Largest Intersecting Families of Almost Linear Posets

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Abstract. The publication of the Erdős-Ko-Rado Theorem in 1961 sparked an interest in classifying the largest intersecting sets of various combinatorial structures such as sets, words, permutations and graphs. In this paper we introduce the idea of intersecting posets: two posets on n points intersect if they share a comparison. It is easily seen that an intersecting set of linear orders on n points has size at most $n!/2$. Let $Y_{k,n}$ be the set of all posets which can be obtained from a linear order on n points by removing the comparison between the k^{th} and $k+1^{st}$ smallest points. We show that an intersecting subset of $Y_{k,n}$ has size at most $n!/4$ and classify the intersecting families attaining this bound. Moreover, we obtain a sharp bound on intersecting subsets of $\bigcup_{k=1}^{n-1} Y_{k,n}$.

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1. Introduction and Definitions

Ever since Erdős, Ko, Rado, Katona and others investigated t-intersecting families of subsets of a set in the 1960s, classifying the largest collections of intersecting subsets of an underlying set has been a standard problem of extremal set theory. Building on the original question, one approach has been to impose additional restrictions on the collection. Another approach is to vary the combinatorial objects which are the elements of the underlying set. Structures which have been considered more recently in this context are sets, permutations, words and partial permutations. In this paper, we introduce the idea of intersecting posets.

Denote the set of the first n natural numbers by $[n]$. A partially ordered set, or *poset* for short, on $[n]$ is a set of ordered pairs, $p \subseteq [n] \times [n]$, which is antisymmetric and transitive: for all $x, y, z \in [n]$ we have $(y, x) \notin p$ whenever $(x, y) \in p$ and $(x, y), (y, z) \in p$ implies $(x, z) \in p$. The ordered pair (x, y) denotes the *comparison* $x < y$, though we often write $x <_p y$ instead of $(x, y) \in p$. The set of all partial orders on $[n]$ is denoted by $\mathcal{P}_n$.

Definition 1.1. Two partial orders $p, q \in \mathcal{P}_n$ *intersect* if they share a comparison, i.e. if there exist $x, y \in [n]$ such that $(x, y) \in p \cap q$. A set of partial orders $\mathcal{F} \subseteq \mathcal{P}_n$ is *intersecting* if every pair of elements of $\mathcal{F}$ is intersecting.

Investigations in this field usually begin by finding the size of the largest intersecting subset of the set of all objects of a given type. Having found a bound, the next natural aim is to classify all intersecting families attaining the bound.

For instance, let $\mathcal{S}_n$ be the set of all permutations on n points. After Deza and Frankl showed in [5] that an intersecting subset of $\mathcal{S}_n$ has size at most $(n-1)!$, Cameron and Ku showed in [3] that an intersecting family attaining this bound must be the coset of a stabiliser of one point.

Similarly, let $\mathcal{A}_k^n$ be the set of all k-partial permutations on $[n]$. These are injections from any k-subset of $[n]$ into $[n]$. A bound for intersecting subsets of $\mathcal{A}_k^n$ was obtained by Ku and Leader in [8]. Then Li and Wang proved in [9] that for any intersecting subset A of $\mathcal{A}_k^n$ attaining this bound, there exist $x, \varepsilon \in [n]$ such that A consists of all k-partial permutations which map x to ε.

The results described above follow the spirit of Erdős-Ko-Rado: they show that, for certain values of the parameters or in general, the largest possible intersecting set consists of all such elements which have a fixed set of image points in common. To generalise this idea, it helps to view this fixing concept as a special case of a saturation process. We will illustrate this by describing the results about t-intersecting subsets of $[n]$ which have size k.

A family F of k-subsets of $[n]$ is *t-intersecting* if $|A \cap B| \geq t$ for all $A, B \in F$. Erdős, Ko and Rado showed in [4] that given k, t and $n \geq n_0(k, t)$, a t-intersecting family of k-subsets of an n-set has size at most $\binom{n-t}{k-t}$. Moreover, if a family attains this bound, then all of its elements contain some *fixed* t-subset of the n-set.

This classical result was complemented 36 years after the publication of [4]: Ahlswede and Khachatrian showed in [1] that for $n < n_0$, we have a different bound and optimal families are obtained by saturation: up to permutations of $[n]$, any t-intersecting family of k-subsets of an n-set attaining the bound must be the collection of k-subsets of $[n]$ containing at least $t + r$ of the first $t + 2r$ points, where r is given explicitly as a function of n, k, t.

In a subsequent article [2], Ahlswede and Khachatrian proved a conjecture of Frankl and Füredi in [6] by showing that the principle of saturation also applies to t-intersecting sets of words of maximal size.

For many combinatorial structures, the generalisation from fixing to saturation does not become relevant before we move from considering 1-intersecting sets to studying t-intersecting families for $t > 1$, and even then it is only optimal for small values of n. We will see that this is not the case for the poset classes considered in this paper.

Classifying all intersecting subsets of $\mathcal{P}_n$ of maximal size in general is likely to be very difficult, so we need to consider subclasses. One approach is to fix the poset, and consider all permutations of the labels. The scenario where the fixed poset is simply a chain of length n is easily dealt with in Section 2: both fixing and saturation are optimal here, and there are many other families attaining the bound.

The main purpose of this paper is to investigate what happens if we remove just one of the comparisons in a linear order. In Section 3 we fix such a poset and take our class to be all permutations of the labels. We partition this class according to whether the individual posets intersect. By describing the blocks of this partition, we obtain a complete classification of intersecting families of maximal size in this class. To get an idea of where these intersecting families lie on the fixing – saturating spectrum, we show that fixing is not optimal in this class whereas, in all but a few marginal cases, a saturation family does attain the bound.

In Section 4 we no longer fix the poset, but consider the union of the classes in the previous section. Here we use the classical method of cyclic orderings to obtain a bound on intersecting subsets of this class. The comparisons on $[n]$ are arranged on a circle, and it is shown that posets in the class are equivalent to intervals on the cyclic orderings. Finally, we show that both fixing and saturating give optimal intersecting families in this class.

2. Linear Orders

Two elements x, y of $[n]$ are *comparable* under the poset p if either $x <_p y$ or $y <_p x$. If all pairs of elements of $[n]$ are comparable under p, we say that p is a *linear* order. For some labelling $\{x_1, x_2, \ldots, x_n\} = [n]$ we may use the notation $x_1 x_2 \ldots x_n$ for the linear order under which $x_1 < x_2 < \cdots < x_n$. The set of all linear orders on $[n]$ is denoted by $\mathcal{L}_n$.

For $\sigma \in \mathcal{L}_n$, denote its *reverse* by $\mathrm{rev}(\sigma)$. That is, if $\sigma = x_1 x_2 \ldots x_n$ then $\mathrm{rev}(\sigma) = x_n x_{n-1} \ldots x_1$. The following proposition classifies the intersecting subsets of $\mathcal{L}_n$ of maximal size.

Proposition 2.1. *If $\mathcal{F} \subseteq \mathcal{L}_n$ is intersecting then $|\mathcal{F}| \le n!/2$. In particular, $\mathcal{F}$ has maximal size if and only if $\mathcal{F}$ is a transversal of*

$$\{\, \{\sigma, \mathrm{rev}(\sigma)\} \mid \sigma \in \mathcal{L}_n \,\}.$$

Proof. It is not hard to see that two linear orders do not intersect if, and only if, one is the reverse of the other. The result follows. □

By making the simple observation that a linear order intersects any other except its reverse, we have succeeded in classifying all intersecting subsets of $\mathcal{L}_n$ of maximal size. In the following section, we investigate what happens if we remove just one of the comparisons in the linear orders: can we classify the intersecting subsets of maximal size in that case?

