# THE TRUE PROSOLUBLE COMPLETION OF A GROUP EXAMPLES AND OPEN PROBLEMS

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Abstract. The *true prosoluble completion* $P\mathcal{S}(\Gamma)$ of a group $\Gamma$ is the inverse limit of the projective system of soluble quotients of $\Gamma$. Our purpose is to describe examples and to point out some natural open problems. We answer the analogue of a question of Grothendieck for profinite completions by providing examples of pairs of non–isomorphic residually soluble groups with isomorphic true prosoluble completions. We also provide a new class of finitely generated examples which answer the original Grothendieck's problem.



## 1. Introduction

A group $\Gamma$ has a *profinite topology*, for which the set $\mathcal{F}$ of normal subgroups of finite index is a basis of neighbourhoods of the identity, and the resulting *profinite completion*, hereafter denoted by $P\mathcal{F}(\Gamma)$. The canonical homomorphism

$$\varphi_{\mathcal{F}} : \Gamma \longrightarrow P\mathcal{F}(\Gamma)$$

is injective if the group $\Gamma$ is *residually finite* (by definition). The notion of profinite completion is relevant in various domains (outside pure group theory), including Galois theory of infinite fields extensions and fundamental groups in algebraic topology [Groth–70]. For the theory of profinite groups, there are many papers and several books available [DiSMS–91], [RibZa–00], [Wilso–98]; see Section 1.1 in [Se–73/94] for a quick introduction and [HallM–50] for an early paper.

Besides $\mathcal{F}$, there are other natural families of normal subgroups of $\Gamma$ which give rise to other "procompletions". The purpose of this report is to consider some variants, with

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special emphasis on the *true prosoluble completion* $P\mathcal{S}(\Gamma)$ associated to the family $\mathcal{S}$ of all normal subgroups of $\Gamma$ with soluble quotients. The corresponding homomorphism

$$\varphi_{\mathcal{S}} : \Gamma \longrightarrow P\mathcal{S}(\Gamma)$$

is injective if the group $\Gamma$ is *residually soluble* (by definition).

On the one hand, we discuss examples including free groups, free soluble groups, wreath products, $SL_d(\mathbf{Z})$ and its congruence subgroups, the Grigorchuk group, and parafree groups; see Sections 6 and 7. On the other hand, we discuss some open problems, of which we would like to point out from the start the following ones.

(i) Let $\Gamma, \Delta$ be two residually finite groups and let $\psi : \Gamma \longrightarrow \Delta$ be a homomorphism such that, at the profinite level, the corresponding homomorphism $P\mathcal{F}(\psi) : P\mathcal{F}(\Gamma) \longrightarrow P\mathcal{F}(\Delta)$ is an isomorphism. How far from an isomorphism can $\psi$ be? The problem goes back to Grothendieck [Groth–70], and is motivated by the need to compare two notions of a fundamental group for algebraic varieties. There are examples with $\psi$ not an isomorphism and $P\mathcal{F}(\psi)$ an isomorphism, with $\Gamma, \Delta$ finitely generated ([PlaTa–86], and Proposition 2 below) and finitely presented [BriGr–04]. If $P\mathcal{F}(\psi)$ is an isomorphism, there are known sufficient conditions[1] on $\Gamma$ and $\Delta$ for $\psi$ to be an isomorphism.

Our main interest in this paper is to discuss the *prosoluble analogue of Grothendieck's problem.* More precisely, let again $\Gamma, \Delta$ be two groups and $\psi : \Gamma \longrightarrow \Delta$ a homomorphism, but assume now that the groups are residually soluble. Assume that, at the true prosoluble level, the corresponding homomorphism $P\mathcal{S}(\psi) : P\mathcal{S}(\Gamma) \longrightarrow P\mathcal{S}(\Delta)$ is an isomorphism. How far from an isomorphism can $\psi$ be? Additional requirements can be added on $\Gamma$ and $\Delta$ (such as finite generation, finite presentation, ...).

The main result of the present note is Proposition 3, which shows an example of a non–isomorphism $\psi$ for which $P\mathcal{S}(\psi)$ is an isomorphism. The group $\Gamma$ is the Grigorchuk group, and $\psi$ is the inclusion of $\Gamma$ in a finitely generated subgroup $\Delta$ of the pro–2–completion of $\Gamma$.

(ii) True prosoluble completions provide a natural setting to turn qualitative statements of the kind "some group $\Gamma$ is not residually soluble" in more precise statements concerning the *true prosoluble kernel* $\mathrm{Ker}(\varphi_{\mathcal{S}} : \Gamma \longrightarrow P\mathcal{S}(\Gamma))$. One example is worked out in (6.G).

(iii) Can one find interesting characterizations of those groups which are true prosoluble completions of residually soluble group? More precisely, let $G$ be a complete Hausdorff topological group such that, for any $g \in G$, $g \neq 1$, there exists an open normal subgroup $N$ of $G$ not containing $g$ such that $G/N$ is soluble; how can it be decided whether $G$ is isomorphic to $P\mathcal{S}(\Gamma)$ for some group $\Gamma$? for some finitely generated group $\Gamma$? for some finitely presented group $\Gamma$?

Other open problems occur in (6.G), (6.H), (7.B), and (7.D).

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[1]Proposition 2 of [PlaTa–90]: if $\Gamma, \Delta$ are finitely generated residually finite groups, if $\Delta$ is a soluble subgroup of $GL_n(\mathbf{C})$ for some $n \geq 1$, and if $P\mathcal{F}(\psi)$ is an isomorphism, then so is $\psi$.

## 2. Completion with respect to a directed set of normal subgroups

In this section, we review some classical constructions and facts. See in particular [Weil–40], with § 5 on projective limits, [Bourb–60], with Chapter 3, § 3, No. 4 on completions and § 7 on projective limits, and [Kelle–55], with Problem Q of Chapter 6 on completions. Defining a topology on a group using a family of subgroups goes back at least to Garrett Birkhoff [Birkh–37, see Pages 52–54] and André Weil.

Let $\Gamma$ be a group. Let $\mathcal{N}$ be a family of normal subgroups of $\Gamma$ which is directed, namely which is such that the intersection of two groups in $\mathcal{N}$ contains always a group in $\mathcal{N}$.

