[
= x_{i+2}^{-1}x_{i+3}^{-1}x_{i+2}^{-1}x_{i+3}x_{i+2}x_i^{-1}x_i
\]
\[
= x_{i+1}^{-1}x_{i+2}^{-1}x_{i+3}^{-1}x_{i+2}^{-1}x_{i+3}x_{i+2}x_i^{-1}x_{i+1}^{-1}x_i
\]
(by multiplying on the left by \(x_i\) and on the right by \(x_i^{-1}\))
and hence
\[
x_{i+1} = x_i x_{i+2} x_i x_{i+3} x_i x_{i+2} x_i.
\]

□

Theorem 3.1.3. The group \(G(n)\) can be generated by the set
\[
\{x_1, x_2, x_4, x_6, \ldots, x_{n-2}, x_n\} \text{ if } n \text{ is even}
\]
and by \(\{x_1, x_2, \ldots, x_{n-3}, x_{n-1}\}\) if \(n\) is odd.

Proof. Consider first the case \(n\) is even. By Lemma 3.1.2, \(x_{2k+1}\) is in the subgroup generated by \(\{x_{2k}, x_{2k+2}, x_{2k+3}\}\) for \(k \in \{1, 2, \ldots, (n-4)/2\}\). It remains to show that \(x_{n-1}\) is in the subgroup generated by \(\{x_{n-2}, x_n, x_1\}\).
We take the product of the relators \(r_{n-1}\) and \(r_{n-2}\) to show that \(x_{n-1}\) can be written as a word in \(\{x_{n-2}, x_n, x_1\}\). Relator \(r_{n-1}\) is
\[
x_n^{-1} x_1 x_{n-1} x_n^{-1} x_1 x_{n-1} x_n^{-1} x_{n-1}
\]
and relator \(r_{n-2}\) is
\[
x_{n-1}^{-1} x_n x_{n-1} x_n x_{n-2} x_{n-1}^{-1} x_{n-2},
\]

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so

\[ 1 = r_{n-1} r_{n-2} \]

\[ = (x_n^{-1} x_1 x_n^{-1} x_1 x_{n-1} x_n^{-1} x_{n-1}) (x_n^{-1} x_n x_{n-1} x_{n-2} x_{n-1} x_{n-2}) \]

\[ = x_n^{-1} x_1 x_n^{-1} x_1 x_n x_{n-2} x_{n-1}^{-1} x_{n-2} \]

\[ = x_{n-2} x_n^{-1} x_1 x_n x_{n-2} x_{n-1}^{-1} \]

and hence

\[ x_{n-1} = x_{n-2} x_n^{-1} x_1 x_n^{-1} x_1 x_n x_{n-2}. \]

Now consider the case \( n \) is odd. By Lemma 3.1.2, \( x_{2k+1} \) is in the subgroup generated by \( \{x_{2k}, x_{2k+2}, x_{2k+3}\} \) for \( k \in \{1, 2, \ldots, (n-3)/2\} \). It remains to show that \( x_n \) is in the subgroup generated by \( \{x_{n-1}, x_1, x_2\} \). We use the same method as before. Relator \( r_n \) is

\[ x_1^{-1} x_2 x_1^{-1} x_2 x_n^{-1} x_1 x_n, \]

so

\[ 1 = r_n r_{n-1} \]

\[ = (x_1^{-1} x_2 x_1^{-1} x_2 x_n^{-1} x_n) (x_1^{-1} x_1 x_n^{-1} x_1 x_n^{-1} x_n^{-1} x_n^{-1} x_{n-1}) \]

\[ = x_1^{-1} x_2 x_1^{-1} x_2 x_1 x_n^{-1} x_n^{-1} x_{n-1} \]

\[ = x_{n-1} x_1^{-1} x_2 x_1 x_n^{-1} x_n^{-1} \]

and hence

\[ x_n = x_{n-1} x_1^{-1} x_2 x_1 x_n^{-1} x_n^{-1}. \]
Theorem 3.1.3 shows that
\[ d(G(n)) \leq \frac{n + 2}{2} \] if \( n \) is even,
and
\[ d(G(n)) \leq \frac{n + 1}{2} \] if \( n \) is odd.

Lemma 3.1.4 The group
\[ \langle x_n, x_1, x_2 \mid x_1^{-1}x_2x_1^{-1}x_2x_nx_1^{-1}x_n \rangle \]

can be generated by the set \( \{x_1^{-1}x_n, x_1^{-1}x_2\} \).

Proof As \( x_1^{-1}x_2x_1^{-1}x_2x_nx_1^{-1}x_n \) can be written as
\[ (x_1^{-1}x_2)^2x_n(x_1^{-1}x_n), \] (3.1)
\( x_n \) is in the subgroup generated by \( \{x_1^{-1}x_n, x_1^{-1}x_2\} \). Also, \( x_1 = x_n(x_1^{-1}x_n)^{-1} \), so \( x_1 \) is in the subgroup generated by \( \{x_1^{-1}x_n, x_1^{-1}x_2\} \), and \( x_2 = x_1(x_1^{-1}x_2) \), so \( x_2 \) is in the subgroup generated by \( \{x_1^{-1}x_n, x_1^{-1}x_2\} \).

By applying the last lemma, we see that \( \{x_1^{-1}x_2, x_4, x_6, \ldots, x_{n-2}, x_1^{-1}x_n\} \) generates \( G(n) \) for \( n \) even. This set has size \( n/2 \). Therefore, for all \( n \),
\[ d(G(n)) \leq \left\lfloor \frac{n + 1}{2} \right\rfloor, \]
which completes the proof of Theorem 3.1.1.
3.1.2 An upper bound for $d(H(n))$

**Theorem 3.1.5** The generator number of the group $H(n)$ is at most $2n/3$.

A proof of Theorem 3.1.5 is not given in [6]. The proof given here is based on the proof of Theorem 3.1.1. The result is a corollary to a theorem that provides a generating set of the appropriate size. The proof uses the fact that, for any $n$, the relators $r_1, \ldots, r_{n-2}$ in the presentation of $H(n)$ are also relators in the corresponding presentation of $H(m)$ for $m \geq n$.

Consider first $H(3)$, which has presentation

$$\langle x_1, x_2, x_3 \mid x_2^{-1}x_3x_2^{-1}x_3x_2^{-2}x_1x_2^{-1}x_1, x_3^{-1}x_1x_3^{-1}x_1x_3^{-2}x_2x_3^{-1}x_2, x_1^{-1}x_2x_1^{-1}x_2x_1^{-2}x_3x_1^{-1}x_3 \rangle.$$  

**Lemma 3.1.6** The generators $x_1, x_2, x_3$ of $H(3)$ are in the subgroup generated by $x_2^{-1}x_1$ and $x_2^{-1}x_3$.

**Proof** We use only the first relator, $r_1$, $x_2^{-1}x_3x_2^{-1}x_3x_2^{-2}x_1x_2^{-1}x_1$, to show that $x_1, x_2,$ and $x_3$ can be written as words in $x_2^{-1}x_1$ and $x_2^{-1}x_3$.

We can write $r_1$ in the form $(x_2^{-1}x_3)^2x_2^{-1}(x_2^{-1}x_1)^2$, so rearranging the relation $1 = r_1$ gives

$$x_2 = (x_2^{-1}x_1)^2(x_2^{-1}x_3)^2. \quad (3.2)$$
Similarly, we can write \( r_1 \) in the form \((x_2^{-1}x_3)^2x_2^{-2}x_1(x_2^{-1}x_1)\), and then use (3.2), to give
\[
1 = r_1 = (x_2^{-1}x_3)^2x_2^{-2}x_1(x_2^{-1}x_1)
\]
\[(x_2^{-1}x_3)^2(x_2^{-1}x_3)^{-2}(x_2^{-1}x_1)^{-2}(x_2^{-1}x_3)^{-2}(x_2^{-1}x_1)^{-2}x_1(x_2^{-1}x_1) \text{ (by (3.2))}
\]
and hence
\[
x_1 = (x_2^{-1}x_1)^2(x_2^{-1}x_3)^2(x_2^{-1}x_1)^2(x_2^{-1}x_1)^{-1},
\]
so
\[
x_1 = (x_2^{-1}x_1)^2(x_2^{-1}x_3)^2(x_2^{-1}x_1).
\] (3.3)

Finally, we re-write \( r_1 \) again, this time in the form \( x_2^{-1}x_3(x_2^{-1}x_3)x_2^{-1}(x_2^{-1}x_1)^2 \).

We substitute for \( x_2 \), giving
\[
1 = r_1 = x_2^{-1}x_3(x_2^{-1}x_3)x_2^{-1}(x_2^{-1}x_1)^2
\]
\[(x_2^{-1}x_3)^{-2}(x_2^{-1}x_1)^{-2}x_3(x_2^{-1}x_3)x_2^{-1}(x_2^{-1}x_1)^{-2} \text{ (by (3.2))}
\]
\[(x_2^{-1}x_3)^{-2}(x_2^{-1}x_1)^{-2}x_3(x_2^{-1}x_3)(x_2^{-1}x_3)^{-2}(x_2^{-1}x_1)^{-2}(x_2^{-1}x_1)^2 \text{ (by (3.2))}
\]
\[(x_2^{-1}x_3)^{-2}(x_2^{-1}x_1)^{-2}x_3(x_2^{-1}x_3)^{-1}
\]
and hence
\[
x_3 = (x_2^{-1}x_1)^2(x_2^{-1}x_3)^3.
\] (3.4)

This lemma proves that \( H(3) \) is generated by \( \{x_2^{-1}x_1, x_2^{-1}x_3\} \).
As \( r_1 \) is a relator in this presentation of \( H(n) \) for \( n \geq 3 \), we can apply Lemma 3.1.6 to \( H(n) \), for all \( n \geq 3 \). Moreover, the result can be generalized, and we show this in the proof of the next result.

**Theorem 3.1.7** Let \( n = 3m + t \) where \( t \in \{0, 1, 2\} \) and \( m \in \mathbb{N} \). Let

\[
A = \{ x_{k+2}^{-1}x_{k+1}, x_{k+2}^{-1}x_{k+3} : k = 0, 1, \ldots, m - 1 \}.
\]

If \( t = 0 \) then \( H(n) \) is generated by \( A \).