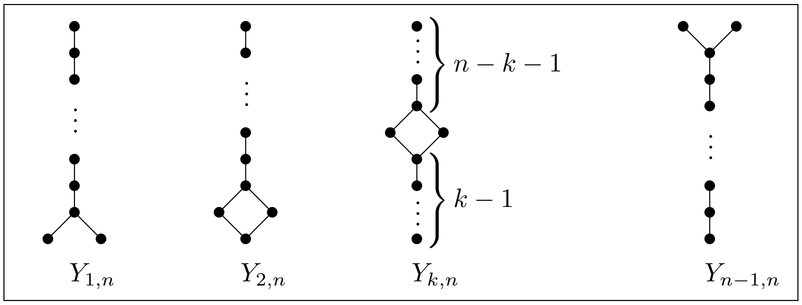

FIGURE 1. Hasse diagrams of elements of $Y_{k,n}$, $1 \leq k \leq n-1$, $n \geq 3$.

3. Fixing the Isomorphism Class

Consider those isomorphism classes in $\mathcal{P}_n$ whose elements are chains with one point replaced by an antichain of size 2: for a linear order $\sigma = x_1 x_2 \dots x_n \in \mathcal{L}_n$, $n \geq 3$, and $1 \leq k \leq n-1$, define

$$y_k(\sigma) = \sigma \setminus \{(x_k, x_{k+1})\}$$

and set

$$Y_{k,n} = \{\, y_k(\sigma) \mid \sigma \in \mathcal{L}_n \,\}.$$

To characterise the intersecting subsets of $Y_{k,n}$, we wish to partition $Y_{k,n}$ in such a way that for any $p \in Y_{k,n}$, the elements of $Y_{k,n}$ which p does not intersect are in the same block as p. So we begin by considering the following set:

$$N(p) = \{\, q \in Y_{k,n} \mid p \cap q = \emptyset \,\}$$

which is the subject of the lemma below. For any poset $p \in \mathcal{P}_n$, we denote by $\mathcal{L}(p)$ the set of linear extensions of p, i.e.

$$\mathcal{L}(p) = \{\, \sigma \in \mathcal{L}_n \mid p \subseteq \sigma \,\}.$$

Lemma 3.1. *If $p \in Y_{k,n}$ then $N(p) = \{\, y_k(\mathrm{rev}(\sigma)) \mid \sigma \in \mathcal{L}(p) \,\}$.*

Proof. Let $p, q \in Y_{k,n}$ be such that p and q do not intersect. Then there exist linear extensions $\hat{p}$ of p and $\hat{q}$ of q such that $\hat{p}$ and $\hat{q}$ do not intersect. But two linear orders do not intersect if, and only if, one is the reverse of the other. Thus $\hat{q} = \mathrm{rev}(\hat{p})$, which gives $q = y_k(\mathrm{rev}(\hat{p}))$. In other words,

$$N(p) \subseteq \{\, y_k(\mathrm{rev}(\sigma)) \mid \sigma \in \mathcal{L}(p) \,\}.$$

Conversely, for $p \in Y_{k,n}$ and $\sigma \in \mathcal{L}(p)$, we have $p \subset \sigma$ and $y_k(\mathrm{rev}(\sigma)) \subset \mathrm{rev}(\sigma)$ by definition. Therefore $p \cap y_k(\mathrm{rev}(\sigma)) = \emptyset$ since $\sigma \cap \mathrm{rev}(\sigma) = \emptyset$. □

To obtain the blocks of the desired partition of $Y_{k,n}$, we keep adding all such posets to $N(p)$ which do not intersect with some poset that is already in $N(p)$: for a set of posets $X \subseteq Y_{k,n}$, define

$$N(X) = \{\, q \in Y_{k,n} \mid p \cap q = \emptyset \text{ for some } p \in X \,\}$$

and set

$$B(p) = \bigcup_{i \in \mathbb{N}} N^i(p).$$

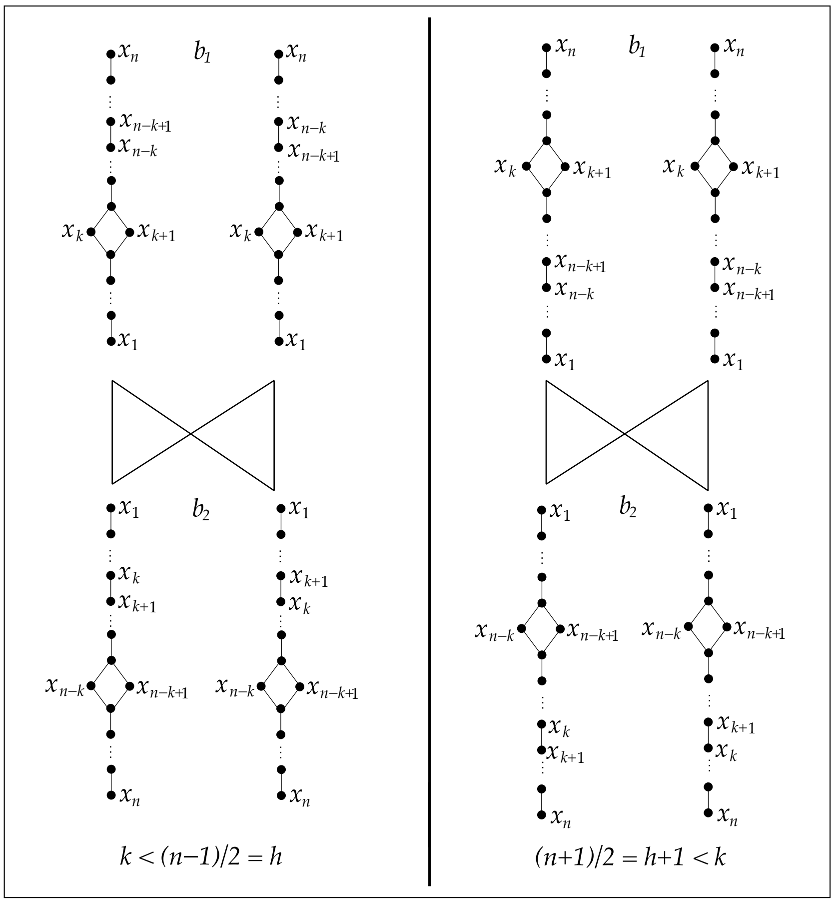

FIGURE 2. elements of $B(p)$ where $p = y_k(x_1 \ldots x_n)$ and $k < h$ on the left, or $k > h+1$ on the right. Posets which do not intersect are joined by a line. On either side, the elements of $b_1(B(p))$ and $b_2(B(p))$ are shown on the top and bottom respectively.

Intuitively, $B(p)$ is obtained from linear extensions of p by successively applying the following operations: taking the reverse, decoupling at the k^{th} level,

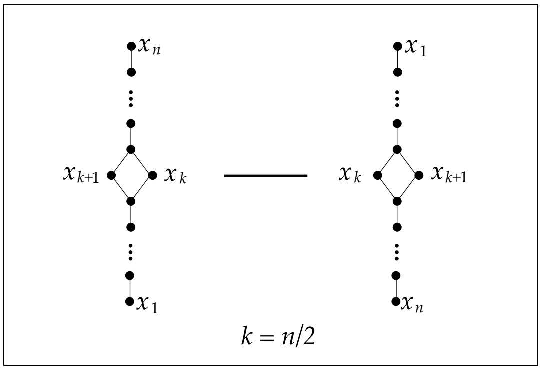

FIGURE 3. elements of $B(p)$ when $k = n/2$ and $p = y_k(x_1 \dots x_n)$. Posets which do not intersect are joined by a line. The set $b_1(B(p))$ merely contains the poset on the left. Similarly, $b_2(B(p))$ consists of the poset on the right.

and swapping at the $(n-k)^{th}$ level. In fact, it is not difficult to convince oneself that when these two levels do not overlap, $B(p)$ contains the posets in Figure 2. If $k = n - k$ then $B(p)$ has only one element other than p, which is obtained by turning p upside down: see Figure 3. A slightly more complicated situation occurs when the k^{th} and $(n-k)^{th}$ level are adjacent; see Figure 4.