(2.A) Denote by $\mathcal{CN}(\Gamma)$ the intersection of all elements in $\mathcal{N}$ (the letter $\mathcal{C}$ stands for "core"), by $\underline{\Gamma}$ the quotient group $\Gamma/\mathcal{CN}(\Gamma)$, and by $\underline{\mathcal{N}}$ the family of normal subgroups of $\underline{\Gamma}$ which are images of groups in $\mathcal{N}$. Then $\underline{\mathcal{N}}$ is a basis of neighbourhoods of the identity for a Hausdorff topology on $\underline{\Gamma}$. The corresponding left and right uniformities have the same Cauchy nets; indeed, for $x, y \in \underline{\Gamma}$ and $\underline{N} \in \underline{\mathcal{N}}$, we have $x^{-1}y \in \underline{N}$ if and only if $xy^{-1} \in \underline{N}$. It follows that $\underline{\Gamma}$ can be completed, say with respect to the left uniformity, to a Hausdorff complete[2] group which is called here the *pro–$\mathcal{N}$–completion of* $\Gamma$ and which is denoted by $P\mathcal{N}(\Gamma)$. The canonical homomorphism

$$\varphi_{\mathcal{N}} \,:\, \Gamma \longrightarrow P\mathcal{N}(\Gamma)$$

has kernel $\mathcal{CN}(\Gamma)$ and image dense in $P\mathcal{N}(\Gamma)$. For $N \in \mathcal{N}$, the projection $\Gamma/\mathcal{CN}(\Gamma) \longrightarrow \Gamma/N$ extends uniquely to a continuous homomorphism

$$p_N \,:\, P\mathcal{N}(\Gamma) \longrightarrow \Gamma/N$$

which is onto.

(2.B) Observe that the following properties are equivalent

(i) $\{1\} \in \mathcal{N}$,
(ii) the topology defined by $\mathcal{N}$ on $\Gamma$ is discrete,
(iii) $P\mathcal{N}(\Gamma) = \Gamma$ and $\varphi_{\mathcal{N}} : \Gamma \longrightarrow P\mathcal{N}(\Gamma)$ is the identity.

Also, the topology on $P\mathcal{N}(\Gamma)$ is discrete if and only if $\mathcal{CN}(\Gamma) \in \mathcal{N}$.

(2.C) Assume moreover that the family $\mathcal{N}$ is countable. We can assume without loss of generality that the elements of $\mathcal{N}$ constitute a nested sequence $N_1 \supset N_2 \supset \cdots$ (otherwise, if $\mathcal{N} = \{\tilde{N}_j\}_{j\geq 1}$, choose inductively $N_{j+1}$ as a group in $\mathcal{N}$ contained in $N_j \cap \tilde{N}_1 \cap \cdots \cap \tilde{N}_{j+1}$). Assume also that $\cap_{i\geq 1} N_i = \{1\}$.

The topology defined by $\mathcal{N}$ on $\Gamma$ is metrisable. Indeed, define first $v_i : \Gamma \longrightarrow \{0, 1\}$ by $v_i(\gamma) = 0$ if $\gamma \in N_i$ and $v_i(\gamma) = 1$ if not. Define next $w : \Gamma \longrightarrow [0, 1]$ by $w(\gamma) =$

[2] Recall that a topological group $G$ is *complete* if both its left and right uniform structures are complete uniform structures, or equivalently if *one* of these structures is a complete uniform structure; see [Bourb–60], Chapter 3, § 3. Let $\Gamma$ be a topological group and let $G$ denote its completion, as a topological space, with respect to the left uniform structure; a sufficient condition for $G$ to be a completion of $\Gamma$ as a topological group is that the left and right uniform structures on $\Gamma$ have the same Cauchy nets; see Theorem 1 of No. 4 in the same book.

$\sum_{i \geq 1} 2^{-i} v_i(\gamma)$. Then the mapping $d : \Gamma \times \Gamma \longrightarrow [0,1]$ defined by $d(\gamma_1, \gamma_2) = w(\gamma_1^{-1}\gamma_2)$ is a left–invariant ultrametric on $\Gamma$ which defines the same topology as $\mathcal{N}$.

(2.D) The quotients

$$\Gamma/N \qquad \text{where } N \in \mathcal{N},$$

and the canonical projections

$$p_{M,N} \,:\, \Gamma/N \longrightarrow \Gamma/M \qquad \text{where } M, N \in \mathcal{N} \text{ are such that } N \subset M,$$

constitute an inverse system of groups of which the inverse limit (also called the projective limit) $\varprojlim \Gamma/N$ can be identified with $P\mathcal{N}(\Gamma)$. The following properties are standard: $P\mathcal{N}(\Gamma)$ is totally disconnected and complete.

For a group $\Gamma$, denote by $Z(\Gamma)$ its center and by $D(\Gamma)$ the subgroup generated by the commutators; for a topological group $G$, denote by $\overline{D}(G)$ the closure of $D(G)$. We have:

$$Z(P\mathcal{N}(\Gamma)) = \varprojlim Z(\Gamma/N) \qquad \text{and} \qquad \overline{D}(P\mathcal{N}(\Gamma)) = \varprojlim D(\Gamma/N)$$

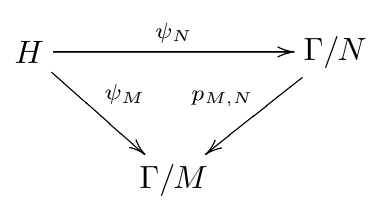
(see [Bourb–82], Appendix I, No. 2). See also (5.C) below.

(2.E) Let $A$ be a partially ordered set which is directed. Consider an inverse system consisting of groups $\Gamma_\alpha$, with $\alpha \in A$, and homomorphisms $p_{\alpha,\beta} : \Gamma_\beta \longrightarrow \Gamma_\alpha$, with $\alpha, \beta \in A$ and $\alpha \leq \beta$. Let $\Gamma = \varprojlim \Gamma_\alpha$ denote the inverse limit. Even when the groups $\Gamma_\alpha$ are not all reduced to $\{1\}$ and the homomorphisms $p_{\alpha,\beta}$ are all onto, the natural homomorphisms $p_\alpha : \Gamma \longrightarrow \Gamma_\alpha$ need not be onto, and indeed the limit $\Gamma$ can be reduced to one element; see [HigSt–54]. (If $A$ is the set of natural integers, with the usual order, it is straightforward to check that the $p_\alpha$ are onto.)