If \( t = 1 \) then \( H(n) \) is generated by \( A \).

If \( t = 2 \) then \( H(n) \) is generated by the set \( A' = A \cup \{ x_n^{-1}x_{n-1} \} \).

**Proof** We use the same argument as that used in the proof of Lemma 3.1.6 to show that \( x_{3k+1}, x_{3k+2}, \) and \( x_{3k+3} \) are in the subgroup generated by \( x_{3k+2}^{-1}x_{3k+1} \) and \( x_{3k+2}^{-1}x_{3k+3} \) for \( k = 0, \ldots, m - 1 \).

We use only the relator \( r_{3k+1} \) to prove the result for each \( k \). Relator \( r_{3k+1} \) is

\[
x_{3k+2}^{-1}x_{3k+3}^{-1}x_{3k+2}^{-1}x_{3k+3}^{-2}x_{3k+2}^{-1}x_{3k+1}^{-1}x_{3k+2}^{-1}x_{3k+1}.
\]

We write this in the form

\[
(x_{3k+2}^{-1}x_{3k+3})^2 x_{3k+2}^{-1} (x_{3k+2}^{-1}x_{3k+1})^2,
\]

so

\[
x_{3k+2} = (x_{3k+2}^{-1}x_{3k+1})^2 (x_{3k+2}^{-1}x_{3k+3})^2.
\] (3.5)
Now we write $r_{3k+1}$ in the form $x_{3k+2}^{-1}(x_{3k+3}^{-1}x_{3k+1}^{-1})x_{3k+2}^{-1}(x_{3k+2}^{-1}x_{3k+1})^2$.

Using $r_{3k+1}$ in this form and using (3.5) we show that $x_{3k+3}$ can be written as a word in $x_{3k+2}^{-1}x_{3k+1}$ and $x_{3k+2}^{-1}x_{3k+3}$.

\[1 = r_{3k+1}\]
\[= x_{3k+2}^{-1}x_{3k+3}(x_{3k+2}^{-1}x_{3k+3})x_{3k+2}^{-1}(x_{3k+2}^{-1}x_{3k+1})^2\]
\[= (x_{3k+2}^{-1}x_{3k+3})^{-2}(x_{3k+2}^{-1}x_{3k+1})^{-2}x_{3k+3}(x_{3k+2}^{-1}x_{3k+3})x_{3k+2}^{-1}(x_{3k+2}^{-1}x_{3k+1})^2\]
\[= (x_{3k+2}^{-1}x_{3k+3})^{-2}(x_{3k+2}^{-1}x_{3k+1})^{-2}x_{3k+3}(x_{3k+2}^{-1}x_{3k+3})^{-2}\]
\[= (x_{3k+2}^{-1}x_{3k+3})^{-2}(x_{3k+2}^{-1}x_{3k+1})^{-2}x_{3k+3}(x_{3k+2}^{-1}x_{3k+3})^{-1}\]

and so

\[x_{3k+3} = (x_{3k+2}^{-1}x_{3k+3})^2(x_{3k+2}^{-1}x_{3k+3})^3.\] (3.6)

Finally, we write $r_{3k+1}$ as $(x_{3k+2}^{-1}x_{3k+3})^2x_{3k+2}^{-1}x_{3k+1}(x_{3k+2}^{-1}x_{3k+1})$, which gives

\[1 = r_{3k+1}\]
\[= (x_{3k+2}^{-1}x_{3k+3})^2x_{3k+2}^{-1}x_{3k+1}(x_{3k+2}^{-1}x_{3k+1})\]
\[= (x_{3k+2}^{-1}x_{3k+3})^2(x_{3k+2}^{-1}x_{3k+3})^{-2}(x_{3k+2}^{-1}x_{3k+1})^{-2}(x_{3k+2}^{-1}x_{3k+3})^{-2}\]
\[= (x_{3k+2}^{-1}x_{3k+3})^{-2}x_{3k+1}(x_{3k+2}^{-1}x_{3k+1})^{-2}x_{3k+1}(x_{3k+2}^{-1}x_{3k+3})^{-2}\]
\[= (x_{3k+2}^{-1}x_{3k+3})^{-2}(x_{3k+2}^{-1}x_{3k+1})^{-2}x_{3k+1}(x_{3k+2}^{-1}x_{3k+3})^{-2}\]

and hence

\[x_{3k+1} = (x_{3k+2}^{-1}x_{3k+3})^2(x_{3k+2}^{-1}x_{3k+3})^2(x_{3k+2}^{-1}x_{3k+1}).\] (3.7)
This completes the proof of the result for the case $t = 0$ because we have shown that generators $x_1, x_2, \ldots, x_n$ are in the subgroup generated by the set $A$.

Now suppose $t = 1$. We have already shown that generators $x_1, \ldots, x_{n-1}$ are in the subgroup generated by $A$. It remains to show that $x_n$ is also in this subgroup. Consider relator $r_n$. This is $x_1^{-1}x_2x_1^{-1}x_2x_1^{-2}x_nx_1^{-1}x_n$. From the relation $1 = r_n$, we have

$$x_n^{-1}x_1x_n^{-1}x_1 = x_1^{-1}x_2x_1^{-1}x_2x_1^{-1}. \quad (3.8)$$

Relator $r_{n-1}$ is

$$x_n^{-1}x_1x_n^{-1}x_1x_n^{-2}x_{n-1}x_n^{-1}x_{n-1},$$

and relator $r_{n-2}$ is

$$x_{n-1}^{-1}x_nx_{n-1}^{-1}x_nx_{n-2}x_{n-1}^{-1}x_{n-2},$$

so

$$1 = r_{n-1}r_{n-2}$$

$$= (x_n^{-1}x_1x_n^{-1}x_1x_n^{-2}x_{n-1}x_n^{-1}x_{n-1})(x_{n-1}^{-1}x_nx_{n-1}^{-1}x_nx_{n-2}x_{n-1}^{-1}x_{n-2})$$

$$= x_n^{-1}x_1x_n^{-1}x_1x_n^{-2}x_{n-1}x_n^{-2}x_{n-2}x_{n-1}x_{n-2}$$

$$= x_1^{-1}x_2x_1^{-1}x_2x_1^{-1}x_n^{-1}x_n^{-2}x_{n-1}^{-1}x_{n-2}^{-1}(\text{by (3.8)})$$

and hence

$$x_n = x_n^{-2}x_n^{-1}x_{n-1}^{-1}x_{n-2}^{-1}x_1^{-1}x_2x_1^{-1}x_2x_1^{-1}, \quad (3.9)$$

so, as required, $x_n$ is in the subgroup generated by $A$. 

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Finally, suppose \( t = 2 \). We have already shown that \( x_1, \ldots, x_{n-2} \) are in the subgroup generated by \( A \). It remains to show that \( x_{n-1} \) and \( x_n \) are in the subgroup generated by \( A' \). We show that \( x_{n-1} \) is in the required subgroup, and then it follows easily that \( x_n \) is also in this subgroup.

Relator \( r_{n-2} \) is

\[
x_{n-1}^{-1}x_nx_{n-1}^{-1}x_nx_n^{-2}x_{n-2}x_{n-1}^{-1}x_n^{-2}
\]

and relator \( r_{n-3} \) is

\[
x_{n-2}^{-1}x_{n-1}^{-1}x_nx_{n-1}^{-1}x_n^{-2}x_{n-3}x_{n-2}^{-1}x_n^{-3}
\]

Therefore we have

\[
1 = r_{n-2}r_{n-3}
\]

\[
= (x_{n-1}^{-1}x_nx_{n-1}^{-1}x_nx_n^{-2}x_{n-2}x_{n-1}^{-1}x_n^{-2})(x_{n-2}^{-1}x_{n-1}^{-1}x_n^{-1}x_{n-2}x_{n-1}^{-1}x_n^{-2}x_{n-3}x_{n-2}^{-1}x_n^{-3})
\]

\[
= x_{n-1}^{-1}x_nx_{n-1}^{-1}x_nx_{n-1}^{-1}x_n^{-2}x_{n-3}x_{n-2}^{-1}x_n^{-3}
\]

\[
= (x_{n-1}^{-1}x_n^{-1})^{-2}x_{n-1}^{-1}x_n^{-1}(x_{n-3}x_{n-2})^{-2}
\]

and hence

\[
x_{n-1} = x_{n-2}^{-1}(x_{n-3}x_{n-2})^{-2}(x_{n-1}^{-1}x_n^{-1})^{-2}
\]  \( (3.10) \)

so \( x_{n-1} \) is in the subgroup generated by \( A' \).

As \( x_n = x_{n-1}(x_n^{-1}x_{n-1})^{-1} \), \( x_n \) is also in the subgroup generated by \( A' \). This completes the proof of Theorem 3.1.7.

\[\square\]
Corollary 3.1.8 \( d(H(n)) \) is at most \( 2n/3 \).

Proof The result follows from Theorem 3.1.7. If \( n \equiv 0 \pmod{3} \) then \( H(n) \) has a generating set of size \( 2n/3 \). If \( n \equiv 1 \pmod{3} \) then \( H(n) \) has a generating set of size \( 2(n - 1)/3 \). If \( n \equiv 2 \pmod{3} \) then \( H(n) \) has a generating set of size \( (2n - 1)/3 \).

\[ \square \]

3.2 Lower bounds for \( d(G(n)) \) and \( d(H(n)) \)

The proof given in [6] of the result for \( d(G(n)) \) is summarised and then a proof of the result for \( d(H(n)) \) is given.