To summarise, we have

$$|B(p)| = \begin{cases} 4 & k \notin \{\frac{n}{2}, \frac{n\pm 1}{2}\} \\ 2 & k = \frac{n}{2} \\ 6 & k = \frac{n\pm 1}{2} \end{cases}$$

and the precise elements of $B(p)$ are given by Lemma 3.2, which is the main auxiliary result enabling us to obtain a bound on the size of intersecting subsets of $Y_{k,n}$.

To state the lemma, note that we already have the decoupling operator y_k, so we need to define a swapping operator. Clearly, a permutation can act on an order by permuting the labels. Let $\sigma \in \mathcal{L}_n$ and define $\tau_i \in \mathcal{S}_n$ to be the transposition swapping the labels of the i^{th} and $i+1^{st}$ smallest points in σ:

$$\tau_i(x_1 x_2 \dots x_n) = x_1 x_2 \dots x_{i-1} x_{i+1} x_i x_{i+2} \dots x_n.$$

Lemma 3.2. *For positive integers k and n with $1 \leq k \leq n-1$, set $h = (n-1)/2$. Let $p \in Y_{k,n}$ and let $\sigma = x_1 \dots x_n \in \mathcal{L}_n$ be such that $p = y_k(\sigma)$.*

1. *If $k \notin \{h, h+1\}$ then*
$$B(p) = \{y_k(\sigma),\ y_k(\mathrm{rev}(\sigma)),\ y_k(\tau_{n-k}(\sigma)),\ y_k(\tau_{n-k}(\mathrm{rev}(\sigma)))\}.$$
2. *If $k \in \{h, h+1\}$ then $B(p) = \{\, y_k(\omega), y_k(\mathrm{rev}(\omega)) \mid \omega \in \Omega_\sigma \,\}$, where*
$$\Omega_\sigma = \{\, x_1 \dots x_{h-1} u\, v\, w\, x_{h+3} \dots x_n \mid \{u, v, w\} = \{x_h, x_{h+1}, x_{h+2}\}\,\}.$$

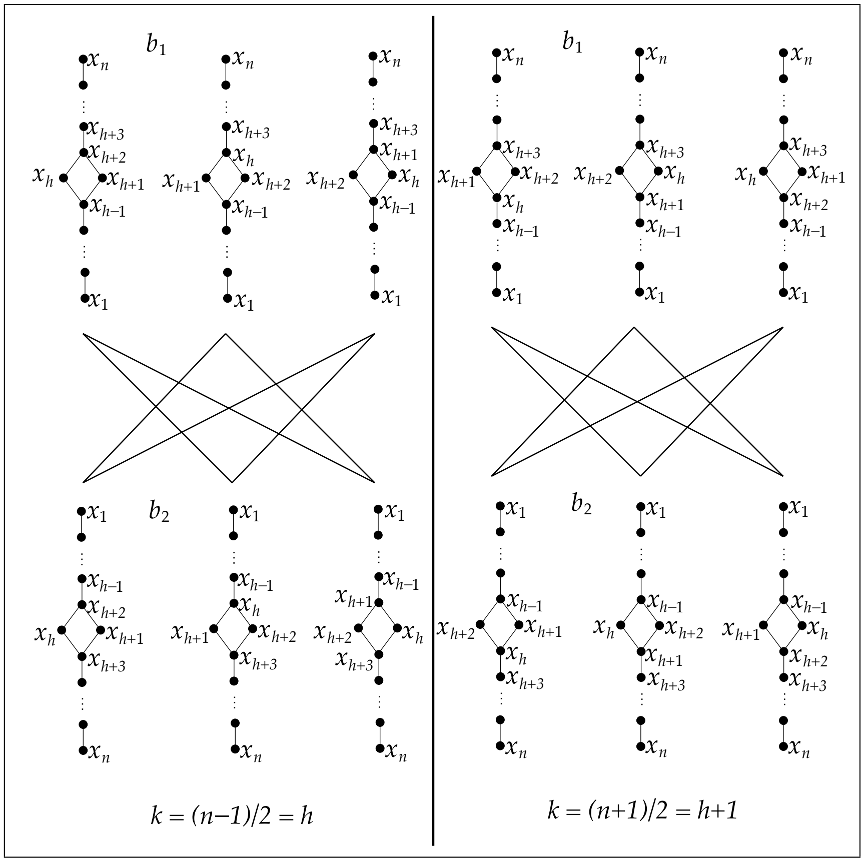

FIGURE 4. elements of $B(p)$ where $p = y_k(x_1 \ldots x_n)$ and $k = h$ or $k = h + 1$. Posets which do not intersect are joined by a line. On either side, the elements of $b_1(B(p))$ and $b_2(B(p))$ are shown on the top and bottom respectively.

Note that the case $k = n/2$ is subsumed in (1) in the statement of Lemma 3.2. In this case, the swapping operator $\tau_{n-k} = \tau_k$ swaps the two incomparable points in elements of $Y_{k,n}$ and hence does not change the poset: we have $B(p) = \{y_k(\sigma),\ y_k(\text{rev}(\sigma))\}$ when $k = n/2$.

A formal proof of Lemma 3.2 is rather tedious, and we omit it here. Referring to Figures 2 - 4 is much more informative for the reader. Moreover, these figures show that $B(p)$ is generated by any of its elements: for any $q \in B(p)$, we have $B(q) = B(p)$ and so

$$\mathcal{B}_{k,n} = \{\, B(p) \mid p \in Y_{k,n} \,\}$$

is a partition of $Y_{k,n}$.

To characterise the intersecting subsets of $Y_{k,n}$ of maximal size, we partition each $B(p)$ in two halves, such that for every $q \in B(p)$, the posets which do not intersect with q are not in the same half as q: see Figures 2 - 4. More formally, for $p \in Y_{k,n}$ with $p = y_k(\sigma)$, $\sigma = x_1 \dots x_n$, set

$$b_1(B(p)) = \begin{cases} \{y_k(\sigma),\ y_k(\tau_{n-k}(\sigma))\} & k \neq \frac{n\pm1}{2} \\ \{\, y_k(\omega) \mid \omega \in \Omega_\sigma \,\} & k = \frac{n\pm1}{2} \end{cases}$$

and

$$b_2(B(p)) = \begin{cases} \{y_k(\mathrm{rev}(\sigma)),\ y_k(\tau_{n-k}(\mathrm{rev}(\sigma)))\} & k \neq \frac{n\pm1}{2} \\ \{\, y_k(\mathrm{rev}(\omega)) \mid \omega \in \Omega_\sigma \,\} & k = \frac{n\pm1}{2} \end{cases}$$

where Ω_σ is given in Lemma 3.2. Note that the partition $\{b_1(B), b_2(B)\}$ of $B = B(p)$ does not depend on the choice of p.

Theorem 3.3. *Let k, n be natural numbers with $1 \leq k < n$ and $n \geq 4$.*

Then $\mathcal{F}$ is an intersecting subset of $Y_{k,n}$ of maximal size if, and only if, $\mathcal{F}$ is the union of a transversal of

$$\{\, \{b_1(B), b_2(B)\} \mid B \in \mathcal{B}_{k,n} \,\}\,.$$

Proof. Let $B \in \mathcal{B}_{k,n}$. It is clear from the definitions of N and B that for any $q \in B$, all posets in $Y_{k,n}$ which do not intersect with q are also in B. Thus if $\{\, F_B \mid B \in \mathcal{B}_{k,n} \,\}$, is a collection of intersecting families with $F_B \subseteq B$, then their union is also intersecting. Conversely, any intersecting subset of $Y_{k,n}$ can be decomposed in this way.