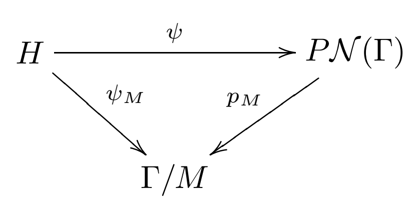
Because of this kind of phenomena, we have chosen here to define pro–$\mathcal{N}$–completions as appropriate topological completions. However, it would be possible to use inverse limits systematically.

## 3. Universal property

(3.A) Let $\Gamma$, $\mathcal{N}$, and $P\mathcal{N}(\Gamma)$ be as in the previous section. Let $H$ be a topological group. Assume that there is given a family $(\psi_N : H \longrightarrow \Gamma/N)_{N \in \mathcal{N}}$ of continuous homomorphisms such that, for $M, N \in \mathcal{N}$ with $N \subset M$, the diagram




commutes. Then there exists a unique continuous homomorphism $\psi : H \longrightarrow P\mathcal{N}(\Gamma)$ such that the diagram



commutes for all $M \in \mathcal{N}$.

The pro–$\mathcal{N}$–completion $P\mathcal{N}(\Gamma)$ is characterized up to unique continuous homomorphism by this universal property.

*Caveat.* Even if $\psi_N$ is onto for each $N \in \mathcal{N}$, $\psi$ needs not be onto; see e.g. (6.A) below.

(3.B) Let $\mathcal{M}$ [respectively $\mathcal{N}$] be a directed family of normal subgroups of a group $\Gamma$ [respectively $\Delta$] and let $\psi : \Gamma \longrightarrow \Delta$ be a group homomorphism. Assume that, for each $N \in \mathcal{N}$, there exists $M \in \mathcal{M}$ such that $\psi(M) \subset N$.

For a given $N \in \mathcal{N}$, the family of homomorphisms $\Gamma/M \longrightarrow \Delta/N$ induced by $\psi$ (the family is indexed by those $M \in \mathcal{M}$ such that $\psi(M) \subset N$) gives rise to a continuous homomorphism $P\mathcal{M}(\Gamma) \longrightarrow \Delta/N$. In turn, these give rise to a continuous homomorphism $P\mathcal{M}(\Gamma) \longrightarrow P\mathcal{N}(\Delta)$.

This will occur in (3.C) and in Section 4.

(3.C) In particular, let $\mathcal{M}$ be a subset of $\mathcal{N}$ which is cofinal, namely such that any $N \in \mathcal{N}$ contains some $M$ in $\mathcal{M}$. Then $\mathcal{CM}(\Gamma) = \mathcal{CN}(\Gamma)$, the topology defined on $\Gamma/\mathcal{CM}(\Gamma)$ by $\mathcal{M}$ coincides with that defined by $\mathcal{N}$, and the continuous homomorphism $P\mathcal{M}(\Gamma) \longrightarrow P\mathcal{N}(\Gamma)$ defined above is an isomorphism.

## 4. Examples of directed sets of normal subgroups

(4.A) The set $\mathcal{F}$ of normal subgroups of finite index in a group $\Gamma$ gives rise to the *profinite completion* $P\mathcal{F}(\Gamma)$ of $\Gamma$. There is a large literature on these completions, alluded to in the introduction.

For a prime number $p$, the set $\mathcal{F}_p$ of normal subgroups of index a power of $p$ in a group $\Gamma$ gives rise to the *pro–p–completion* $P_{\hat{p}}(\Gamma)$. See [DiSMS–91] and [SaSeS–00]. Since $\mathcal{F}_p \subset \mathcal{F}$, there is a canonical homomorphism

$$P\mathcal{F}(\Gamma) \longrightarrow P_{\hat{p}}(\Gamma)$$

by (2.A). The resulting homomorphism $P\mathcal{F}(\Gamma) \longrightarrow \prod_p P_{\hat{p}}(\Gamma)$ is sometimes an isomorphism, as it is the case for $\mathbf{Z}$, and sometimes not, as it is the case for $\bigoplus_{n\geq 5} \mathrm{Alt}(n)$, or for any non–trivial direct sum of non abelian finite simple groups.

(4.B) The set $\mathcal{S}$ of normal subgroups with soluble quotients gives rise to the *true prosoluble completion* $P\mathcal{S}(\Gamma)$ of $\Gamma$; it is the main subject of the present note.

The *prosoluble completion* of the literature refers usually to the family $\mathcal{FS}$ of normal subgroups with *finite* soluble quotients (an exception is [CocHa], where our $P\mathcal{S}(\Gamma)$ is called the "pro–solvable completion" of $\Gamma$, and where other variants appear). Since $\mathcal{FS} \subset \mathcal{F}$ and $\mathcal{FS} \subset \mathcal{S}$, there are canonical homomorphisms

$$P\mathcal{F}(\Gamma) \longrightarrow P\mathcal{FS}(\Gamma) \qquad \text{and} \qquad P\mathcal{S}(\Gamma) \longrightarrow P\mathcal{FS}(\Gamma)$$

by (3.B). See (6.A) and (6.B) for examples.

The adjective "true" should not mislead the reader: for some groups, for example for an infinite cyclic group, the true prosoluble completion is "much smaller" than the prosoluble completion.

(4.C) Similarly, we distinguish the *true pronilpotent completion* $P\mathcal{N}i(\Gamma)$ from the pronilpotent completion $P\mathcal{F}\mathcal{N}i(\Gamma)$ of the literature. There are again canonical homomorphisms

$$P\mathcal{F}(\Gamma) \longrightarrow P\mathcal{F}\mathcal{N}i(\Gamma) \qquad \text{and} \qquad P\mathcal{S}(\Gamma) \longrightarrow P\mathcal{N}i(\Gamma) \longrightarrow P\mathcal{F}\mathcal{N}i(\Gamma)$$

by (3.B).