3.2.1 A lower bound for \( d(G(n)) \)

Theorem 3.2.1 \( d(G(n)) \) is bounded below by

\[
\frac{\log 24}{\log 60} \left\lfloor \frac{n - 6}{4} \right\rfloor.
\]

Let \( G \) be a group with generator number \( d \). The number of homomorphisms from \( G \) to \( A_5 \) is at most \( 60^d \). The proof uses this fact to give a lower bound for \( d(G(n)) \).
A special homomorphism from $G(n)$ to $A_5$ is defined to be a homomorphism $\phi$ that satisfies

(S1) $x_1^3 \phi = 1$,
(S2) $x_1 \phi \neq 1$,
(S3) $x_n^3 \phi = 1$,
(S4) $x_n \phi \neq 1$,
(S5) $(x_n^{-1} x_1)^2 \phi = 1$,
(S6) $x_1 x_2 \phi = 1$,
(S7) $x_{n-1} x_n \phi = 1$.

If $\xi(G(n))$ is the number of special homomorphisms from $G(n)$ to $A_5$, then

$$\xi(G(n)) \leq 60^d(G(n)),$$

so

$$\frac{\log \xi(G(n))}{\log 60} \leq d(G(n)).$$

We need to find a lower bound for $\xi(G(n))$. In Lemma 5 in [6], it is proved that given non-trivial elements $a, b$ of order 3 in $A_5$ such that $(a^{-1}b)^2 = 1$, there are at least 24 pairs $(c, d)$ of non-trivial elements of $A_5$ such that

$$c^3 = d^3 = (c^{-1}d)^2 = 1 \quad \text{and} \quad (a^{-1}c)^2 = (d^{-1}b)^2 = 1.$$  \hspace{1cm} (3.11)
Given any special homomorphism $\phi$ from $G(n)$ to $A_5$, define a map

$$\psi : G(n + 4) \rightarrow A_5$$

by

$$y_i \psi = x_i \phi \text{ for } i \in \{1, 2, \ldots, n\}$$

$$y_{n+1} \psi = c, \ y_{n+2} \psi = c^{-1}, \ y_{n+3} \psi = d^{-1}, \ y_{n+4} = d$$

where $y_1, y_2, \ldots, y_{n+4}$ are the generators of $G(n+4)$, and $c, d$ satisfy (3.11). This map is shown to be a special homomorphism. Therefore, for each special homomorphism from $G(n)$ to $A_5$ there are at least 24 distinct special homomorphisms $\psi$ from $G(n+4)$ to $A_5$ such that $y_i \psi = x_i \phi$ for $i \in \{1, \ldots, n\}$.

One special homomorphism from $G(n)$ to $A_5$ is exhibited for $n = 6, 7, 8, 9$, so by induction, applying the previous result, there are at least $24^{\lfloor \frac{n-6}{4} \rfloor}$ special homomorphisms from $G(n)$ to $A_5$. Therefore $\xi(G(n))$ is bounded below by $24^{\lfloor \frac{n-6}{4} \rfloor}$, so we have

$$\log 24 \left\lfloor \frac{n-6}{4} \right\rfloor \leq d(G(n)).$$

□

3.2.2 A lower bound for $d(H(n))$

**Theorem 3.2.2** The generator number of $H(n)$ is bounded below by

$$\frac{\log 16}{\log 60} \left\lfloor \frac{n-6}{4} \right\rfloor.$$
The proof of the lower bound for \( d(H(n)) \) is similar in method to the proof given of the lower bound for \( d(G(n)) \).

**Lemma 3.2.3** Let \( a \) be a non-trivial element of order 5 in \( A_5 \). There are exactly 16 pairs \((c, d)\) of non-trivial elements of \( A_5 \) such that

\[
c^5 = d^5 = (c^{-1}d)^2 = 1 \quad \text{and} \quad (a^{-1}c)^2 = (d^{-1}a)^2 = 1.
\]

**Proof** There are two conjugacy classes of 5-cycles in \( A_5 \), with representatives \((1 2 3 4 5)\) and \((1 2 3 5 4)\), so for any 5-cycle \( a \) in \( A_5 \) there is an inner automorphism mapping \( a \) to \((1 2 3 4 5)\) if \( a \in (1 2 3 4 5) \) and \( a \) to \((1 2 3 5 4)\) if \( a \in (1 2 3 5 4) \). Therefore we only need to consider two cases. GAP [8] was used to verify the result for each case; see the GAP code in Appendix A. In the first case \( a \) was taken to be \((1 2 3 4 5)\) and in the second case \( a \) was taken to be \((1 2 3 5 4)\).

\[\square\]

**Lemma 3.2.4** Let \( a, b \) be non-trivial elements in \( A_5 \) such that \( a^5 = b^5 = 1 \) and \((a^{-1}b)^2 = 1\). There are exactly 16 pairs \((c, d)\) of non-trivial elements in \( A_5 \) such that \( c^5 = d^5 = (c^{-1}d)^2 = 1 \) and \((a^{-1}c)^2 = (d^{-1}b)^2 = 1\).

**Proof** For each 5-cycle \( a \) in \( A_5 \) there are six choices in \( A_5 \) for \( b \) such that \((a^{-1}b)^2 = 1\). Let \( a \in A_5 \) be an element of order 5. Suppose \( a \in \mathcal{C} \), for
Then there is an inner automorphism $\sigma$ of $A_5$ such that $a\sigma = c$. Let $b$ be such that $(a^{-1}b)^2 = 1$. Then as $\sigma$ is a homomorphism, $b\sigma$ is such that $(a^{-1}\sigma b\sigma)^2 = 1$, so without loss of generality, we just need to consider the two cases $a = (1 2 3 4 5)$ and $a = (1 2 3 5 4)$. In each case GAP was used to verify the lemma for each $b$ such that $(a^{-1}b)^2 = 1$. See the GAP code in Appendix A.

\[\square\]

We use the result of Lemma 3.2.4 to prove the next result. As in the proof of the lower bound for $d(G(n))$, the lower bound for $d(H(n))$ is proved by constructing homomorphisms from $H(n+4)$ to $A_5$ out of certain homomorphisms from $H(n)$ to $A_5$. The generators of $H(n)$ are denoted by $x_1, x_2, \ldots, x_n$ and the generators of $H(n + 4)$ are denoted by $y_1, y_2, \ldots, y_{n+4}$. The particular homomorphisms from $H(n)$ to $A_5$ are slightly different from those used in the proof for $G(n)$. In this case, a homomorphism $\phi$ from $H(n)$ to $A_5$ is called special if it satisfies conditions (S2) and (S4)-(S7) above, together with

(S1') $x_1^5 \phi = 1$,

(S3') $x_n^5 \phi = 1$.

**Lemma 3.2.5** Let $\phi : H(n) \to A_5$ be a special homomorphism. There are at least 16 distinct special homomorphisms $\psi$ from $H(n + 4)$ to $A_5$ such that $y_i \psi = x_i \phi$ for $i \in \{1, 2, \ldots, n\}$. 

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Proof Let $a = x_n \phi$ and $b = x_1 \phi$. As $\phi$ is special, $a^5 = (x_n \phi)^5 = x_n^5 \phi = 1$, and $a$ is non-trivial. Similarly, $b^5 = 1$, and $b$ is non-trivial. Also,

$$(a^{-1}b)^2 = (x_n \phi^{-1} x_1 \phi)^2 = (x_n^{-1} x_1)^2 \phi = 1,$$

so we can apply Lemma 3.2.4, and conclude that there are 16 pairs $(c, d)$ of elements of $A_5$ such that

$$c^5 = d^5 = (a^{-1}c)^2 = (c^{-1}d)^2 = (d^{-1}b)^2 = 1. \quad (3.12)$$

For each pair $(c, d)$, define $\psi : H(n + 4) \to A_5$ by

$$y_i \psi = x_i \phi \text{ for } i \in \{1, 2, \ldots, n\}$$

$$y_{n+1} \psi = c, \quad y_{n+2} \psi = c^{-1}, \quad y_{n+3} \psi = d^{-1}, \quad y_{n+4} = d.$$

We show first that $\psi$ is a homomorphism, and then show that it is special.

By the Substitution test ([3] p.29), if $v(y_i \psi, y_{i+1} \psi, y_{i+2} \psi) = 1$ for $i$ in the set \{1, \ldots, n + 4\}, then $\psi$ is a homomorphism.

For $i \in \{1, 2, \ldots, n - 2\}$, as $\phi$ is a homomorphism, we have

$$v(y_i \psi, y_{i+1} \psi, y_{i+2} \psi) = v(x_i \phi, x_{i+1} \phi, x_{i+2} \phi)$$

$$= (v(x_i, x_{i+1}, x_{i+2})) \phi$$

$$= 1 \phi$$

$$= 1.$$
The remaining relators are considered separately.

\[ v(y_{n-1} \psi, y_n \psi, y_{n+1} \psi) = v(x_{n-1} \phi, a, c) \]

\[ = a^{-1}ca^{-1}ca^{-2}(x_{n-1} \phi)a^{-1}(x_{n-1} \phi) \]

\[ = (a^{-1}c)^2a^{-2}(x_{n-1} \phi)a^{-1}(x_{n-1} \phi) \]

\[ = a^{-2}(x_{n-1} \phi)a^{-1}(x_{n-1} \phi) \quad \text{(by (3.12))} \]

\[ = a^{-2}(x_{n-1} \phi)a^{4}(x_{n-1} \phi) \quad \text{(by (3.12))} \]

\[ = a^{-2}(x_{n-1} \phi)(x_n \phi)a^3(x_{n-1} \phi) \quad \text{(since } a = x_n \phi) \]

\[ = a^{-2}(x_{n-1} x_n \phi)a^3(x_{n-1} \phi) \]

\[ = a(x_{n-1} \phi) \quad \text{(by (S7))} \]

\[ = a(x_{n-1} \phi)a^5 \]

\[ = a(x_{n-1} \phi)(x_n \phi)a^4 \]

\[ = a^5 \]

\[ = 1 \]

\[ v(y_n \psi, y_{n+1} \psi, y_{n+2} \psi) = v(a, c, c^{-1}) \]

\[ = c^{-1}c^{-1}c^{-1}c^{-2}ac^{-1}a \]

\[ = c^{-5}(c^{-1}a)^2 \]

\[ = (c^{-1}a)^2 \quad \text{(by (3.12))} \]