Indeed, let $\mathcal{F} \subseteq Y_{k,n}$ be intersecting and suppose $\mathcal{F}$ has maximal size. Then we must have

$$\mathcal{F} = \bigcup_{B \in \mathcal{B}_{k,n}} F_B$$

where for each $B \in \mathcal{B}_{k,n}$, the set $F_B = \mathcal{F} \cap B$ is an intersecting subset of B of maximal size. So to prove the proposition, we need to demonstrate that for $B \in \mathcal{B}_{k,n}$, if F is an intersecting subset of B of maximal size, then either $F = b_1(B)$ or $F = b_2(B)$.

We begin by showing that $b_1(B)$ and $b_2(B)$ are intersecting and maximal in terms of set inclusion. Let $p \in Y_{k,n}$, $B = B(p)$ and $\sigma = x_1 \dots x_n \in \mathcal{L}_n$ such that $p = y_k(\sigma)$. We begin by considering the case $k \neq (n\pm1)/2$ and let $\pi \in \{\sigma, \mathrm{rev}(\sigma)\}$. The transposition τ_i only replaces a single comparison in π by its reverse, so provided $|y_k(\pi)| \geq 2$, then $y_k(\pi)$ must intersect $y_k(\tau_i(\pi))$. Similarly, the operator y_k only removes a single comparison from π, so $|y_k(\pi)| \geq 2$ is satisfied whenever $|\pi| \geq 3$. Since $\pi \in \mathcal{L}_n$, we have $|\pi| \geq 3$ for all $n \geq 3$. We have shown

$$n \geq 3,\ k \neq \frac{n\pm1}{2},\ q \in b_i(B) \implies N(q) \subseteq b_j(B),\ \{i,j\} = \{1,2\}, \tag{1}$$

in other words, $b_1(B)$ and $b_2(B)$ are intersecting.

Now suppose $k = (n \pm 1)/2$. This clearly requires n to be odd, so we have $n \geq 5$, which guarantees

$$(x_1, x_n) \in \bigcap_{\omega \in \Omega_\sigma} y_k(\omega), \quad (x_n, x_1) \in \bigcap_{\omega \in \Omega_\sigma} y_k(\mathrm{rev}(\omega)).$$

Thus again, $b_1(B)$ and $b_2(B)$ are intersecting for all $B \in \mathcal{B}_{k,n}$.

Recall that posets are only added to B if they do not intersect with some element of B. Indeed, it is clear from Figures 2 - 4 that

$$q \in b_i(B) \implies \exists q' \in b_j(B) \text{ such that } q \cap q' = \emptyset, \ \{i, j\} = \{1, 2\}.$$

Therefore both $b_1(B)$ and $b_2(B)$ are maximal under set inclusion as intersecting subsets of B.

It remains to be shown that $b_1(B)$ and $b_2(B)$ have maximal size among intersecting subsets of B. If $k = n/2$ then this follows immediately from Figure 3. If $k \notin \left\{\frac{n-1}{2}, \frac{n}{2}, \frac{n+1}{2}\right\}$ then it follows from Figure 2 that (1) becomes

$$q \in b_i(B) \implies b_j(B) = N(q), \ \{i, j\} = \{1, 2\}.$$

Thus any intersecting subsets of B must be contained in either $b_1(B)$ or $b_2(B)$, as required.

Finally, let $k = \frac{n \pm 1}{2}$ and let F be an intersecting subset of B of size $|F| \geq |b_i(B)| = 3$. Suppose, for a contradiction, that F is not contained in $b_i(B)$, $i = 1, 2$. By the pigeonhole principle, $|F \cap b_i(B)| = 2$ for some $i \in \{1, 2\} = \{i, j\}$; say $F \cap b_i(B) = \{q_1, q_2\}$. But then it is clear from Figure 4 that any element of $b_j(B)$ does not intersect with at least one of q_1, q_2. Thus $F \cap b_j(B) = \emptyset$, which contradicts $|F| \geq |b_i(B)| = 3$. □

Corollary 3.4. *Let $\mathcal{F}$ be an intersecting subset of $Y_{k,n}$. Then $|\mathcal{F}| \leq n!/4$.*

Proof. Any intersecting $\mathcal{F} \subseteq Y_{k,n}$ of maximal size must be the union of a transversal of $\{\, \{b_1(B), b_2(B)\} \mid B \in \mathcal{B}_{k,n} \,\}$ by Theorem 3.3. Clearly for any $B \in \mathcal{B}_{k,n}$, both $b_1(B)$ and $b_2(B)$ have half the size of B. Thus

$$|\mathcal{F}| = \sum_{B \in \mathcal{B}_{k,n}} \frac{|B|}{2} = \frac{|Y_{k,n}|}{2} = \frac{n!}{4}$$

as required. □

We say that an intersecting subset $\mathcal{F}$ of a class of posets $\mathcal{X}$ is *optimal* if it attains the bound for the size of an intersecting subset of $\mathcal{X}$. As was mentioned in the introduction, fixing and saturating are two common forms in which solutions to extremal problems occur. For the classes $Y_{k,n}$, we can see that saturation is optimal unless k is around $n/2$, but fixing is never optimal:

Remark 3.5. The fix-family

$$F(k, n) = \{\, p \in Y_{k,n} \mid x_1 <_p x_2 \,\}$$

is not optimal in $Y_{k,n}$ for any $x_1, x_2 \in [n]$: by Theorem 3.3, to show that $F(k,n)$ does not attain the bound of Corollary 3.4, it suffices to find $B \in \mathcal{B}_{k,n}$ such that neither $b_1(B)$ nor $b_2(B)$ are entirely contained in $F(k,n)$.

Let $p \in Y_{k,n}$ such that $x_1 ||_p x_2$. By the definition of $Y_{k,n}$, such a p exists for any $k \in [n-1]$. Then p cannot be an element of $F(k,n)$, so $b_1(B(p)) \not\subset F(k,n)$ since $p \in b_1(B(p))$. Let $h = \frac{n-1}{2}$.

- If $h \le k \le h+1$ then it is easily seen from Figures 3 and 4 that $b_2(B(p))$ must contain an element q with $x_1 ||_q x_2$, which implies $q \notin F(k,n)$ by the definition of $F(k,n)$.
- Otherwise, it follows from Figure 2 that there exists $q \in b_2(B(p))$ with $x_1 >_q x_2$ so again, $q \notin F(k,n)$.

Hence fix-families are not optimal in $Y_{k,n}$. On the other hand, Propositions 3.6 and 3.7 show that provided the k^{th} and $(n-k)^{th}$ level do not interact, we can find saturation families which are optimal.

Proposition 3.6. *For positive integers k, h, n with $k < n = 2h+1$, let $v_n \in \mathcal{P}_n$ be the poset $v_n = \{\, (i,n) \mid i = 1, \dots, n-1 \,\}$ and define*

$$G(k,n) = \{\, p \in Y_{k,n} \mid \text{either } |p \cap v_n| \ge h+1 \text{ or } |p \cap v_n| = h,\ 1 <_p n \,\}.$$

Then $G(k,n)$ is an intersecting subset of $Y_{k,n}$ of size

$$|G(k,n)| = \begin{cases} (n-1)\cdot(n-1)!/4 & k \in \{h, h+1\} \\ n!/4 & \text{otherwise} \end{cases}.$$

Proof. Firstly, observe that $G(k,n)$ is intersecting: set

$$\begin{aligned} A_{k,n} &= \{\, p \in Y_{k,n} \mid |p \cap v_n| \ge h+1 \,\}, \\ B_{k,n} &= \{\, p \in Y_{k,n} \mid 1 <_p n,\ |p \cap v_n| = h \,\}, \end{aligned}$$

so $G(k,n) = A_{k,n} \cup B_{k,n}$, and let $p, q \in G(k,n)$. We need to show that p and q have non-empty intersection.