(4.D) The set $\mathcal{A}$ of normal subgroups with amenable quotients gives rise to the *proamenable completion* $P\mathcal{A}(\Gamma)$ of $\Gamma$. By (3.B), there are canonical homomorphisms from $P\mathcal{A}(\Gamma)$ to $P\mathcal{F}(\Gamma)$ and to $P\mathcal{S}(\Gamma)$. The related notion of residual amenability occurs for example in [Clair–99] and [EleSz].

(4.E) If $\mathcal{N}$ is any of the classes appearing in Examples (4.A) to (4.D) above, $P\mathcal{N}$ is a functor. This means that any homomorphism $\psi : \Gamma \longrightarrow \Delta$ factors through $\Gamma/\mathcal{CN}(\Gamma) \longrightarrow \Delta/\mathcal{CN}(\Delta)$ and then extends to a continuous homomorphism

$$P\mathcal{N}(\psi) \,:\, P\mathcal{N}(\Gamma) \longrightarrow P\mathcal{N}(\Delta).$$

As described in the introduction for $\mathcal{F}$ and $\mathcal{S}$, the *pro–$\mathcal{N}$–analogue of Grothendieck's problem* is to find examples of homomorphisms $\psi : \Gamma \longrightarrow \Delta$, with $\Gamma, \Delta$ residually $\mathcal{N}$ and possibly subjected to some extra conditions (such as finite generation or finite presentation), such that $\psi$ *is not* an isomorphism and such that $P\mathcal{N}(\psi)$ *is one.*

(4.F) The following two properties are equivalent:

(a) for any pair $N_1, N_2$ of normal subgroups not reduced to $\{1\}$ in $\Gamma$, we have $N_1 \cap N_2 \neq \{1\}$;
(b) there does *not* exist normal subgroups $N, N_1, N_2 \neq \{1\}$ of $\Gamma$ such that $N = N_1 \times N_2$;

(we leave it as an exercise to the reader to check this). For $\Gamma$ with these properties, the family $\mathcal{N}^{\neq 1}$ of all normal subgroups distinct from $\{1\}$ could be an interesting family. The topology defined by $\mathcal{N}^{\neq 1}$ is the *pronormal topology* of [GlSoS].

(4.G) Let $p$ be a prime number. Consider a group $\Gamma$ which is residually a finite $p$–group, and its embedding $\varphi_{\hat{p}} : \Gamma \longrightarrow P_{\hat{p}}(\Gamma)$ in its pro–$p$–completion.

Viewed as an abstract group, $P_{\hat{p}}(\Gamma)$ is residually finite. Indeed any totally disconnected compact group has this property. This is for example a straightforward consequence of Corollary 1 in Chapter 3, § 4, No. 6 of [Bourb–60], namely of the fact that, in a totally discontinuous locally compact group, any neighbourhood of the identity contains an open subgroup.

Assume now that $\Gamma$ is finitely generated. It is a theorem of Serre (Theorem 4.3.5 in [Wilso–98]) that any subgroup of finite index in a finitely generated pro–$p$–group, in particular in $P_{\hat{p}}(\Gamma)$, is open. Therefore, any such finite index subgroup in $P_{\hat{p}}(\Gamma)$ has index a power of $p$; in particular, $P_{\hat{p}}(\Gamma)$ is residually a finite $p$–group. Consequently, there is an embedding of the group $P_{\hat{p}}(\Gamma)$ in its own pro–$p$–completion $P_{\hat{p}}\left(P_{\hat{p}}(\Gamma)\right)$. It follows that this embedding is an isomorphism onto (see for example Theorem 1.2.5 in [Wilso–98]). From now on, we will identify $P_{\hat{p}}(\Gamma)$ and $P_{\hat{p}}\left(P_{\hat{p}}(\Gamma)\right)$ by this isomorphism.

**Proposition 1.** *Let $\Gamma$ be a finitely generated group which is residually a finite $p$–group. Let $\Delta$ be a finitely generated subgroup of $P_{\hat{p}}(\Gamma)$ containing $\Gamma$. Then the inclusion $\psi : \Gamma \longrightarrow \Delta$ induces an isomorphism $P_{\hat{p}}(\psi)$ from $P_{\hat{p}}(\Gamma)$ onto $P_{\hat{p}}(\Delta)$.*

*Proof.* Denote by $j$ the inclusion of $\Delta$ in $P_{\hat{p}}(\Gamma)$. The sequence of homomorphisms

$$\Gamma \overset{\psi}{\longrightarrow} \Delta \overset{j}{\longrightarrow} P_{\hat{p}}(\Gamma) \overset{P_{\hat{p}}(\psi)}{\longrightarrow} P_{\hat{p}}(\Delta)$$

induces a sequence

$$P_{\hat{p}}(\Gamma) \overset{P_{\hat{p}}(\psi)}{\longrightarrow} P_{\hat{p}}(\Delta) \overset{P_{\hat{p}}(j)}{\longrightarrow} P_{\hat{p}}(\Gamma) \overset{P_{\hat{p}}(\psi)}{\longrightarrow} P_{\hat{p}}(\Delta)$$

where the composition $P_{\hat{p}}(j)P_{\hat{p}}(\psi)$ is the identity of $P_{\hat{p}}(\Gamma)$ and the composition $P_{\hat{p}}(\psi)P_{\hat{p}}(j)$ is the identity of $P_{\hat{p}}(\Delta)$. It follows that the homomorphism $P_{\hat{p}}(\psi)$ is both one–to–one and onto. □

If $\Gamma$ is infinite and countable, $P_{\hat{p}}(\Gamma)$ is uncountable, and it is always possible to find $\Delta$ such that the embedding $\psi$ is proper. Thus, Proposition 1 is an answer to the *pro–p–analogue of Grothendieck's problem.*

It is easy to provide examples of situations in which the groups $\Gamma$ and $\Delta$ are not abstractly isomorphic.

(4.H) The analogous result for profinite completions follows by the same argument from the recent solution by Nikolov and Segal of a conjecture by Serre, according to which any subgroup of finite index in a finitely generated profinite group is open [NikSe03]. In particular, for any finitely generated group $\Gamma$, the natural homomorphism of $P\mathcal{F}(\Gamma)$ to $P\mathcal{F}(P\mathcal{F}(\Gamma))$ is an isomorphism, by which we can and do identify these two groups.