\[ = 1 \quad \text{(by (3.12))} \]
\[ v(y_{n+1} \psi, y_{n+2} \psi, y_{n+3} \psi) = v(c, c^{-1}, d^{-1}) \]
\[ = cd^{-1}cd^{-1}c^{-2}ccc \]
\[ = (cd^{-1})^2c^5 \]
\[ = (cd^{-1})^2 \quad \text{(by (3.12))} \]
\[ = 1 \quad \text{(by (3.12))} \]

\[ v(y_{n+2} \psi, y_{n+3} \psi, y_{n+4} \psi) = v(c^{-1}, d^{-1}, d) \]
\[ = ddddd^2c^{-1}dc^{-1} \]
\[ = d^5(dc^{-1}) \]
\[ = 1 \quad \text{(by (3.12))} \]

\[ v(y_{n+3} \psi, y_{n+4} \psi, y_1 \psi) = v(d^{-1}, d, b) \]
\[ = d^{-1}bd^{-1}bd^{-2}d^{-1}d^{-1}d^{-1}d^{-1} \]
\[ = (d^{-1}b)^2d^{-5} \]
\[ = 1 \quad \text{(by(3.12))} \]
\[ v(y_{n+4}\psi, y_1\psi, y_2\psi) = v(d, b, x_2\phi) \]
\[ = b^{-1}(x_2\phi)b^{-1}(x_2\phi)b^{-2}db^{-1}d \]
\[ = b^{-1}(x_2\phi)b^{-1}(x_2\phi)b^{-1}(b^{-1}d) \]
\[ = b^{-1}(x_2\phi)b^{-1}(x_2\phi)b^{-1} \quad \text{(by (3.12))} \]
\[ = b^4(x_2\phi)b^4(x_2\phi)b^{-1} \quad \text{(by (3.12))} \]
\[ = b^3(x_1\phi)(x_2\phi)b^3(x_2\phi)b^{-1} \quad \text{(since } b = x_1\phi) \]
\[ = b^3(x_1x_2\phi)b^3(x_1x_2\phi)b^{-1} \]
\[ = b^5 \quad \text{(by (S6))} \]
\[ = 1. \]

Therefore \( \psi \) is a homomorphism. It remains to show that \( \psi \) is special:

(S1') \( y_1^5\psi = (y_1\psi)^5 = (x_1\phi)^5 = x_1^5\phi = 1, \)

(S2) \( y_1\psi = x_1\phi \neq 1, \)

(S3') \( y_{n+4}^5\psi = (y_{n+4}\psi)^5 = d^5 = 1, \)

(S4) \( y_{n+4}\psi = d \neq 1, \)

(S5) \( (y_{n+4}^{-1}y_1)^2\psi = (y_{n+4}\psi)(y_1\psi)(y_{n+4}\psi)(y_1\psi) = d^{-1}bd^{-1}b = 1, \)

(S6) \( y_1y_2\psi = (y_1\psi)(y_2\psi) = (x_1\phi)(x_2\phi) = x_1x_2\phi = 1, \)

(S7) \( y_{n+3}y_{n+4}\psi = (y_{n+3}\psi)(y_{n+4}\psi) = d^{-1}d = 1. \)
This completes the proof of Lemma 3.2.5.

\[\square\]

Let \( \xi(H(n)) \) be the number of special homomorphisms from \( H(n) \) to \( A_5 \). Then \( \xi(H(n)) \leq 60^{d(H(n))} \), so

\[
\frac{\log \xi(H(n))}{\log 60} \leq d(H(n)). \tag{3.13}
\]

\textbf{GAP} was used to construct special homomorphisms from \( H(n) \) to \( A_5 \) for \( n \in \{8, 9, 10, 11\} \). By taking \( x_n \phi \) to be \( (1 2 3 4 5) \) and \( x_1 \phi \) to be in the set \( \{(1 2 3 4 5), (1 4 3 2 5)\} \), then (S1'), (S2),(S3'),(S4) and (S5) are satisfied. Setting \( x_2 \phi \) to be \( (x_1 \phi)^{-1} \) and \( x_{n-1} \phi \) to be \( (x_n \phi)^{-1} \) ensures that if \( \phi \) is a homomorphism, it is special. \textbf{GAP} was used to find possibilities for \( x_i \phi \) for \( i \in \{3, \ldots, n-2\} \). We exhibit one special homomorphism from \( H(n) \) to \( A_5 \) for \( n = 8 \) and \( n = 9 \). As the method of proof that the given maps are special homomorphisms is the same for each \( n \), the details of the proof are only given for the case \( n = 8 \).

\textbf{Lemma 3.2.6} The map \( \phi : H(8) \to A_5 \), given by \( x_1 \phi = (1 4 3 2 5) \), \( x_2 \phi = (1 5 2 3 4) \), \( x_3 \phi = (1 5 4 3 2) \), \( x_4 \phi = (1 2 3 4 5) \), \( x_5 \phi = (1 2 3 4 5) \), \( x_6 \phi = (1 5 4 3 2) \), \( x_7 \phi = (1 5 4 3 2) \), and \( x_8 \phi = (1 2 3 4 5) \) is a special homomorphism.

\textbf{Proof} We show first that \( v(x_i \phi, x_{i+1} \phi, x_{i+2} \phi) = 1 \) for \( i \in \{1, 2, \ldots, 8\} \).
\[ v(x_1\phi, x_2\phi, x_3\phi) = (1\ 4\ 3\ 2\ 5)(1\ 5\ 4\ 3\ 2)(1\ 4\ 3\ 2\ 5)(1\ 5\ 4\ 3\ 2)(1\ 4\ 3\ 2\ 5) \]
\[ = (1\ 4\ 3\ 2\ 5)(1\ 5\ 4\ 3\ 2)(1\ 4\ 3\ 2\ 5)(1\ 5\ 4\ 3\ 2) \]
\[ = 1 \]

\[ v(x_2\phi, x_3\phi, x_4\phi) = (1\ 2\ 3\ 4\ 5)(1\ 2\ 3\ 4\ 5)(1\ 2\ 3\ 4\ 5)(1\ 2\ 3\ 4\ 5)(1\ 2\ 3\ 4\ 5) \]
\[ = (1\ 2\ 3\ 4\ 5)(1\ 5\ 2\ 3\ 4)(1\ 2\ 3\ 4\ 5)(1\ 5\ 2\ 3\ 4) \]
\[ = 1 \]

\[ v(x_3\phi, x_4\phi, x_5\phi) = (1\ 5\ 4\ 3\ 2)(1\ 2\ 3\ 4\ 5)(1\ 5\ 4\ 3\ 2)(1\ 2\ 3\ 4\ 5)(1\ 5\ 4\ 3\ 2) \]
\[ = (1\ 5\ 4\ 3\ 2)(1\ 2\ 3\ 4\ 5)(1\ 5\ 4\ 3\ 2)(1\ 2\ 3\ 4\ 5) \]
\[ = 1 \]

\[ v(x_4\phi, x_5\phi, x_6\phi) = (1\ 5\ 4\ 3\ 2)(1\ 5\ 4\ 3\ 2)(1\ 5\ 4\ 3\ 2)(1\ 5\ 4\ 3\ 2)(1\ 5\ 4\ 3\ 2) \]
\[ = (1\ 5\ 4\ 3\ 2)(1\ 2\ 3\ 4\ 5)(1\ 5\ 4\ 3\ 2)(1\ 2\ 3\ 4\ 5) \]
\[ = 1 \]
\[ v(x_5 \phi, x_6 \phi, x_7 \phi) = (1 2 3 4 5)(1 5 4 3 2)(1 2 3 4 5)(1 5 4 3 2)(1 2 3 4 5) \]
\[ = (1 2 3 4 5)(1 5 4 3 2)(1 2 3 4 5)(1 5 4 3 2) \]
\[ = 1 \]

\[ v(x_6 \phi, x_7 \phi, x_8 \phi) = (1 2 3 4 5)(1 2 3 4 5)(1 2 3 4 5)(1 2 3 4 5)(1 2 3 4 5) \]
\[ = (1 2 3 4 5)(1 5 4 3 2)(1 2 3 4 5)(1 5 4 3 2) \]
\[ = 1 \]

\[ v(x_7 \phi, x_8 \phi, x_1 \phi) = (1 5 4 3 2)(1 4 3 2 5)(1 5 4 3 2)(1 4 3 2 5)(1 5 4 3 2) \]
\[ = (1 5 4 3 2)(1 4 3 2 5)(1 5 4 3 2)(1 4 3 2 5) \]
\[ = 1 \]

\[ v(x_8 \phi, x_1 \phi, x_2 \phi) = (1 5 2 3 4)(1 5 2 3 4)(1 5 2 3 4)(1 5 2 3 4)(1 5 2 3 4) \]
\[ = (1 5 2 3 4)(1 2 3 4 5)(1 5 2 3 4)(1 2 3 4 5) \]
\[ = 1 \]

so \( \phi \) is a homomorphism.

Now we show that \( \phi \) is special. Clearly (S1'), (S2), (S3') and (S4) hold, and
(S5) \((x_6^{-1}x_1)^2 = ((x_6\phi)^{-1}x_1\phi)^2 = ((15432)(14325))^2 = 1,\)

(S6) \(x_1x_2\phi = x_1\phi x_2\phi = (14325)(15234) = 1\)

(S7) \(x_5x_6\phi = x_5\phi x_6\phi = (15432)(12345) = 1,\)

so \(\phi\) is a special homomorphism.

\[\square\]

**Lemma 3.2.7** The map \(\phi : H(9) \rightarrow A_5\) given by
\(x_1\phi = x_9\phi(12345), x_2\phi = x_8\phi = (15432), (x_3\phi)^{-1} = x_4\phi = (12345), (x_5\phi)^{-1} = x_7\phi = (153), x_7\phi = (153),\) is a special homomorphism.

We exhibit 16 special homomorphisms from \(H(10)\) to \(A_5\) and from \(H(11)\) to \(A_5\). The proofs that these maps are special homomorphisms are similar to the proof for \(n = 8\), so they are omitted.

