Enumeration of Semigroups of Order 9

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This was an open mathematical problem for 14 years. There is a large number of solutions, and the problem has many symmetries.

The solution involved

- the development of an enumeration formula for most of the solutions;
- computer search for the remaining solutions.
### Basic definitions

**Definition (Semigroup)**

A set $S$ with a binary operation $\circ$ satisfying

- **A1** $(x \circ y) \circ z = x \circ (y \circ z)$ \hspace{1cm} associativity

It may also be true that

- **A2** $x \circ e = e \circ x = x$ \hspace{1cm} identity element $e$
- **A3** $x \circ x^{-1} = e$ \hspace{1cm} inverse elements

Examples: $(\mathbb{N}, +), (\mathbb{Z}_n, \ast)$

**Definition (Band)**

A semigroup in which all elements are idempotents, i.e.

- **A4** $x \circ x = x$
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<thead>
<tr>
<th>order</th>
<th># groups</th>
<th># semigroups</th>
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[Forsythe ’54]

[Motzkin, Selfridge ’55]

[Plemmons ’66]

[Jürgensen, Wick ’76]

[Satoh, Yama, Tokizawa ’94]
The problem

For a given number $n \in \mathbb{N}$ find all structural types of semigroups of order $n$.

A bijection $\sigma : S \rightarrow T$ is an anti-isomorphism if

$$(xy)^\sigma = y^\sigma x^\sigma$$

for all $x, y \in S$.

**Definition**

Two semigroups are *equivalent* if they are isomorphic or anti-isomorphic.
Nilpotent semigroups

Of the 1,843,120,128 semigroups of order 8 . . .

- . . . less than 0.1% are monoids.
- . . . 99.5% are semigroups with a zero element.
- . . . 99.5% are nilpotent semigroups.
- . . . 99.4% are 3-nilpotent semigroups.

Definition

A semigroup $S$ is nilpotent if $|S^r| = 1$ for some $r \in \mathbb{N}$.  

$(S^r = \{s_1 \circ s_2 \circ \cdots \circ s_r \mid s_i \in S\})$

A nilpotent semigroup is $r$-nilpotent if $r \in \mathbb{N}$ is the smallest number such that $|S^r| = 1$.

Theorem (Kleitman, Rothschild, Spencer ’76)

Asymptotically ’almost all’ finite semigroups are 3-nilpotent.
Let \( n \geq 2 \). Denote \([n] = \{1, \ldots, n\}\).

Define a semigroup \( S \) on \([n]\) as follows:

- Take a proper subset \( B \) of \([n]\) and set \( A = [n] \setminus B \).
- Choose an element \( z \in B \).
- Choose a function \( \psi : A \times A \to B \).
- For \( x, y \in A \) define \( x \circ y = \psi(x, y) \).
- Define \( x \circ y = z \) in any other case.
Example

\[ n = 7, B = \{1, 3, 5, 7\} \Rightarrow A = \{2, 4, 6\}, z = 7 \]

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4 & 7 & 3 & 7 & 7 & 7 & 1 & 7 \\
5 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
6 & 7 & 5 & 7 & 3 & 7 & 1 & 7 \\
7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\
\end{array}
\]
The number of 3-nilpotent semigroups

Theorem

The number \( P(n, r) \) of isomorphism types of 3-nilpotent semigroups of order \( n \) with \( |B| = r \) equals

\[
\sum_{(j) \vdash n-r, \ (k) \vdash r-1} \left( \prod_{i=1}^{n-r} j_i! i^{i_j} \right)^{-1} \left( \prod_{i=1}^{r-1} k_i! i^{k_i} \right)^{-1} \prod_{i_1, i_2=1}^{n-r} \left( 1 + \sum_{d \mid \text{lcm}(i_1, i_2)} d k_d \right)^{j_1 j_2 \text{gcd}(i_1, i_2)}
\]

Thus the number of non-isomorphic 3-nilpotent semigroups of order \( n \) equals

\[
\sum_{r=2}^{n-1} P(n, r) - P(n - 1, r - 1).
\]
Find all semigroups of order $n$ that are not 3-nilpotent.

Formulate a *Constraint Satisfaction Problem* (CSP) $L$:

- **Variables:** $T(i, j), 1 \leq i, j \leq n$
- **Domains:** $\{1, 2, \ldots, n\}$
- **Constraints:**
  \[
  T(i, T(j, k)) = T(T(i, j), k), 1 \leq i, j, k \leq n,
  \exists i, j, k : T(i, T(j, k)) \neq "0",
  T \leq T^g, g \in S_n \times C_2
  \]

We use the Minion CSP solver to obtain solutions for (variations on) $L$.

**Problem:**
There are $2n!$ symmetry breaking (SB) constraints.
Case split

The action on tables induces an action on diagonals. We construct a set of orbit representatives.

For every diagonal $D$ formulate a CSP based on $L$.

- Add constraints: $T(i, i) = D(i), 1 \leq i \leq n$,
- Replace SB constraints: $T \leq T^g, g \in \text{Stab}_{S_n}(D) \times C_2$

What we care about in a case split:

- different cases have non-equivalent solutions,
- all cases together have the solutions of $L$,
- most instances have fewer symmetries,
- the time and space complexities are small enough for us to be able to solve the instances on our compute servers.
All semigroups of order 9

3-nilpotent

Not 3-nilpotent

Bands

Diagonals not allowing 3-nilpotent solutions

Nilpotent

Diagonals allowing 3-nilpotent solutions

Not nilpotent
A case for bands

**Theorem**

Let $B$ be a band with $\mathcal{D}$-classes $\{E_{\alpha} : \alpha \in Y\}$. Then each $E_{\alpha}$ is a rectangular band and $Y$ forms a semilattice (the join being defined by $E_{\alpha} E_{\beta} \subseteq E_{\alpha \beta}$).

Formulate CSPs for non-equivalent sets of rectangular bands indexed by semilattices.
### Results

<table>
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<th>order</th>
<th># semigroups</th>
<th># 3-nilpotent semigroups</th>
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<td>52,989,400,714,478</td>
<td>52,966,239,062,973</td>
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Challenges

It took 30 months to design and implement the case split searches. These will now run in a few days on a single computer to give the non-3-nilpotent semigroups of order 9.

The same approach would require several years search for order 10 semigroups.

The challenges now are to:

- re-model certain CSP instances to utilise knowledge of the possible generators of solution semigroups;
- distribute and/or parallelise the search across a large number of processors;
- deal with the $2 \times 10!$ symmetries effectively and efficiently.