**Proposition 2.** *Let $\Gamma$ be a finitely generated group which is residually finite. Let $\Delta$ be a finitely generated subgroup of $P\mathcal{F}(\Gamma)$ containing $\Gamma$. Then the inclusion $\psi : \Gamma \longrightarrow \Delta$ induces an isomorphism $P\mathcal{F}(\psi)$ from $P\mathcal{F}(\Gamma)$ onto $P\mathcal{F}(\Delta)$.*

Observe again that, as soon as the inclusion $\psi$ is not an isomorphism, Proposition 2 is an answer to the *original Grothendieck's problem*, for finitely generated groups. This solution is different from those we know in the literature ([PlaTa–86], [PlaTa–89], and [BriGr–04]).

## 5. True prosoluble completions

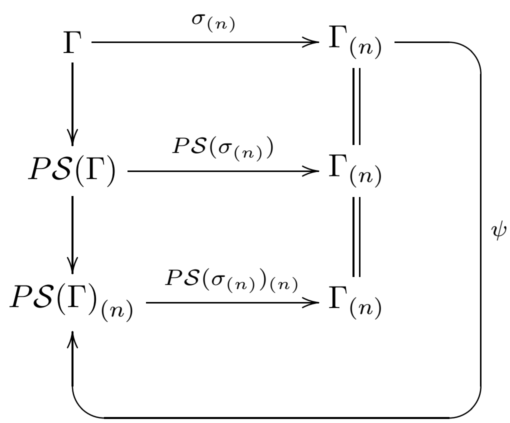
(5.A) An obvious obstruction to the residual solubility of a group $\Gamma$ is the existence of a perfect subgroup not reduced to one element. Any group $\Gamma$ contains a unique maximal perfect subgroup, that we denote by $P_\Gamma$; this follows from the fact that a subgroup generated by two perfect subgroups is itself perfect. Observe that $P_\Gamma$ is contained in the intersection $D^\infty(\Gamma)$ of all the groups in the derived series of $\Gamma$, but the inclusion can be strict. (In fact, $P_\Gamma$ is the intersection of all the groups in the so–called transfinite derived series of $\Gamma$, but this transfinite series can be as long, without repetition, as the cardinality of $\Gamma$ allows; see [Malc'–49].)

(5.B) For a topological group $G$ and an integer $n \geq 0$, we denote by $\overline{D^n}(G)$ the $n^{\text{th}}$ group of the topological derived series, defined inductively by $\overline{D^0}(G) = G$ and $\overline{D^{n+1}}(G) =$

$\overline{[\overline{D^n}(G), \overline{D^n}(G)]}$, where $\overline{[A,B]}$ stands for the closure of the subgroup of $G$ generated by commutators $a^{-1}b^{-1}ab$ with $a \in A$ and $b \in B$. We denote by $G_{(n)}$ the quotient $G/\overline{D^n}(G)$ and by $\sigma_{(n)} : G \longrightarrow G_{(n)}$ the canonical projection.

Recall that a topological group $G$ is "topologically soluble", namely such that $\overline{D^n}(G) = \{1\}$ for $n$ large enough, if and only if it is soluble, namely such that the $n^{\text{th}}$ term $D^n(G)$ of its ordinary derived series is reduced to $\{1\}$ for $n$ large enough (see for example Chapter III, § 9, No. 1 in [Bourb–72]).

(5.C) For a group $\Gamma$ and an integer $n \geq 0$, we claim that $P\mathcal{S}(\Gamma)_{(n)}$ is canonically isomorphic to $\Gamma_{(n)}$. This follows from contemplation of the commutative diagram




where the existence and canonicity of $\psi$ follow from the universal property of the projection $\sigma_{(n)}$. In other words, as the morphism $\Gamma \longrightarrow P\mathcal{S}(\Gamma)_{(n)}$ has a range which is soluble of degree $n$, this morphism factors through $\Gamma_{(n)}$.

(5.D) In a group $\Gamma$, the family $\mathcal{S}$ of all normal subgroups with soluble quotients and the countable family $(D^n(\Gamma))_{n\geq 0}$ define the same procompletion $P\mathcal{S}(\Gamma)$, by Item (3.C).

It follows from (2.C) that the topology on $P\mathcal{S}(\Gamma)$ can be defined by a metric.

Recall that the family $\mathcal{S}$ can be uncountable. P. Hall has shown that this is the case for $\Gamma$ a free group of rank 2; see [HallP–50], and the exposition in [BruBW–79].

(5.E) For a topological group $G$ and an integer $j \geq 0$, we denote by $\overline{C^j}(G)$ the $j^{\text{th}}$ group of the topological lower central series, defined inductively by $\overline{C^1}(G) = G$ and $\overline{C^{j+1}}(G) = \overline{[G, \overline{C^j}(G)]}$ for $j \geq 1$. Then $G$ is nilpotent if and only if $\overline{C^j}(G) = \{1\}$ for $j$ large enough. (Compare with (5.B).)

The nilpotent quotients $P\mathcal{N}i(\Gamma)/\overline{C^j}(P\mathcal{N}i(\Gamma))$ and $\Gamma/C^j(\Gamma)$ are isomorphic for all $j \geq 1$. (Compare with (5.C).)

The topology on $P\mathcal{N}i(\Gamma)$ can be defined by a metric. (Compare with (5.D).)

## 6. Examples

(6.A) Let us show that the canonical homomorphism $P\mathcal{S}(\Gamma) \longrightarrow P\mathcal{F}\mathcal{S}(\Gamma)$ needs not be onto.

First, consider an infinite cyclic group: $\Gamma = \mathbf{Z}$. Since $\mathbf{Z}$ is soluble, $\varphi_{\mathcal{S}} : \mathbf{Z} \longrightarrow P\mathcal{S}(\mathbf{Z})$ is an isomorphism onto. As any finite quotient of $\mathbf{Z}$ is soluble (indeed abelian), the prosoluble completion of $\mathbf{Z}$ coincides with its profinite completion. Hence

$$\mathbf{Z} \, = \, P\mathcal{S}(\mathbf{Z}) \, \not\approx \, P\mathcal{FS}(\mathbf{Z}) \, \approx \, P\mathcal{F}(\mathbf{Z}) \, \approx \, \prod_p \mathbf{Z}_p$$

where $\mathbf{Z}_p$ denotes the ring of $p$–adic integers and where the product is over all rational primes.