Let \(\phi_j : H(10) \rightarrow A_5\) be given by
\(x_1\phi_j = (12345), x_2\phi_j = (15432), x_9\phi_j = (15432), x_{10}\phi_j = (12345), x_4\phi_j = (x_3\phi_j)^{-1}, x_6\phi_j = (x_5\phi_j)^{-1},\) and
\(x_8\phi_j = (x_7\phi_j)^{-1}\) for \(j \in \{1, 2, \ldots, 16\}\) where
\[x_3\phi_1 = (12435), x_5\phi_1 = (12453), \text{ and } x_7\phi_1 = (14532),\]
\[x_3\phi_2 = (12435), x_5\phi_2 = (12453), \text{ and } x_7\phi_2 = (12543),\]
\[x_3\phi_3 = (12435), x_5\phi_3 = (14352), \text{ and } x_7\phi_3 = (14325),\]
\[
x_3 \phi_4 = (12435), \ x_5 \phi_4 = (14352), \text{ and } x_7 \phi_4 = (14532),
\]
\[
x_3 \phi_5 = (12435), \ x_5 \phi_5 = (15243), \text{ and } x_7 \phi_5 = (15423),
\]
\[
x_3 \phi_6 = (12435), \ x_5 \phi_6 = (15243), \text{ and } x_7 \phi_6 = (12543),
\]
\[
x_3 \phi_7 = (12435), \ x_5 \phi_7 = (14235), \text{ and } x_7 \phi_7 = (15423),
\]
\[
x_3 \phi_8 = (12435), \ x_5 \phi_8 = (14235), \text{ and } x_7 \phi_8 = (14325),
\]
\[
x_3 \phi_9 = (13452), \ x_5 \phi_9 = (14523), \text{ and } x_7 \phi_9 = (14532),
\]
\[
x_3 \phi_{10} = (13452), \ x_5 \phi_{10} = (14523), \text{ and } x_7 \phi_{10} = (15423),
\]
\[
x_3 \phi_{11} = (13452), \ x_5 \phi_{11} = (14352), \text{ and } x_7 \phi_{11} = (14532),
\]
\[
x_3 \phi_{12} = (13452), \ x_5 \phi_{12} = (14352), \text{ and } x_7 \phi_{12} = (14325),
\]
\[
x_3 \phi_{13} = (13452), \ x_5 \phi_{13} = (13425), \text{ and } x_7 \phi_{13} = (15342),
\]
\[
x_3 \phi_{14} = (13452), \ x_5 \phi_{14} = (13425), \text{ and } x_7 \phi_{14} = (14325),
\]
\[
x_3 \phi_{15} = (13452), \ x_5 \phi_{15} = (13542), \text{ and } x_7 \phi_{15} = (15423),
\]
\[
x_3 \phi_{16} = (13452), \ x_5 \phi_{16} = (13542), \text{ and } x_7 \phi_{16} = (15342).
\]

Let \( \phi : H(11) \to A_5 \) be given by \( x_8 \phi_j = x_1 \phi_j \) and \( x_9 \phi = x_2 \phi_j = (x_1 \phi_j)^{-1} \),
\( x_4 \phi = x_{11} \phi_j \) and \( x_3 \phi_j = x_{10} \phi_j = (x_{11} \phi_j)^{-1} \), \( x_6 \phi_j = 1 \), and \( x_7 \phi_j = (x_5 \phi_j)^{-1} \),
where

\[
x_1 \phi_1 = (14325), \ x_5 \phi_1 = (124), \text{ and } x_{11} \phi_1 = (12345),
\]

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$x_1\phi_2 = (14325)$, $x_5\phi_2 = (245)$, and $x_{11}\phi_2 = (12345)$,

$x_1\phi_3 = (12345)$, $x_5\phi_3 = (254)$, and $x_{11}\phi_3 = (14325)$,

$x_1\phi_4 = (12345)$, $x_5\phi_4 = (142)$, and $x_{11}\phi_4 = (14325)$,

$x_1\phi_5 = (12435)$, $x_5\phi_5 = (254)$, and $x_{11}\phi_5 = (15432)$,

$x_1\phi_6 = (12435)$, $x_5\phi_6 = (253)$, and $x_{11}\phi_6 = (15432)$,

$x_1\phi_7 = (13452)$, $x_5\phi_7 = (153)$, and $x_{11}\phi_7 = (15432)$,

$x_1\phi_8 = (13452)$, $x_5\phi_8 = (253)$, and $x_{11}\phi_8 = (15432)$,

$x_1\phi_9 = (13245)$, $x_5\phi_9 = (142)$, and $x_{11}\phi_9 = (15432)$,

$x_1\phi_{10} = (13245)$, $x_5\phi_{10} = (134)$, and $x_{11}\phi_{10} = (15432)$,

$x_1\phi_{11} = (15234)$, $x_5\phi_{11} = (254)$, and $x_{11}\phi_{11} = (15432)$,

$x_1\phi_{12} = (15234)$, $x_5\phi_{12} = (142)$, and $x_{11}\phi_{12} = (15432)$,

$x_1\phi_{13} = (12354)$, $x_5\phi_{13} = (153)$, and $x_{11}\phi_{13} = (15432)$,

$x_1\phi_{14} = (12354)$, $x_5\phi_{14} = (143)$, and $x_{11}\phi_{14} = (15432)$,

$x_1\phi_{15} = (15324)$, $x_5\phi_{15} = (125)$, and $x_{11}\phi_{15} = (14253)$,

$x_1\phi_{16} = (15324)$, $x_5\phi_{16} = (123)$, and $x_{11}\phi_{16} = (14253)$.

The next lemma completes the proof of Theorem 3.2.2.
Lemma 3.2.8 For $n \geq 8$, $\xi(H(n)) = 16^{\lfloor (n-6)/4 \rfloor}$.

Proof Let $n = 6 + 4k + l$ where $l \in \{0, 1, 2, 3\}$ and $k \in \mathbb{N}$. As $\frac{n-6}{4} = \frac{l}{4} + k$, we just need to show that there are $16^k$ special homomorphisms from $H(n)$ to $A_5$. We use induction and consider the cases $l = 0, 1, 2, 3$ separately. We have already shown that the result is true for $n = 8$ and $n = 9$.

Case 1: $l = 0$.

Base Step Let $k = 1$. We have already shown that there are 16 special homomorphisms from $H(10)$ to $A_5$.

Inductive Step Suppose that there are $16^k$ special homomorphisms from $H(6 + 4k)$ to $A_5$. By Lemma 3.2.5, for each of these special homomorphisms there are at least 16 special homomorphisms from $H(6 + 4k + 4)$ to $A_5$ i.e. there are at least $16^{k+1}$ special homomorphisms from $H(6 + 4(k + 1))$ to $A_5$.

Case 2: $l = 1$.

Base Step Let $k = 1$. There are 16 special homomorphisms from $H(11)$ to $A_5$.

Inductive Step Suppose that there are $16^k$ special homomorphisms from $H(6 + 4k + 1)$ to $A_5$. By Lemma 3.2.5, for each of these special homomorphisms there are at least 16 special homomorphisms from $H(6 + 4k + 1 + 4)$ to $A_5$ i.e. there are at least $16^{k+1}$ special homomorphisms from $H(6 + 4(k + 1) + 1)$
to $A_5$.

**Case 3: $l = 2$.**

**Base Step** Let $k = 1$, so $n = 12$. We have shown that there is a special homomorphism from $H(8)$ to $A_5$, so by Lemma 3.2.5 there are at least 16 special homomorphisms from $H(12)$ to $A_5$.

**Inductive Step** Suppose that there are $16^k$ special homomorphisms from $H(6+4k+2)$ to $A_5$. By Lemma 3.2.5, for each of these special homomorphisms there are at least 16 special homomorphisms from $H(6 + 4k + 2 + 4)$ to $A_5$, i.e. there are at least $16^{k+1}$ special homomorphisms from $H(6 + 4(k + 1) + 2)$ to $A_5$.

**Case 4: $l = 3$.**

**Base Step** Let $k = 1$, so $n = 13$. We have shown that there is a special homomorphism from $H(9)$ to $A_5$, so by Lemma 3.2.5 there are at least 16 special homomorphisms from $H(13)$ to $A_5$.

**Inductive Step** Suppose that there are $16^k$ special homomorphisms from $H(6+4k+3)$ to $A_5$. By Lemma 3.2.5, for each of these special homomorphisms there are at least 16 special homomorphisms from $H(6 + 4k + 3 + 4)$ to $A_5$, i.e. there are at least $16^{k+1}$ special homomorphisms from $H(6 + 4(k + 1) + 3)$ to $A_5$. This completes the proof of Lemma 3.2.8.
Therefore, by (3.13)

\[ d \geq \frac{\log 16}{\log 60} \left\lfloor \frac{n - 6}{4} \right\rfloor. \]
Chapter 4

A smaller upper bound for the generator number of groups in a family of cyclically presented groups

It is indicated in [6] that the upper bound for \(d(H(n))\) can be reduced. ACE [9] was used experimentally within GAP to find smaller generating sets for \(H(n)\) for \(n \leq 15\). As every third relator had been used to write sets of three generators in terms of two new generators in the proof of Newman’s upper bound, I looked for patterns between generating sets for \(H(n)\) with \(n\) increasing in steps of four.

For example, I found that \(H(4)\) can be generated by \(\{x_2^{-1}x_1, x_2^{-1}x_3\}\), \(H(8)\) can be generated by the set \(\{x_2^{-1}x_1, x_2^{-1}x_3, x_6^{-1}x_5, x_6^{-1}x_7\}\) and \(H(12)\) can be generated by the set \(\{x_2^{-1}x_1, x_2^{-1}x_3, x_6^{-1}x_5, x_6^{-1}x_7, x_{10}^{-1}x_9, x_{10}^{-1}x_{11}\}\).
$H(5)$ can be generated by the set $\{x_2^{-1}x_1, x_2^{-1}x_3, x_5^{-1}x_4\}$, $H(9)$ can be generated by the set $\{x_2^{-1}x_1, x_2^{-1}x_3, x_6^{-1}x_5, x_6^{-1}x_7, x_9^{-1}x_8\}$ and $H(13)$ can be generated by $\{x_2^{-1}x_1, x_2^{-1}x_3, x_6^{-1}x_5, x_6^{-1}x_7, x_{10}^{-1}x_9, x_{10}^{-1}x_{11}, x_{13}^{-1}x_{12}\}$.