If $p, q \in A_{k,n}$ then they both contain at least $h+1$ elements of the $2h$-element set v_n, which guarantees $|p \cap q| \ge 2$. If $p, q \in B_{k,n}$ then $(1,n) \in p \cap q$. If, without loss of generality, $p \in A_{k,n}$ and $q \in B_{k,n}$ then

$$|p \cap v_n| + |q \cap v_n| \ge (h+1) + h = n > |v_n| = n-1.$$

Thus, again by the pigeonhole principle, p, q and v_n must share at least one comparison. In conclusion, $G(k,n)$ is intersecting.

Now let $p \in A_{k,n}$ and let $\sigma = x_1 \dots x_n \in \mathcal{L}_n$ such that $p = y_k(\sigma)$. Then p must contain at least $h+1$ distinct comparisons (a,n) for some $a \in [n-1]$. If $k \ne h+1$ then clearly $|p \cap v_n| \ge h+1$ if, and only if, $n \in \{x_{h+2}, \dots, x_n\}$. Thus

$$|A_{k,n}| = (n-(h+1))\cdot(n-1)!/2 = (n-1)\cdot(n-1)!/4, \qquad k \ne h+1.$$

If $k = h+1$ and $n \in \{x_{h+1}, x_{h+2}\}$, then there are only h elements below n with respect to p, and so $p \notin A_{k,n}$. Thus we must have $n \in \{x_{h+3}, \dots, x_n\}$, giving

$$|A_{k,n}| = (n-(h+2))\cdot(n-1)!/2 = (n-3)\cdot(n-1)!/4, \qquad k = h+1.$$

Now let $p \in B_{k,n}$ and $\sigma = x_1 \dots x_n \in \mathcal{L}_n$ such that $p = y_k(\sigma)$. Then p contains precisely h comparisons of the form (a, n) for some $a \in [n-1]$. If $k = h$ then such a p does not exist. For $k \neq h$ we have $|p \cap v_n| = h$ if, and only if, $x_{h+1} = n$. The other condition for $p \in B_{k,n}$ is $1 <_p n$ which is now clearly equivalent to $1 \in \{x_1, \dots, x_h\}$.

If $k = h+1$, then $Y_{k,n}$ has $(n-1)!$ elements p with $x_{h+1} = n$ and precisely half of them have $1 \in \{x_1, \dots, x_h\}$. If $k \notin \{h, h+1\}$, then $Y_{k,n}$ has $(n-1)!/2$ elements p with $x_{h+1} = n$, and precisely half of these have $1 \in \{x_1, \dots, x_h\}$.

In summary, we have

$$|B_{k,n}| = \begin{cases} (n-1)!/2 & k = h+1 \\ 0 & k = h \\ (n-1)!/4 & \text{otherwise} \end{cases}$$

and, since $A_{k,n}$ and $B_{k,n}$ are disjoint,

$$|G(k,n)| = |A_{k,n}| + |B_{k,n}| = \begin{cases} (n-1) \cdot (n-1)!/4 & k \in \{h, h+1\} \\ n!/4 & \text{otherwise} \end{cases}$$

as required. □

For n even we have the following proposition, which has an analogous and shorter proof than the above proposition.

Proposition 3.7. *For positive integers k, n with $k < n$ and n even, let $v_n \in \mathcal{P}_n$ be the poset $v_n = \{\, (i, n) \mid i = 1, \dots, n-1 \,\}$, and define*

$$G(k,n) = \{\, p \in Y_{k,n} \mid |p \cap v_n| \geq n/2 \,\}.$$

Then $G(k,n)$ is an intersecting subset of $Y_{k,n}$ of size

$$|G(k,n)| = \begin{cases} (n-2) \cdot (n-1)!/4 & \text{if } k = n/2 \\ n!/4 & \text{otherwise} \end{cases}.$$

4. Almost Linear Posets

We now wish to study maximal intersecting subsets of the class $\mathcal{M}_n$ of all posets that are one comparison away from being linear. It is easy to see that a linear order on n points contains $r_n = n(n-1)/2$ comparisons, and so we define $\mathcal{M}_n$ as

$$\mathcal{M}_n = \{\, p \in \mathcal{P}_n \mid |p| = r_n - 1 \,\}.$$

Our first lemma relates $\mathcal{M}_n$ and the posets $Y_{k,n}$ from the previous section.

Lemma 4.1. $\mathcal{M}_n = \bigcup_{i=1}^{n-1} Y_{i,n}$.

Proof. Clearly, each $p \in Y_{k,n}$ contains precisely $r_n - 1$ comparisons. Conversely, any element p of $\mathcal{M}_n$ can be obtained from some linear order σ by removing one comparison, say (a, b). Suppose there exists $c \in [n]$ with $a <_\sigma c <_\sigma b$. Since p contains all comparisons of σ other than (a, b), we have $a <_p c$ and $c <_p b$ which implies $a <_p b$, a contradiction. Thus such a c cannot exist, which implies $p \in Y_{k,n}$ for some $k \in [n-1]$. □

In extremal combinatorics, one standard method of obtaining bounds for the size of 1-intersecting sets of certain combinatorial structures is to arrange elements of the ground set on a circle. This method was first introduced by Katona in [7] who arranged the elements of $[n]$ on a circle to give a simple proof of the bound in the Erdős-Ko-Rado Theorem for $t = 1$. Recently, a nice exposition of this method was given by Li and Wang in [9] to classify intersecting sets of partial permutations of maximal size.

In our context, we wish to arrange the comparisons (a, b) for distinct $a, b \in [n]$ on a circle, so let

$$\mathrm{Comp}_n = \{\, (a, b) \mid a \neq b,\ a, b \in [n] \,\}\,.$$

Definition 4.2. Given a linear order $\sigma = x_1 x_2 \dots x_n \in \mathcal{L}_n$, let $\alpha(\sigma, x_j)$ be the sequence of comparisons

$$x_1 < x_j,\ \ x_2 < x_j,\ \ \dots,\ \ x_{j-1} < x_j.$$

Denote by $c(\sigma)$ the cyclic arrangement of the comparisons on $[n]$ obtained as follows: on one half of the circle, we have

$$\alpha(\sigma, x_2),\ \ \alpha(\sigma, x_3),\ \ \dots,\ \ \alpha(\sigma, x_n)$$

clockwise in that order. (Note $\alpha(\sigma, x_1)$ is an empty sequence.) Moreover, for all comparisons $x < y \in \mathrm{Comp}_n$, we have $y < x$ directly opposite $x < y$ on $c(\sigma)$.

We collect these cyclic arrangements together in

$$\mathcal{C}_n := \{\, c(\sigma) \mid \sigma \in \mathcal{L}_n \,\}$$

and make some further, intuitive definitions.

- An *interval* A on a cyclic arrangement $c \in \mathcal{C}_n$ is a sequence of elements of Comp_n which are consecutive on c. Sometimes A will refer to the set containing the elements of the sequence; it will be clear from the context whether we consider A as a set or as a sequence.

  We say that A has *length* $|A|$ and an l-interval is simply an interval of length l. Finally, we make the convention that all intervals are read clockwise.
- Note that for $\sigma \in \mathcal{L}_n$, the circle $c(\mathrm{rev}(\sigma))$ cannot be obtained from $c(\sigma)$ by combinations of rotations and reflections. For instance, note from Figure 5 that $[(1, 4), (2, 4)]$ is an interval in $c(1234)$, whereas any interval in $c(\mathrm{rev}(1234))$ containing $(1, 4)$ and $(2, 4)$ must also contain $(2, 3)$.
- Given a set of comparisons $X \subseteq \mathrm{Comp}_n$ and a partial order $p \in \mathcal{P}_n$, we say that X *defines* p if p is the transitive closure of X.