Here is another family of examples. For $k \geq 2$ and $d \geq 1$, denote by $F(k,d)$ the quotient of a non–abelian free group on $k$ generators by the $d$th term of its derived series (the *free soluble group* of class $d$ with $k$ generators). This group is obviously soluble and infinite, and it is known to be residually finite (Theorem 6.3 in [Gruen–57]). Hence $F(k,d) = P\mathcal{S}(F(k,d))$ embeds properly in its profinite completion $P\mathcal{F}(F(k,d))$, and the latter coincides with $P\mathcal{FS}(F(k,d))$.

(6.B) Similarly, the canonical homomorphisms $P\mathcal{S}(\Gamma) \longrightarrow P\mathcal{FS}(\Gamma)$ needs not be injective.

Consider a wreath product $\Gamma = S \wr T$ where $S$ is soluble non–abelian and where $T$ is soluble infinite. Theorem 3.1 in [Gruen–57] establishes that $\Gamma$ is not residually finite, so that in particular the morphism $\varphi_{\mathcal{FS}} : \Gamma \longrightarrow P\mathcal{FS}(\Gamma)$ has kernel not reduced to $\{1\}$. But $\Gamma$ is soluble, and therefore isomorphic to $P\mathcal{S}(\Gamma)$.

For a finitely generated group $\Gamma$, it is a theorem of P. Hall that $\Gamma/D^2(\Gamma)$ is residually finite [HallP–59], so that we have an embedding

$$\Gamma/D^2(\Gamma) \, \approx \, P\mathcal{S}(\Gamma)/\overline{D^2}(P\mathcal{S}(\Gamma)) \, \longrightarrow \, P\mathcal{F}\left(\Gamma/D^2(\Gamma)\right).$$

In particular, a finitely generated group $\Gamma$ which is soluble of class 2 always embeds in $P\mathcal{F}(\Gamma)$.

However, there exists a finitely generated soluble group of class 3 which is non–Hopfian [HallP–61]. In particular, a finitely generated group $\Gamma$ which is soluble of class at least 3 does not always embed in $P\mathcal{F}(\Gamma) = P\mathcal{FS}(\Gamma)$. A finitely *presented* soluble group which is non–Hopfian is described in [Abels–79].

(6.C) For each integer $k \geq 2$, let $F_k$ denote the nonabelian free group on $k$ generators. This is a residually soluble group (indeed it is residually a finite $p$–group for any prime $p$, by a result of Iwasawa, see No. 6.1.9 in [Robin–82]), and therefore embeds in its true prosoluble completion $P\mathcal{S}(F_k)$.

Since the abelianized groups $F_k/[F_k, F_k] \approx \mathbf{Z}^k$ are pairwise non–isomorphic, the true prosoluble completions $P\mathcal{S}(F_k)$ are pairwise non–isomorphic by (2.D) or (5.C).

(6.D) For $d \geq 3$, the group $SL_d(\mathbf{Z})$ is perfect (this is an immediate consequence of the following fact: any matrix in $SL_d(\mathbf{Z})$ is a product of elementary matrices; see for example Theorem 22.4 in [MacDu–46]). It follows that $P\mathcal{S}(SL_d(\mathbf{Z}))$ is reduced to one element and that $P\mathcal{S}(GL_d(\mathbf{Z}))$ is the group of order two.

Consider however an integer $d \geq 2$, a prime $p$, and the congruence subgroup

$$\Gamma_d(p) \, = \, \mathrm{Ker}\left(SL_d(\mathbf{Z}) \longrightarrow SL_d(\mathbf{Z}/p\mathbf{Z})\right),$$

which is a subgroup of finite index in $SL_d(\mathbf{Z})$. Then $\Gamma_d(p)$ is residually a finite $p$–group, and therefore embeds in $P\mathcal{S}(\Gamma_d(p))$; see for example Proposition 3.3.15 in [RibZa–00], since their proof written for $d = 2$ carries over to any $d \geq 2$.

In particular, the property for a group $\Gamma$ to embed up to finite kernel in its prosoluble completion is *not* stable by finite index.

(6.E) Let $\Gamma$ be the first Grigorchuk group, which is infinite and residually a finite 2–group. This group was introduced in [Grigo–80]; see also Chapter VIII in [Harpe–00].

All proper quotients of $\Gamma$ are finite 2–groups (and are consequently finite nilpotent groups). $\Gamma$ is not soluble, for example because each term of its derived series is of finite index (the indices are computed in Section 13 of [Grigo–00]). It follows that $\Gamma$ embeds properly in its true prosoluble completion $P\mathcal{S}(\Gamma)$, which is canonically isomorphic to its profinite completion, its prosoluble completion, and its pro–2–completion:

$$\Gamma \subset P\mathcal{S}(\Gamma) \,=\, P\mathcal{FS}(\Gamma) \,=\, P\mathcal{F}(\Gamma) \,=\, P_{\hat{2}}(\Gamma).$$

(6.F) Groups which are residually soluble and not soluble include Baumslag–Solitar groups $\langle a, t \mid ta^p t^{-1} = a^q \rangle$ for $|p|\,, |q| \geq 2$ [RapVa–89], positive one–relator groups [Baums–71], non–trivial free products[3] of soluble groups, and various just non-soluble groups [BruSV–99]. Also, some free products of soluble groups amalgamated over a cyclic group are residually soluble and not soluble [Kahro1].

These are potential examples for further investigation of the properties of true prosoluble completions.

(6.G) For groups which are not residually soluble, it is a natural problem to find out properties of the true prosoluble kernel

$$\mathrm{Ker}(\Gamma \longrightarrow P\mathcal{S}(\Gamma)) \,=\, \bigcap_{n=0}^{\infty} D^n(\Gamma).$$

For example, consider the one–relator group[4]

$$\Gamma \,=\, \langle a, b; a = [a, a^b] \rangle$$

which appears in [Baums–69] and [Baums–71]. Baumslag has shown that the profinite kernel of $\Gamma$ coincides with the derived group $D(\Gamma)$, which is also the smallest normal subgroup of $\Gamma$ containing $a$, and which is not reduced to one element by the Magnus Freiheitssatz. As $D(\Gamma)$ is perfect, the true prosoluble kernel of $\Gamma$ coincides also with $D(\Gamma)$. Since $\Gamma/D(\Gamma) \approx \mathbf{Z}$, we have $\mathbf{Z} \approx P\mathcal{S}(\Gamma) \not\approx P\mathcal{F}(\Gamma) \approx \prod_p \mathbf{Z}_p$; compare with (6.A).