$H(6)$ can be generated by the set $\{x_2^{-1}x_1, x_2^{-1}x_3, x_5^{-1}x_4\}$, $H(10)$ can be generated by the set $\{x_2^{-1}x_1, x_2^{-1}x_3, x_6^{-1}x_5, x_6^{-1}x_7, x_9^{-1}x_8\}$ and $H(14)$ can be generated by $\{x_2^{-1}x_1, x_2^{-1}x_3, x_6^{-1}x_5, x_6^{-1}x_7, x_{10}^{-1}x_9, x_{10}^{-1}x_{11}, x_{13}^{-1}x_{12}\}$.

$H(7)$ can be generated by $\{x_2^{-1}x_1, x_2^{-1}x_3, x_6^{-1}x_5, x_6^{-1}x_7\}$, and $H(11)$ can be generated by the set $\{x_2^{-1}x_1, x_2^{-1}x_3, x_6^{-1}x_5, x_6^{-1}x_7, x_{10}^{-1}x_9, x_{10}^{-1}x_{11}\}$.

These results suggested the following upper bound.

For $n \geq 3$, $d(H(n)) \leq (n+1)/2$.

The bound given above is a corollary to a theorem in which generating sets of the required size are produced. The proof of this theorem is similar in method to the proof of the upper bound in Theorem 3.1.5. We begin the proof with two lemmas.

**Lemma 4.0.9** Let $n = 4m + t$ where $t \in \{0, 1, 2, 3\}$. For $k = 0, 1, \ldots, m - 1$, the generators $x_{4k+1}, x_{4k+2}$, and $x_{4k+3}$ of $H(n)$ can be written as words in $y_k = x_{4k+2}^{-1}x_{4k+1}$ and $z_k = x_{4k+2}^{-1}x_{4k+3}$.

**Proof** For each $k$, the result is proved using only the relator $r_{4k+1}$. Relator $r_{4k+1}$ is $x_{4k+2}^{-1}x_{4k+3}x_{4k+2}^{-1}x_{4k+3}x_{4k+2}^{-1}x_{4k+3}x_{4k+2}^{-1}x_{4k+1}x_{4k+2}^{-1}x_{4k+1}$, which can be written as
\[ z_k^2 x_{4k+2}^{-1} y_k^2, \] so

\[ x_{4k+2} = y_k^2 z_k^2. \] (4.1)

If instead we only substitute \( y_k \) for the second of the \( x_{4k+2}^{-1} x_{4k+1} \) in \( r_{4k+1} \), we get the relation

\[
1 = r_{4k+1} \\
= z_k^2 x_{4k+2} x_{4k+1} y_k \\
= z_k^2 z_k^{-2} y_k^2 x_{4k+1} y_k \quad \text{(by (4.1))}, \\
= y_k^2 z_k^{-2} y_k^{-2} x_{4k+1} y_k
\]

and so

\[ x_{4k+1} = y_k^2 z_k^2 y_k. \] (4.2)

It remains to show that \( x_{4k+3} \) can also be written as a word in \( y_k \) and \( z_k \). If we substitute \( z_k \) only for the second \( x_{4k+2}^{-1} x_{4k+3} \) in the relation \( 1 = r_{4k+1} \), we get

\[
1 = x_{4k+2}^{-1} x_{4k+3} z_k x_{4k+2} y_k^2 \\
= z_k^{-2} y_k^{-2} x_{4k+3} z_k z_k^{-2} y_k^{-2} y_k^2 \quad \text{(by (4.1))} \\
= z_k^{-2} y_k^{-2} x_{4k+3} z_k^{-1}
\]

and hence

\[ x_{4k+3} = y_k^2 z_k^3. \] (4.3)
Lemma 4.0.10 Let $n = 4m + t$ where $t \in \{0, 1, 2, 3\}$. For $k = 0, 1, \ldots, m-2$, the generator $x_{4k+4}$ of $H(n)$ can be written as a word in $y_k = x^{-1}_{4k+2}x_{4k+1}$, $z_k = x^{-1}_{4k+2}x_{4k+3}$, $y_{k+1} = x^{-1}_{4(k+1)+2}x_{4(k+1)+1}$, and $z_{k+1} = x^{-1}_{4(k+1)+2}x_{4(k+1)+3}$.

Proof Relator $r_{4k+4}$ is $x_{4k+5}^{-1}x_{4k+6}^{-1}x_{4k+5}x_{4k+6}x_{4k+5}^{-2}x_{4k+4}x_{4k+4}^{-1}x_{4k+4}$. Therefore

\[ 1 = r_{4k+4} \]
\[ = x_{4k+5}^{-1}x_{4k+6}^{-1}x_{4k+5}x_{4k+6}x_{4k+5}^{-2}x_{4k+4}x_{4k+4}^{-1}x_{4k+4} \]
\[ = (x_{4k+5}^{-1}x_{4k+6}x_{4k+5})^{-2}x_{4k+5}^{-1}x_{4k+4}x_{4k+4}^{-1}x_{4k+4} \]
\[ = (x_{4k+5}^{-1}x_{4k+5})^{-2}x_{4k+5}^{-1}(x_{4k+4}x_{4k+5})^{-2} \]
\[ = y_{k+1}^{-2}x_{4k+5}^{-1}(x_{4k+4}x_{4k+5})^{-2} \quad \text{(by definition of } y_{k+1}) \]
\[ = y_{k+1}^{-2}y_{k+1}z_{k+1}y_{k+1}z_{k+1}(x_{4k+4}x_{4k+5})^{-2} \quad \text{(by (4.2))} \]

and hence

\[ (x_{4k+5}^{-1}x_{4k+5})^{-2} = y_{k+1}^{-2}z_{k+1}y_{k+1}^{-2}. \] \hfill (4.4)

Relator $r_{4k+3}$ is $x_{4k+4}^{-1}x_{4k+5}^{-1}x_{4k+5}x_{4k+6}x_{4k+5}^{-2}x_{4k+4}x_{4k+4}^{-1}x_{4k+3}x_{4k+4}$, which can be written as

\[ (x_{4k+4}^{-1}x_{4k+5})^{-2}x_{4k+4}^{-1}(x_{4k+3}x_{4k+4})^{-2} \]

and relator $r_{4k+2}$ is $x_{4k+3}^{-1}x_{4k+4}^{-1}x_{4k+4}x_{4k+3}x_{4k+4}^{-2}x_{4k+4}x_{4k+4}^{-1}x_{4k+4}x_{4k+3}$, which can be written as

\[ (x_{4k+3}^{-1}x_{4k+4})^{-2}x_{4k+3}^{-1}(x_{4k+2}x_{4k+3})^{-2}. \]

We use the product of these two relators to prove that $x_{4k+4}$ is a word in the
required set.

\[ 1 = r_{4k+3}r_{4k+2} \]

\[ = (x_{4k+4}^{-1}x_{4k+5}^{-1}x_{4k+4}^{-1}x_{4k+3}^{-1}x_{4k+4}^{-1}x_{4k+3}^{-1}x_{4k+2}^{-1}x_{4k+3}^{-1})^{-2} \]

\[ = (x_{4k+4}^{-1}x_{4k+5}^{-1}x_{4k+4}^{-1}x_{4k+3}^{-1}x_{4k+2}^{-1})^{-2} (by (4.4)) \]

\[ = y_k^{-3}z_k^{-2}y_k^{-2}z_k^{-3}y_k^{-2}z_k^{-2} \]

Therefore

\[ x_{4k+4} = z_k^{-3}y_k^{-2}z_k^{-2}y_k^{-2}z_k^{-2}y_k^{-1}. \] (4.5)

\[ \square \]

**Theorem 4.0.11** Let \( n = 4m + t \) where \( t \in \{0,1,2,3\} \) and \( m \geq 2 \). Let \( S = \{y_k, z_k : k = 0, 1, \ldots, m - 1\} \).

If \( t = 0 \) then \( H(n) \) is generated by \( S \).

If \( t = 1 \) then \( H(n) \) is generated by the set \( S' = S \cup \{x_n^{-1}x_{n-1}\} \).

If \( t = 2 \) then \( H(n) \) is generated by the set \( S'' = S \cup \{x_n^{-1}x_{n-2}\} \).

If \( t = 3 \) then \( H(n) \) is generated by the set \( S''' = S \cup \{x_n^{-1}x_{n-2}, x_{n-1}^{-1}x_n\} \).

**Proof** The cases \( t = 0,1,2,3 \) are considered separately, but similar methods are used throughout. We use the relators of the presentation to show that
the generators $x_1, x_2, \ldots, x_n$ of $H(n)$ are in the subgroup generated by the required set.

Suppose $t = 0$. By Lemmas 4.0.9 and 4.0.10 the generators $x_1, x_2, \ldots, x_{n-1}$ are in the subgroup generated by the elements of $S$. We just need to show that $x_n$ is also in this subgroup.