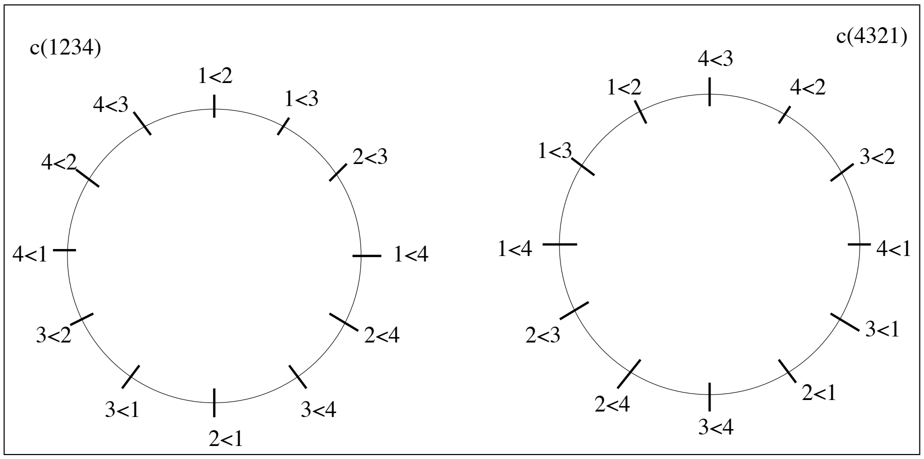

FIGURE 5. $c(1234)$ and $c(\text{rev}(1234))$.

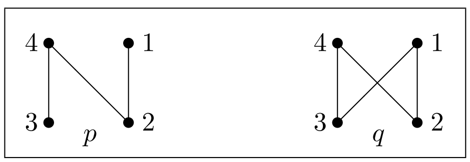

FIGURE 6. posets p and q are respectively defined by the 3-interval and 4-interval starting clockwise at $2 < 4$ in $c(1234)$.

Observe that there are $|\operatorname{Comp}_n| = n(n-1) = 2r_n$ points on each circle $c \in \mathcal{C}_n$. No interval on c longer than r_n can define an order, since it contains both (a, b) and (b, a) for some $a, b \in [n]$.

Example 4.3. Consider the 3-interval starting clockwise at $2 < 4$ in $c(1234)$ (see Figure 5): $A = [2 < 4, 3 < 4, 2 < 1]$. As a set, A is the poset p whose Hasse diagram is shown in Figure 6.

Now consider the 4-interval starting clockwise at $2 < 4$ in $c(1234)$. We have p together with the comparison $3 < 1$, which gives the poset q whose Hasse diagram is depicted in Figure 6 as well.

Note that neither of these posets contain any comparisons other than the ones from the interval which originally defined them. The following proposition shows that this is true in general: any interval of length up to r_n not only defines an order, but coincides precisely with some order p.

Proposition 4.4. *If A is an interval of length at most $r_n = n(n-1)/2$ in some $c = c(\sigma) \in \mathcal{C}_n$, then A is in fact a poset, i.e. $A \in \mathcal{P}_n$.*

Proof. Let S be the semicircle of $c = c(\sigma)$ defining σ and let R be the semicircle complementing S. We need to show that A is closed under transitivity, so suppose (x, y), $(y, z) \in A$ for some $x, y, z \in [n]$.

Case 1: $x <_\sigma y$, $y <_\sigma z$.
Since σ is transitive, we have $x <_\sigma z$. Now $y <_\sigma z$ implies that reading clockwise, (x, y) comes before both of (y, z) and (x, z) in S. Similarly, since $x <_\sigma y$ we must have (x, z) before (y, z) in S by the definition of $\alpha(\sigma, z)$. In other words, S contains the subinterval

$$(x, y), \dots, (x, z), \dots, (y, z). \tag{2}$$

Since A is an interval and (x, y), $(y, z) \in A$, it follows that A must contain either the interval (2) or the interval

$$(y, z), \dots, (x, y). \tag{3}$$

Now (2) is contained in S, and so can have length at most $|S| = r_n$. Since $|c| = 2r_n$, this means that (3) is strictly longer than r_n and thus cannot be contained in A. Hence A contains (2) and so $(x, z) \in A$.

Case 2: $x \not<_\sigma y$, $y \not<_\sigma z$.
This is settled in an analogous manner to Case 1.

Case 3: $x <_\sigma y$, $y \not<_\sigma z$.
Since σ is linear, we must have $z <_\sigma y$. We do not know how x and z compare under σ, but since both x and z are less than y under σ, the comparison relating x and z must come before both of (x, y), (z, y) in S: one of the intervals

$$(x, z), \dots, (x, y), \dots, (z, y), \tag{4}$$

$$(z, x), \dots, (z, y), \dots, (x, y) \tag{5}$$

must be contained in S.

Suppose $x <_\sigma z$ and consider the semicircles on c clockwise *ending* at (x, y) and (y, x) respectively, call them D_1 and D_2. Since (4) is a subinterval of the r_n-interval S, we have $(x, z) \in D_1$ and $(z, y) \in D_2$. Since (y, z) occurs directly opposite (z, y) on c, $(z, y) \in D_2$ implies $(y, z) \in D_1$. Thus the r_n-interval D_1 contains the subinterval $(y, z), \dots, (x, z), \dots, (x, y)$, which implies $(x, z) \in A$ as required. This situation is illustrated in Figure 7 (Case 3.a).

Case 3.b, when $z <_\sigma x$, is very similar to Case 3.a: here (5) is a subinterval of S. Thus if we draw a line on c connecting (y, z) and (z, y), we must have (x, y) and (x, z) on the same semicircle with respect to that line, because reverse comparisons are opposite each other on c. Since A cannot cover more than half of c, this implies $(x, z) \in A$.

Case 4: $x \not<_\sigma y$, $y <_\sigma z$
This is settled in an analogous manner to Case 3.

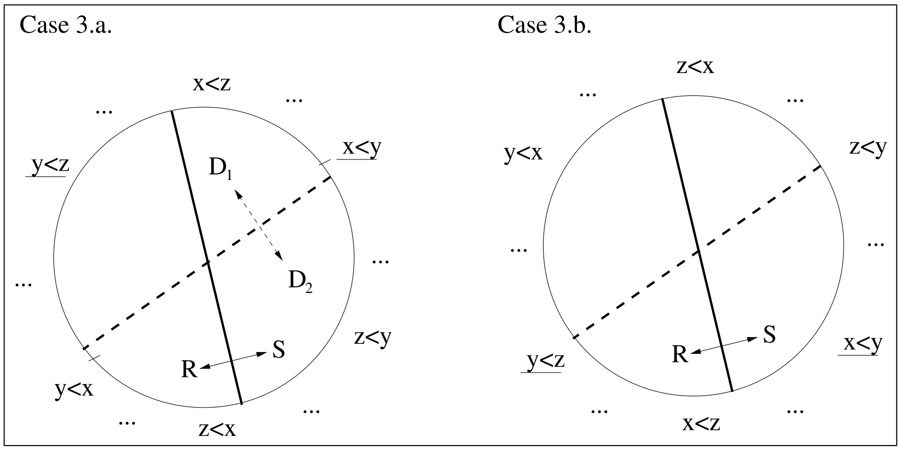

FIGURE 7. possibilities for $c(\sigma)$ in Case 3. Elements of A are underlined.

We have shown that A is closed under transitivity. Moreover, $|A| \leq r_n$ means that A cannot cover both of two opposite points on the circle $c(\sigma)$. Hence $A \in \mathcal{P}_n$, as required. □

Definition 4.5. We say that $p \in \mathcal{P}_n$ and $c \in \mathcal{C}_n$ are *compatible*, denoted $p \prec c$, if the set of all comparisons of p can be found as an interval on c.

For any $c \in \mathcal{C}_n$, $a, b \in [n]$ and $1 \leq l \leq r_n$, denote by $[(a,b)]_c^l$ the interval of length l starting clockwise at (a,b) in c.

To obtain a bound on intersecting subsets of $\mathcal{M}_n$, we need to investigate the posets arising from elements of $\mathcal{C}_n$ in some more detail.