Examples to investigate include:

- other one–relator groups which are not residually soluble;

[3] Recall of a particular case of a result of Gruenberg, see Section 4 in [Gruen–57]: any free product of residually soluble groups is residually soluble. By "non–trivial" free product, we mean that the free product has at least two factors, and that it is not the free product of two groups of order two.

[4] Recall that $[x, y] = x^{-1}y^{-1}xy$ and $x^y = y^{-1}xy$.

- ∘ wreath products as in (6.B);
- ∘ free products of nilpotent groups with amalgamation (see for example Proposition 7 in [Kahro1]);
- ∘ parafree groups (see (7.C));
- ∘ meta–residually soluble groups, namely groups having a residually soluble normal subgroup with residually soluble quotient (see Theorem 2 in [Kahro2]); particular cases: free–by–free groups which are not residually soluble.

(6.H) Does there exist a discrete group $\Gamma$ which on the one hand is residually finite and residually soluble, namely for which both homomorphisms $\Gamma \longrightarrow P\mathcal{F}(\Gamma)$, $\Gamma \longrightarrow P\mathcal{S}(\Gamma)$ are injective, and which on the other hand is such that the homomorphism $\Gamma \longrightarrow P\mathcal{FS}(\Gamma)$ is NOT injective?

## 7. Solution of the true prosoluble analogue of Grothendieck's problem

(7.A) Consider the Grigorchuk group $\Gamma$ and the inclusion $\varphi_{\hat{2}} : \Gamma \longrightarrow P_{\hat{2}}(\Gamma)$ of $\Gamma$ in its pro–2–completion. We have already observed in (6.E) that the canonical homomorphism $P\mathcal{F}(\Gamma) \longrightarrow P_{\hat{2}}(\Gamma)$ of (4.A) is an isomorphism onto. Let $\Delta$ be a finitely generated subgroup of $P_{\hat{2}}(\Gamma)$ which contains $\Gamma$, and let $\psi : \Gamma \longrightarrow \Delta$ denote the inclusion. We know from Propositions 1 and 2 that both $P_{\hat{2}}(\psi) : P_{\hat{2}}(\Gamma) \longrightarrow P_{\hat{2}}(\Delta)$ and $P\mathcal{F}(\psi) : P\mathcal{F}(\Gamma) \longrightarrow P\mathcal{F}(\Delta)$ are isomorphisms onto. In particular, the canonical homomorphism $P\mathcal{F}(\Delta) \longrightarrow P_{\hat{2}}(\Delta)$ of (4.A) is also an isomorphism onto.

We know moreover that, in $P_{\hat{2}}(\Gamma)$, any proper normal subgroup has index finite and a power of 2 (Theorem 12.3.31 in [LeGMc–02]). [Though it is useless for our argument here, let us point out that $P_{\hat{2}}(\Gamma)$ coincides with the closure of $\Gamma$ in the compact automorphism group of the rooted dyadic tree on which $\Gamma$ acts in the usual way (Theorem 9 in [Grigo–00]).]

**Lemma.** *With the notation above, a quotient group of $\Delta$ is finite if and only if it is soluble.*

*Proof.* Let $N$ be a normal subgroup of $\Delta$ such that $\Delta/N$ is finite. Since $P\mathcal{F}(\Delta) \approx P\mathcal{F}(\Gamma) \approx P_{\hat{2}}(\Gamma) \approx P_{\hat{2}}(\Delta)$, the quotient $\Delta/N$ is a finite 2–group. In particular, it is a nilpotent group, a fortiori a soluble group.

For the converse, we proceed by contradiction and assume that there exists a normal subgroup $N$ in $\Delta$ such that the quotient $\Delta/N$ is soluble and infinite. Since $\Delta$ is finitely generated, Zorn's lemma implies that there exists a normal subgroup $M$ in $\Delta$ containing $N$ such that $\Delta/M$ is just infinite.

Let $\Delta/M = D^0 \supset D^1 \supset D^2 \supset \cdots$ denote the derived series of the just infinite soluble group $\Delta/M$; denote by $k$ the smallest integer such that $D^{k+1}$ is of infinite index in $\Delta/M$. Then $D^{k+1} = \{1\}$ because $\Delta/M$ is just infinite. The group $D^k$ is abelian and of finite index in $\Delta/M$; it is therefore finitely generated. The torsion subgroup of $D^k$ is normal in $\Delta/M$; it is of infinite index in $D^k$, and thus also in $\Delta/M$; hence $D^k$ has no torsion. Let $\Delta_1$ denote the inverse image of $D^k$ in $\Delta$; it is a normal subgroup of finite index containing $M$. The quotient $\Delta_1/M$ is finitely generated and free abelian, say $\Delta_1/M \approx \mathbf{Z}^d$ for some $d \geq 1$.

Let $Q_2$ be the subgroup of $\Delta_1/M$ generated by the cubes, and denote by $\Delta_2$ its inverse image in $\Delta_1$; observe that $\Delta_2$ is of finite index in $\Delta_1$, and therefore also in $\Delta$, and that $\Delta_1/\Delta_2 \approx (\mathbf{Z}/3\mathbf{Z})^d$. Let $\Delta_3$ be the intersection of all the conjugates of $\Delta_2$ in $\Delta$; observe that $\Delta_3$ is of finite index in $\Delta_2$, and therefore also in $\Delta$, and that it is also normal in $\Delta$. Then $\Delta/\Delta_3$ is a finite quotient of $\Delta$ of order

$$[\Delta : \Delta_1] \times [\Delta_1 : \Delta_2] \times [\Delta_2 : \Delta_3] \;=\; [\Delta : \Delta_1] \times 3^d \times [\Delta_2 : \Delta_3].$$

In particular, 3 divides the order of $\Delta/\Delta_3$.

We have obtained a contradiction because, since $P_{\hat{2}}(\Delta) \approx P\mathcal{F}(\Delta)$, any finite quotient of $\Delta$ is a 2–group. This ends the proof. □

We have shown the following result.