Relator $r_n$ is $x_1^{-1}x_2x_1^{-1}x_2x_1^{-2}x_nx_1^{-1}x_n$. Therefore

$$1 = x_1^{-1}x_2x_1^{-1}x_2x_1^{-2}x_nx_1^{-1}x_n$$
$$= (x_2^{-1}x_1)^{-2}x_1^{-1}x_2x_1^{-2}x_nx_1^{-1}x_n$$
$$= (x_2^{-1}x_1)^{-2}x_1^{-1}(x_n^{-1}x_1)^{-2}$$
$$= y_0^{-2}x_1^{-1}(x_n^{-1}x_1)^{-2}$$
$$= y_0^{-2}z_0^{-2}y_0^{-2}(x_n^{-1}x_1)^{-2} \quad \text{(by (4.2))}$$
$$= y_0^{-3}z_0^{-2}y_0^{-2}(x_n^{-1}x_1)^{-2}$$

and hence

$$(x_n^{-1}x_1)^2 = y_0^{-3}z_0^{-2}y_0^{-2}. \quad (4.6)$$

Relator $r_{n-1}$ is $x_n^{-1}x_1x_n^{-1}x_1x_n^{-2}x_{n-1}x_n^{-1}x_{n-1}$, which can be written as

$$(x_n^{-1}x_1)^2x_n^{-1}(x_{n-1}^{-1}x_n)^{-2}$$

and $r_{n-2}$ is $x_n^{-1}x_nx_{n-1}x_n^{-1}x_{n-1}x_{n-2}x_{n-1}^{-1}x_{n-2}$, which can be written as

$$(x_n^{-1}x_n)^2x_{n-1}^{-1}(x_{n-2}^{-1}x_{n-1})^{-2}.$$
We use the product of these relators and (4.6) to show that $x_n$ is in the subgroup generated by $S$.

$$1 = r_{n-1}r_{n-2}$$

$$= (x_n^{-1}x_1)^2x_n^{-1}(x_{n-1}^{-1}x_n)^{-2}(x_{n-1}^{-1}x_n)^2x_{n-1}^{-1}(x_{n-2}^{-1}x_{n-1})^{-2}$$

$$= (x_n^{-1}x_1)^2x_n^{-1}x_{n-1}^{-1}(x_{n-2}^{-1}x_{n-1})^{-2}$$

$$= y_0^{-2} z_0^{-2} y_0^{-2} x_n^{-1} x_{n-1}^{-1}(x_{n-2}^{-1}x_{n-1})^{-2} \quad \text{(by (4.6))}$$

$$= y_0^{-2} z_0^{-2} y_0^{-2} x_n^{-1} y_{m-1}^{-1} y_{m-1}^{-1}(x_{n-2}^{-1}x_{n-1})^{-2} \quad \text{(by (4.3))}$$

$$= y_0^{-2} z_0^{-2} y_0^{-2} x_n^{-1} y_{m-1}^{-1} y_{m-1}^{-1} z_{m-1}$$

and hence

$$x_n = z_{m-1}^{-2} y_{m-1}^{-2} z_{m-1}^{-2} y_0^{-2} z_0^{-2} y_0^{-2} \in \langle S \rangle.$$

Now suppose $t = 1$. By Lemmas 4.0.9, and 4.0.10, the generators $x_1, x_2, \ldots, x_{n-2}$ are in the subgroup generated by $S$, and hence in the subgroup generated by $S'$. We need to show that $x_{n-1}$ and $x_n$ are in the subgroup generated by $S'$.

First we write relators $r_{n-2}$ and $r_{n-3}$ in a suitable form:

$r_{n-2}$ is $x_{n-1}^{-1}x_n x_{n-1}^{-1}x_n x_{n-1}^{-2}x_n^{-1}x_{n-2}^{-1}x_{n-1}^{-1}x_n^{-2}$, which can be written as

$$(x_n^{-1}x_{n-1})^{-2} x_n^{-1} (x_{n-2}^{-1}x_{n-1})^{-2} \quad (4.7)$$

and $r_{n-3}$ is $x_{n-2}^{-1}x_n^{-1}x_{n-2}^{-1}x_{n-1}^{-2}x_{n-3}^{-1}x_{n-2}^{-1}x_{n-3}$, which can be written as

$$(x_{n-2}^{-1}x_{n-1})^{-2} x_{n-2}^{-1} (x_{n-3}^{-1}x_{n-2})^{-2}. \quad (4.8)$$

We use the product of these relators to show that $x_{n-1}$ is a word in the
elements of $S'$.

$$1 = r_{n-2} r_{n-3}$$

$$= (x_n^{-1} x_{n-1})^{-2} x_{n-1}^{-1} (x_{n-2}^{-1} x_{n-1})^{-2} (x_{n-3}^{-1} x_{n-2})^{-2}$$

(by (4.7) & (4.8))

$$= (x_n^{-1} x_{n-1})^{-2} x_{n-1}^{-1} x_{n-2}^{-1} (x_{n-1}^{-1} x_{n-2})^{-2}$$

$$= (x_n^{-1} x_{n-1})^{-2} x_{n-1}^{-1} z_{n-1}^{-2}$$

$$= (x_n^{-1} x_{n-1})^{-2} x_{n-1}^{-1} z_{n-1}^{-2} y_{n-1}^{-2} z_{n-1}^{-2}$$

(by (4.3))

and hence

$$x_{n-1} = z_{n-1}^{-3} y_{n-1}^{-2} z_{n-1}^{-2} (x_n^{-1} x_{n-1})^{-2} \in \langle S' \rangle.$$  (4.9)

Relator $r_n$ is $x_1^{-1} x_2^{-1} x_1^{-2} x_2^{-1} x_n^{-1} x_n$. This can be written as $y_0^{-2} x_1^{-1} (x_n^{-1} x_1)^{-2}$, giving

$$(x_n^{-1} x_1)^2 = y_0^{-2} x_1^{-1}.$$  (4.10)

Relator $r_{n-1}$ is $x_n^{-1} x_1 x_n^{-1} x_1 x_n^{-2} x_{n-1} x_n^{-1} x_{n-1}$, which can be written in the form

$$(x_n^{-1} x_1)^2 x_n^{-1} (x_n^{-1} x_{n-1})^2.$$  (4.11)

We use relator $r_{n-1}$ in this form and relator $r_{n-2}$ in the form (4.7) to show that $x_n$ is in the subgroup generated by $S'$.

$$1 = r_{n-1} r_{n-2}$$

$$= (x_n^{-1} x_1)^2 x_n^{-1} (x_{n-1}^{-1} x_n)^2 (x_n^{-1} x_{n-1})^{-2} x_{n-1}^{-1} (x_{n-2} x_{n-1})^{-2}$$

$$= (x_n^{-1} x_1)^2 x_n^{-1} x_{n-1}^{-1} (x_{n-2} x_{n-1})^{-2}$$

$$= y_0^{-2} x_1^{-1} x_n^{-1} x_{n-1}^{-1} (x_{n-2} x_{n-1})^{-2}$$

(by (4.10))
and hence
\[ x_n = x_{n-1}^{-1}(x_{n-2}^{-1}x_{n-1})^{-2}y_0^{-2}x_1^{-1} \in \langle S' \rangle. \] (4.12)

Suppose \( t = 2 \). By Lemmas 4.0.9 and 4.0.10, the generators \( x_1, x_2, \ldots, x_{n-3} \) are in the subgroup generated by \( S'' \). We now need to show that \( x_{n-2}, x_{n-1}, \) and \( x_n \) are also in this subgroup. Once again we do this by writing certain relators in an appropriate form and taking their product.

Relator \( r_{n-3} \) is \( x_{n-2}^{-1}x_{n-1}^{-1}x_{n-2}^{-1}x_{n-3}^{-2}x_{n-3}^{-1}x_{n-2}x_{n-3} \), which can be written in the form
\[ (x_{n-1}x_{n-2})^{-2}x_{n-2}^{-1}(x_{n-3}x_{n-2})^{-2} \] (4.13)
and relator \( r_{n-4} \) is \( x_{n-3}^{-1}x_{n-2}^{-1}x_{n-3}^{-2}x_{n-4}^{-1}x_{n-3}^{-1}x_{n-4} \), which can be written in the form
\[ (x_{n-3}x_{n-2})^{-2}x_{n-3}^{-1}(x_{n-4}^{-1}x_{n-3})^{-2} \] (4.14)

Therefore
\[ 1 = r_{n-3}r_{n-4} \]
\[ = (x_{n-1}x_{n-2})^{-2}x_{n-2}^{-1}(x_{n-3}x_{n-2})^{-2}(x_{n-3}x_{n-2})^{-2}x_{n-3}^{-1}(x_{n-4}^{-1}x_{n-3})^{-2} \]
\[ = (x_{n-1}x_{n-2})^{-2}x_{n-2}^{-1}x_{n-3}^{-1}(x_{n-4}^{-1}x_{n-3})^{-2} \]
and hence
\[ x_{n-2} = x_{n-3}^{-1}(x_{n-4}^{-1}x_{n-3})^{-2}(x_{n-1}x_{n-2})^{-2} \in \langle S'' \rangle. \] (4.15)

As \( x_{n-1} = x_{n-2}(x_{n-1}x_{n-2})^{-1} \), \( x_{n-1} \) is in the subgroup generated by \( S'' \).
By (4.12) $x_n = x_{n-1}(x_{n-2}^{-1}x_{n-1})^{-2}y_0^{-2}x_1^{-1}$ and hence $x_n \in \langle S'' \rangle$.

Finally, suppose $t = 3$. By Lemmas 4.0.9 and 4.0.10, generators $x_1, \ldots, x_{n-4}$ are in the subgroup generated by $S'''$. We need to show that $x_{n-3}, x_{n-2}, x_{n-1}$, and $x_n$ are also in this subgroup.

As $r_{n-2}$ is $r_{4m+1}$, the same argument as that used in the proof of Lemma 4.0.9 shows that $x_{n-2}, x_{n-1}$, and $x_n$ are in the subgroup generated by $x_{n-1}^{-1}x_{n-2}$ and $x_{n-1}^{-1}x_n$, which is contained in the subgroup generated by $S'''$. It remains to show that $x_{n-3}$ is in this subgroup.