Definition 4.6. For $\sigma = x_1x_2\ldots x_n \in \mathcal{L}_n$ and $1 \leq i < j \leq n$, define a set of comparisons $\lambda(\sigma, i, j)$ as follows:

(a) the chain $x_{j+1} < x_{j+2} < \cdots < x_{n-1} < x_n$ is preserved from σ;
(b) the chain on $x_1, x_2, \ldots, x_{j-1}$ is reversed: in $\lambda(\sigma, i, j)$ we have $x_{j-1} < x_{j-2} < \cdots < x_2 < x_1$;
(c) all elements of (b) are less than all elements of (a); and
(d) $x_i < x_j < x_{i-1}$ when $i > 1$, and $x_1 < x_j < x_{j+1}$ if $i = 1$.

It is not difficult to see that this description defines the linear order depicted in Figure 8.

Proposition 4.7. *Let* $\sigma = x_1x_2\ldots x_n \in \mathcal{L}_n$ *and let* $1 \leq i < j \leq n$. *Then*

$$[(x_i, x_j)]_{c(\sigma)}^{r_n} = \lambda(\sigma, i, j); \quad \textit{and}$$
$$[(x_j, x_i)]_{c(\sigma)}^{r_n} = \mathrm{rev}(\lambda(\sigma, i, j)).$$

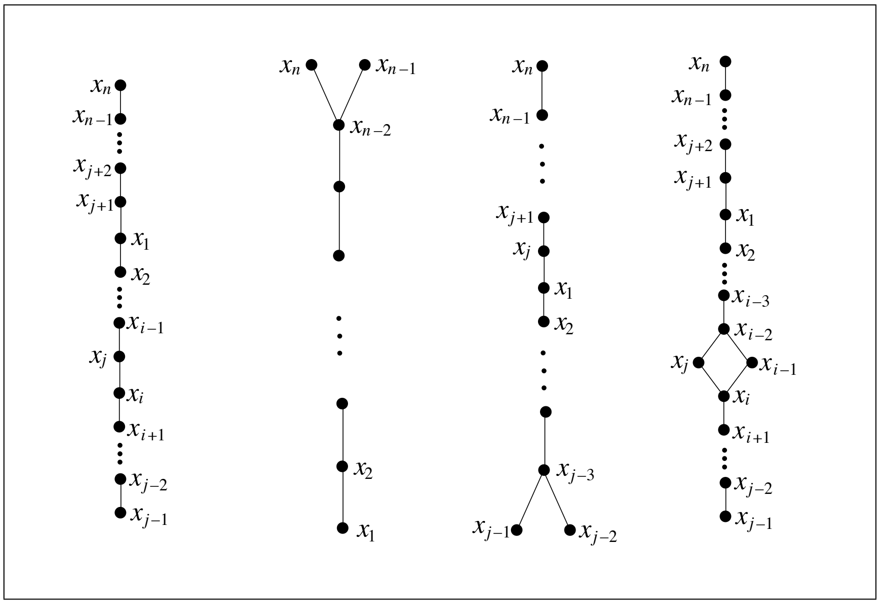

FIGURE 8. for $\sigma = x_1 \ldots x_n \in \mathcal{L}_n$ we have, from left to right, Hasse diagrams of posets $\lambda(\sigma, i, j)$, $\mu(\sigma, 1, 2)$, $\mu(\sigma, 1, j)$ for $j \geq 3$, and $\mu(\sigma, i, j)$ for $1 < i < j \leq n$.

Proof. Let $r = r_n$ and $c = c(\sigma)$. By Proposition 4.4, any r-interval on a cyclic ordering is equivalent to some linear order on $[n]$. The second claim, namely $[(x_j, x_i)]_c^r = \text{rev}(\lambda(\sigma, i, j))$, will follow from the first, since for $a, b \in [n]$, it is clear that

$$(a, b) \in [(x_j, x_i)]_c^r \iff (b, a) \in [(x_i, x_j)]_c^r.$$

So to prove the proposition, it suffices to show that $[(x_i, x_j)]_c^r = \lambda(\sigma, i, j)$.

Now it follows from the definition of the cyclic ordering $c(\sigma)$ that $[(x_i, x_j)]_c^r$ consists of the following comparisons:

(i) the intervals $\alpha(\sigma, x_{j+1}), \ldots, \alpha(\sigma, x_n)$,
(ii) the reverse of each comparison in the intervals $\alpha(\sigma, x_1), \ldots, \alpha(\sigma, x_{j-1})$,
(iii) the comparisons $(x_i, x_j), (x_{i+1}, x_j), \ldots, (x_{j-1}, x_j)$,
(iv) and the comparisons $(x_j, x_{i-1}), \ldots, (x_j, x_2), (x_j, x_1)$.

Note that (i) implies (a) and (c) in Definition 4.6, and (ii) implies (b). We have (x_i, x_j) by (iii) and (x_j, x_{i-1}) by (iv) for $i > 1$. Finally, $x_j < x_{j+1}$ follows from (i), so we conclude that $(i) - (iv)$ also imply (d) in Definition 4.6. Thus $[(x_i, x_j)]_c^r$ contains $\lambda(\sigma, i, j)$, which means they must be equal since both are linear orders on $[n]$. □

In the next proposition we will show how elements of $Y_{k,n}$, and hence elements of $\mathcal{M}_n$, arise as intervals of cyclic arrangements. For $\sigma = x_1 \dots x_n \in \mathcal{L}_n$ and $1 \le i < j \le n$, define posets $\mu(\sigma, i, j) \in \mathcal{P}_n$ by their Hasse diagrams in Figure 8.

Remark 4.8. Observe that for any $\sigma \in \mathcal{L}_n$, we have

$$\begin{aligned} \mu(\sigma,1,2) &\in Y_{n-1,n}, \\ \mu(\sigma,1,j) &\in Y_{1,n}, \quad 3 \le j \le n, \\ \mu(\sigma,i,j) &\in Y_{j-i+1,n}, \quad 1 < i < j \le n. \end{aligned}$$

Note also that the statement of Proposition 4.9 below is only concerned with intervals of $c(\sigma)$ starting at elements of σ. The remaining ones can be determined by recalling that as a poset, the l-interval starting at (x_j, x_i) is isomorphic to the reverse of the l-interval starting at (x_i, x_j).

Proposition 4.9. *Let $\sigma = x_1 x_2 \dots x_n \in \mathcal{L}_n$ and $1 \le i < j \le n$. Then*

$$[(x_i, x_j)]_{c(\sigma)}^{r_n - 1} = \mu(\sigma, i, j).$$

Proof. Set $r = r_n$, $c = c(\sigma)$, and let p be the poset equivalent to $[(x_i, x_j)]_c^{r-1}$.

Suppose firstly that $i > 1$ and recall that there are precisely $2r$ points on c. Since (x_{i-1}, x_j) is the comparison clockwise preceding (x_i, x_j) on c, the last point in $[(x_i, x_j)]_c^r$ is (x_j, x_{i-1}). Considering intervals as sets, we therefore have

$$p = [(x_i, x_j)]_c^{r-1} = [(x_i, x_j)]_c^r \setminus \{(x_j, x_{i-1})\}.$$

But $[(x_i, x_j)]_c^r = \lambda(\sigma, i, j)$ by Proposition 4.7, and so

$$p = \lambda(\sigma, i, j) \setminus \{(x_j, x_{i-1})\}.$$

It is easy to see from Figure 8 that the previous equation implies $p = \mu(\sigma, i, j)$.

The case $i = 1$ is very similar; we need to show that $p = \mu(\sigma, 1, j)$. When $j = 2$, clearly $[(x_1, x_j)]_c^{r-1}$ contains all elements of σ except (x_{n-1}, x_n), so $p = \mu(\sigma, 1, 2)$.