**Proposition 3.** *Let* $\Gamma$ *be the Grigorchuk group, let* $\Delta$ *be a finitely generated subgroup of the pro–2–completion of* $\Gamma$ *which contains* $\Gamma$ *as a proper subgroup, and let* $\psi$ *denote the inclusion of* $\Gamma$ *in* $\Delta$. *Then the morphism*

$$P\mathcal{S}(\psi) \,:\, P\mathcal{S}(\Gamma) \;\longrightarrow\; P\mathcal{S}(\Delta)$$

*induced by* $\psi$ *on the true prosoluble completions is an isomorphism.*

For example, let $\delta$ be an element of infinite order in $P_{\hat{2}}(\Gamma)$. Such elements exist, from direct inspection, or as a consequence of Theorem 10 in [Grigo–00]. A possible choice for $\Delta$ is the group generated by $\Gamma$ and $\delta$. Then $\psi$ "is far from" an isomorphism, since $\Gamma$ is a 2–group and $\Delta$ is not a torsion group.

(7.B) The following question is due to Said Sidki. Does there exist a residually soluble extension $\tilde{\Gamma}$ of the Grigorchuk group and some homomorphism $\tilde{\psi}$ from $\tilde{\Gamma}$ into some residually soluble group $\tilde{\Delta}$ such that $\tilde{\psi}$ is not an isomorphism and $P\mathcal{S}(\tilde{\psi})$ is an isomorphism? This would provide a "finitely presented analogue" of Proposition 3. An interesting HNN-extension of $\Gamma$ appears in [Grigo–96] and [Grigo–98].

(7.C) Let us describe how Problem (i) from the introduction has been studied for true pronilpotent completions.

Recall that a group $\Gamma$ is *parafree* if it is residually nilpotent and if there exists a free group $F$ such that $F/C^j(F)$ and $\Gamma/C^j(\Gamma)$ are isomorphic for all $j \geq 1$. Nonfree parafree groups have been discovered by G. Baumslag [Baums–67]; later papers include [Baums–68] and [Baums–05].

Let $\Gamma$ be a parafree group with finitely generated abelianization. Let $F$ be as above; observe that $F$ is finitely generated. Choose a subset $T$ of $\Gamma$ of which the canonical image freely generates the free abelian group $\Gamma/C^2(\Gamma)$; observe that $T$ is finite. (Be careful: $T$ need not generate $\Gamma$.) Let $S$ be a free set of generators of $F$ such that the canonical image of $S$ and $T$ are in bijection with each other through the given isomorphism $F/C^2(F) \approx \Gamma/C^2(\Gamma)$, and let $\varphi : S \longmapsto T$ be a compatible bijection. Then $\varphi$ extends to a homomorphism, again denoted by $\varphi$, from $F$ to $\Gamma$, and this $\varphi$ induces the given isomorphism from $F/C^2(F)$ onto $\Gamma/C^2(\Gamma)$. A group homomorphism with range a nilpotent group $A$ is onto if and only if its composition with the abelianization $A \longrightarrow A/C^2(A)$ is onto (see e.g. [Bourb–70], Corollary 4, Page A I.70); it follows that the homomorphism $\varphi_{(j)}$ from $F/C^j(F)$ to $\Gamma/C^j(\Gamma)$ induced by $\varphi$ is onto for all $j \geq 1$. Since the group $F/C^j(F) \approx \Gamma/C^j(\Gamma)$ is Hopfian for all $j \geq 1$ (any finitely generated residually finite group is Hopfian, by Mal'cev theorem [Mal'c–40]), it follows that $\varphi_{(j)}$ is an isomorphism for all $j \geq 1$. We have shown:

**Observation.** *A residually nilpotent group* $\Gamma$ *with finitely generated abelianization is parafree if and only if there exist a free group* $F$ *of finite rank and a homomorphism* $\varphi : F \longrightarrow \Gamma$ *which induces an isomorphism* $P\mathcal{N}i(\varphi) : P\mathcal{N}i(F) \longrightarrow P\mathcal{N}i(\Gamma)$ *on the true pronilpotent completions.*

Note that, in case the set $T$ is moreover generating for $\Gamma$, the group $\Gamma$ itself is free; see Problem 2 on Pages 346–347 of [MaKaS–66]. However, as G. Baumslag discovered, there are pairs $(\Gamma, F)$ as in the observation with $\Gamma$ not free and generated by $k+1$ elements, and with $F$ free of rank $k$, for each $k \geq 2$.

There is a related example on Page 173 of [Stall–65]. Let $F_2$ be the free group on two generators $x$ and $y$. Set $y' = yxyx^{-1}y^{-1}$. Let $\Gamma$ be the subgroup of $F_2$ generated by $x$ and $y'$. Then $\Gamma$ is a proper subgroup of $F_2$, because $y \notin \Gamma$ (though $\Gamma$ is isomorphic to $F_2$). The inclusion of $\Gamma$ in $F_2$ provides an isomorphism $\Gamma/C^j(\Gamma) \longrightarrow F_2/C^j(F_2)$ for all $j \geq 1$, and therefore an isomorphism from $P\mathcal{N}i(\Gamma)$ onto $P\mathcal{N}i(F_2)$.

(7.D) In [Baums–68], there are examples of pairs $(F, \Gamma)$ of groups with the following properties: $F$ is finitely generated and free, both $F$ and $\Gamma$ are residually soluble, the quotients $F/D^j(F)$ and $\Gamma/D^j(\Gamma)$ are isomorphic for all $j \geq 0$, nevertheless $F$ and $\Gamma$ are *not* isomorphic (indeed $\Gamma$ is not finitely generated).

However, nothing like the observation of Item (7.C) holds for the quotients by the groups of the derived series. We do not know if there exist a finitely generated free group $F$, a group $\Gamma$ which is residually soluble and not free, and a homomorphism $\psi : F \longrightarrow \Gamma$ such that $\psi$ induces an isomorphism $F/D^j(F) \longrightarrow \Gamma/D^j(\Gamma)$ for all $j \geq 1$, or equivalently such that $P\mathcal{S}(\psi) : P\mathcal{S}(F) \longrightarrow P\mathcal{S}(\Gamma)$ is an isomorphism.

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