Relator $r_{n-5}$ is $x_{n-4}^{-1}x_{n-3}^{-1}x_{n-4}^{-1}x_{n-3}^{-2}x_{n-5}^{-1}x_{n-4}^{-1}x_{n-5}$, which can be written in the form

$$\left(x_{n-4}^{-1}x_{n-3}\right)^2x_{n-4}^{-1}(x_{n-5}^{-1}x_{n-4})^{-2}.$$

(4.16)

Therefore

$$1 = r_{n-4}r_{n-5}$$

$$= (x_{n-3}^{-1}x_{n-2})^2x_{n-3}^{-1}(x_{n-4}^{-1}x_{n-3})^{-2}(x_{n-4}^{-1}x_{n-3})^2x_{n-4}^{-1}(x_{n-5}^{-1}x_{n-4})^{-2}$$

(by (4.14) & (4.16))

$$= (x_{n-3}^{-1}x_{n-2})^2x_{n-3}^{-1}x_{n-4}^{-1}(x_{n-5}^{-1}x_{n-4})^{-2}$$

$$= (x_{n-1}^{-1}x_{n-2})^{-2}x_{n-2}^{-1}x_{n-3}^{-1}x_{n-4}^{-1}(x_{n-5}^{-1}x_{n-4})^{-2}$$

(by (4.13))

and hence

$$x_{n-3}^{-1} = x_{n-4}^{-1}(x_{n-5}^{-1}x_{n-4})^{-2}(x_{n-1}^{-1}x_{n-2})^{-2}x_{n-2}^{-1} \in \langle S''' \rangle,$$

which completes the proof of Theorem 4.0.11.

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Corollary 4.0.12  For $n \geq 3$ $d(H(n))$ is at most $(n + 1)/2$.

Proof  For $n \geq 8$ the result follows from Theorem 4.0.11. The result is true for $H(3)$ by Lemma 3.1.6. The generating sets for $H(4)$, $H(5)$, $H(6)$, and $H(7)$ given at the beginning of the chapter were proved using GAP and the code used is in Appendix B.
Chapter 5

Remarks

5.1 Remark

There are more special homomorphisms from $G(n)$ to $A_5$, $n \in \{6, 7, 8, 9\}$ and from $H(n)$ to $A_5$ for $n \in \{8, 9, 10, 11\}$, so it is possible that the lower bounds for $d(G(n))$ and $d(H(n))$ could be increased.

5.2 Remark

Generating sets for $H(n)$ of the required size are not given in [6]. There are generating sets other than the ones given in Theorem 3.1.7. For example, $H(6)$ is generated by the set $\{x_2^{-1}x_1, x_2^{-1}x_3, x_4, x_5\}$. 
5.3 Remark

Experiments using ACE within GAP, like those used to find a smaller upper bound for $d(H(n))$ did not produce smaller generating sets for $G(n)$ for small $n$, but the search was not extensive.
Appendix A

GAP code for Lemma 3.2.3

The following GAP code was used to verify Lemma 3.2.3.

\[
\text{A := AlternatingGroup(5);}
\]
\[
\text{a := (1,2,3,4,5);}
\]
\[
\text{scount := 0;}
\]
\[
\text{for c in A do}
\]
\[
\text{for d in A do}
\]
\[
\text{if c^5 = Identity(A) and}
\]
\[
\text{d^5 = Identity(A) and}
\]
\[
\text{(a^-1*c)^2 = Identity(A) and}
\]
\[
\text{(c^-1*d)^2 = Identity(A) and}
\]
\[
\text{(d^-1*a)^2 = Identity(A) then}
\]
\[
\text{scount := scount + 1;}
\]
\[
\text{Print( "solution number ", scount,"\n ", c, " ", d, "\n ");}
\]
\[
\text{fi;}
\]
\[
\text{od;}
\]
\[
\text{od;}
\]

The code was also run with

\[
\text{a := (1,2,3,5,4);}
\]

in place of the second line.
GAP code for Lemma 3.2.4

The following GAP code was used to verify Lemma 3.2.4.

```
A := AlternatingGroup(5);
a := (1,2,3,4,5);
for b in [(1,2,3,4,5), (1,4,5,3,2), (1,5,4,2,3),
(1,5,3,4,2), (1,2,5,4,3), (1,4,3,2,5)]
    do
    Print("For input \n ", a, " ", b, " with a^-1*b= ", a^-1*b);
    scount := 0;
    for c in A do
        for d in A do
            if c^5 = Identity(A) and
               d^5 = Identity(A) and
               (a^-1*c)^2 = Identity(A) and
               (c^-1*d)^2 = Identity(A) and
               (d^-1*b)^2 = Identity(A)
                then
                scount := scount +1;
                Print("solution number ", scount, "\n ", c, " ", d, "\n ");
            fi;
        od;
    od;
    od;
```

This code was run again with

```
a := (1,2,3,5,4);
```

and the second line and

```
for b in [(1,2,3,5,4), (1,5,4,3,2), (1,5,3,2,4),
(1,4,5,2,3), (1,2,4,5,3), (1,4,3,5,2)]
```

as the third line.
Appendix B

Code using ACE within GAP

The following GAP codes were used to find smaller generating sets for $H(n)$, for $n = 4, 5, 6, 7$. The codes use ACE within GAP. Similar codes were used to find smaller generating sets for $H(n)$ for $n \leq 15$.

```
f:=FreeGroup("a", "b", "c", "d");
a:=f.1;
b:=f.2;
c:=f.3;
d:=f.4;
rels:=[b^-1*c*b^-1*c*b^-2*a*b^-1*a, c^-1*d*c^-1*d*c^-2*b*c^-1*b,
d^-1*a*d^-1*a*d^-2*c*d^-1*c, a^-1*b*a^-1*b*a^-2*d*a^-1*d];
G:=f/rels;
s:=G.1;
t:=G.2;
u:=G.3;
v:=G.4;
H:=Subgroup(G, [t^-1*s, t^-1*u]);
RequirePackage("ace");
TCENUM:=ACETCENUM;;
Index(G,H : hard:= true, workspace:=10^-7);
```

```
f:=FreeGroup("a", "b", "c", "d", "e");
a:=f.1;
b:=f.2;
c:=f.3;
d:=f.4;
e:=f.5;
rels:=[b^-1*c*b^-1*c*b^-2*a*b^-1*a, c^-1*d*c^-1*d*c^-2*b*c^-1*b,
d^-1*e*d^-1*e*d^-2*c*d^-1*c, e^-1*a*e^-1*a*e^-2*d*e^-1*d,}
\[a^{-1}b\cdot a^{-1}b\cdot a^{-2}\cdot e\cdot a^{-1}e;\]
\[G := f/rels;\]
\[s := G.1;\]
\[t := G.2;\]
\[u := G.3;\]
\[v := G.4;\]
\[w := G.5;\]
\[H := \text{Subgroup}(G, [t^{-1} \cdot s, t^{-1} \cdot u, w^{-1} \cdot v]);\]
\[\text{RequirePackage("ace");}\]
\[\text{TCENUM := ACETCENUM;};\]
\[\text{Index}(G,H : \text{hard := true, workspace := }10^7);\]

\[f := \text{FreeGroup("a", "b", "c", "d", "e", "g");}\]
\[a := f.1;\]
\[b := f.2;\]
\[c := f.3;\]
\[d := f.4;\]
\[e := f.5;\]
\[g := f.6;\]
\[\text{rels := [b^{-1} \cdot c \cdot b^{-1} \cdot c \cdot b^{-2} \cdot a \cdot b^{-1} \cdot a,}\]
\[c^{-1} \cdot d \cdot c^{-1} \cdot d \cdot c^{-2} \cdot b \cdot c^{-1} \cdot b,\]
\[d^{-1} \cdot e \cdot d^{-1} \cdot e \cdot d^{-2} \cdot c \cdot d^{-1} \cdot c,\]
\[e^{-1} \cdot g \cdot e^{-1} \cdot g \cdot e^{-2} \cdot d \cdot e^{-1} \cdot d,\]
\[g^{-1} \cdot a \cdot g^{-1} \cdot a \cdot g^{-2} \cdot e \cdot g^{-1} \cdot e,\]
\[a^{-1} \cdot b \cdot a^{-1} \cdot b \cdot a^{-2} \cdot g \cdot a^{-1} \cdot g];\]
\[G := f/rels;\]
\[s := G.1;\]
\[t := G.2;\]
\[u := G.3;\]
\[v := G.4;\]
\[w := G.5;\]
\[x := G.6;\]
\[H := \text{Subgroup}(G, [t^{-1} \cdot s, t^{-1} \cdot u, w^{-1} \cdot v]);\]
\[\text{RequirePackage("ace");}\]
\[\text{TCENUM := ACETCENUM;};\]
\[\text{Index}(G,H : \text{hard := true, workspace := }10^7);\]

\[f := \text{FreeGroup("a", "b", "c", "d", "e", "g", "h");}\]
\[a := f.1;\]
\[b := f.2;\]
\[c := f.3;\]
\[d := f.4;\]
\[e := f.5;\]
\[g := f.6;\]
\[h := f.7;\]
\[\text{rels := [b^{-1} \cdot c \cdot b^{-1} \cdot c \cdot b^{-2} \cdot a \cdot b^{-1} \cdot a,}\]
\[c^{-1} \cdot d \cdot c^{-1} \cdot d \cdot c^{-2} \cdot b \cdot c^{-1} \cdot b,\]
\[d^{-1} \cdot e \cdot d^{-1} \cdot e \cdot d^{-2} \cdot c \cdot d^{-1} \cdot c,\]
\[e^{-1} \cdot g \cdot e^{-1} \cdot g \cdot e^{-2} \cdot d \cdot e^{-1} \cdot d,\]
\[g^{-1} \cdot h \cdot g^{-1} \cdot h \cdot g^{-2} \cdot e \cdot g^{-1} \cdot e,\]
\[h^{-1} \cdot a \cdot h^{-1} \cdot a \cdot h^{-2} \cdot g \cdot h^{-1} \cdot g,\]
\[a^{-1} \cdot b \cdot a^{-1} \cdot b \cdot a^{-2} \cdot h \cdot a^{-1} \cdot h];\]
\[G := f/rels;\]
\[s := G.1;\]
\[t := G.2;\]
\[ \begin{align*} 
u & := G.3; \\
v & := G.4; \\
w & := G.5; \\
x & := G.6; \\
y & := G.7; \\
H & := \text{Subgroup}(G, [t^{-1}s, t^{-1}u, x^{-1}y]); \\
& \text{RequirePackage("ace");} \\
& \text{TCENUM} := \text{ACET\{ENUM};} \\
& \text{Index}(G,H : \text{hard} := \text{true, workspace} := 10^{-7}); \\
\end{align*} \]
Bibliography