When $j > 2$, the comparison preceding (x_1, x_j) in c is (x_{j-2}, x_{j-1}) and so

$$p = [(x_1, x_j)]_c^r \setminus \{(x_{j-1}, x_{j-2})\} = \lambda(\sigma, 1, j) \setminus \{(x_{j-1}, x_{j-2})\}$$

by Proposition 4.7. Reconsidering Figure 8, we see that removing the comparison (x_{j-1}, x_{j-2}) from $\lambda(\sigma, 1, j)$ gives $\mu(\sigma, 1, j)$. □

Proposition 4.10. *Let $c \in \mathcal{C}_n$ and $1 \le k \le n-1$. Then c is compatible with n elements of $Y_{k,n}$.*

Proof. Let $\sigma = x_1 x_2 \dots x_n \in \mathcal{L}_n$ be such that $c = c(\sigma)$. By Proposition 4.9, we obtain elements of $Y_{k,n}$ from c simply by picking appropriate starting points of $(r_n - 1)$-intervals. Denote by $p(a, b)$ the poset equivalent to the $(r_n - 1)$-interval on c starting at (a, b).

Case $1 < k < n - 1$.

We begin by considering intervals starting at elements of σ. By Proposition 4.9 and Remark 4.8, we have $p(x_i, x_j) \in Y_{k,n}$ if, and only if, $k = j - i + 1$ for some $1 < i < j \leq n$. For given i, we therefore have $j = k + i - 1 \leq n$, which implies $i \leq n - k + 1$. Thus there are $|\{2, 3, \ldots, n - k + 1\}| = n - k$ values of i that determine intervals of c which start at elements of σ and are equivalent to elements of $Y_{k,n}$.

Since $Y_{k,n} \cong \mathrm{rev}(Y_{n-k,n})$, it follows by symmetry that there are $n-(n-k) = k$ such intervals whose starting points are not in σ. Thus c is compatible with a total of $n - k + k = n$ elements of $Y_{k,n}$.

Case $k = 1$ or $k = n - 1$.

Consider the case $k = 1$. Since the isomorphism classes $Y_{k,n}$ are only defined for $n \geq 3$, Remark 4.8 and Proposition 4.9 tell us that $p(x_i < x_j) \in Y_{1,n}$ if, and only if, $i = 1$ and $3 \leq j \leq n$. Hence there are $n - 2$ intervals of c starting at elements of σ which are equivalent to elements of $Y_{1,n}$.

The remaining ones are precisely those intervals opposite the (r_n-1)-intervals on c starting at elements of σ and equivalent to elements of $Y_{n-1,n}$. Using Remark 4.8 and Proposition 4.9 again, we have $p(x_i, x_j) \in Y_{n-1,n}$ if, and only if, either $i = 1$ and $j = 2$ or $j-i+1 = n-1$ for some $1 < i < j \leq n$. In the second situation, we have $j = n+i-2 \leq n$ which implies $i \leq 2$, forcing $i = 2$ and $j = n+i-2 = n$. We have shown that there are two $(r_n - 1)$-intervals on c starting at elements of σ and equivalent to elements of $Y_{n-1,n}$. Reversing these gives elements of $Y_{1,n}$. In summary, then, c is compatible with a total of $n - 2 + 2 = n$ elements of $Y_{1,n}$.

The case $k = n - 1$ follows by symmetry. □

Theorem 4.11. *Let $\mathcal{M}_n$ be the class of posets on n points that are one comparison away from being linear, i.e. $\mathcal{M}_n = \{\, p \in \mathcal{P}_n \mid |p| = n(n-1)/2 - 1 \,\}$. If $\mathcal{F} \subseteq \mathcal{M}_n$ is intersecting then*

$$|\mathcal{F}| \leq \frac{(n-2)(n+1)(n-1)!}{4}$$

and equality implies that for each $c \in \mathcal{C}_n$, all elements of $\mathcal{F}$ compatible with c have a fixed comparison in common.

Proof. Let $\mathcal{F} \subseteq \mathcal{M}_n$ be intersecting. We count pairs consisting of a poset in $\mathcal{F}$ and a cyclic ordering in $\mathcal{C}_n$ which are compatible with one another:

$$|\,\{\, (p, c) \mid p \in \mathcal{F}, c \in \mathcal{C}_n, p \prec c \,\}\,|.$$

Elements of $\mathcal{C}_n$ are obtained from each other by relabellings of $[n]$, and the same is true for elements of $Y_{k,n}$ for fixed k. Thus by Proposition 4.10, each poset in $Y_{k,n}$ is compatible with $n \cdot |\mathcal{C}_n|/|Y_{k,n}|$ elements of $\mathcal{C}_n$. Since $|Y_{k,n}| = |\mathcal{L}_n|/2 = n!/2$ for all $k \in [n-1]$, each poset in $Y_{k,n}$ is therefore compatible with

$$\frac{n \cdot |\mathcal{C}_n|}{n!/2} = \frac{2|\mathcal{C}_n|}{(n-1)!} \tag{6}$$

elements of $\mathcal{C}_n$. Since $\mathcal{F} \subseteq \mathcal{M}_n = \bigcup_{k=1}^{n-1} Y_{k,n}$ by Lemma 4.1, and since (6) is independent of k, we conclude that each element of $\mathcal{F}$ is compatible with $2|\mathcal{C}_n|/(n-$

1)! elements of $\mathcal{C}_n$. Thus

$$|\{ (p,c) \mid p \in \mathcal{F}, c \in \mathcal{C}_n, p \prec c \}| = |\mathcal{F}| \cdot \frac{2|\mathcal{C}_n|}{(n-1)!}. \tag{7}$$

Conversely, fix $c \in \mathcal{C}_n$ and set $F_c = \{ p \in \mathcal{F} \mid p \prec c \}$. Let $v, w \in F_c$ be elements of minimal intersection; that is,

$$|v \cap w| \leq |p \cap q|$$

for all $p, q \in F_c$. Note that $|v \cap w| \geq 1$ since $F_c \subseteq \mathcal{F}$ is intersecting. In what follows we consider different cases relating $v \cap w$ with other elements of F_c in order to derive a bound for $|F_c|$.

Case 1: $(v \cap w) \subseteq p$ for all $p \in F_c$.

Recall that all elements of F_c are intervals on c, all of which have length $r_n - 1$ since $F_c \subseteq \mathcal{M}_n$. The fact that $v \cap w$ is contained in all elements of F_c implies that, for any $(a,b) \in (v \cap w)$,

$$|F_c| \leq |\{ p \in \mathcal{P}_n \mid p \prec c,\, a <_p b \}| = r_n - 1. \tag{8}$$

Case 2: there exists $p \in F_c$ with $(v \cap w) \not\subseteq p$.

Denote the set of all such p by A_c:

$$A_c = \{ p \in F_c \mid (v \cap w) \not\subseteq p \}.$$

Let $p \in A_c$ and suppose $v \cap w \cap p \neq \emptyset$. All posets compatible with c correspond to *intervals* on c: if an order compatible with c contains two comparisons which are 'points' on c, then it must contains all 'points' in between. Thus the fact that p contains some element of $v \cap w$ implies that for either $q = v$ or $q = w$, we must have $(p \cap q) \subset (v \cap w)$. But this contradicts the definition of v and w, and so for all $p \in A_c$ we have

$$v \cap w \cap p = \emptyset. \tag{9}$$

Recall that intervals are read clockwise and assume, without loss of generality, that v comes clockwise before w. Now $F_c \subseteq \mathcal{F}$ is intersecting, and so (9) implies that all $p \in A_c$ 'cover the gap' between v and w on c: the elements of A_c intersect w at their starting point and v at their end point. Let z be the element of A_c which has minimal intersection with v as shown in Figure 9: for all $p \in A_c$ we have

$$|z \cap v| \leq |p \cap v|. \tag{10}$$

Before we consider two subcases, note that v, w and z together cover c, so

$$\begin{aligned} 2r_n = |c| &= |v| + |w| + |z| - |v \cap w| - |w \cap z| - |z \cap v| \\ &= 3(r_n - 1) - |v \cap w| - |w \cap z| - |z \cap v| \\ &\leq 3(r_n - 1) - |w \cap z| - 2, \end{aligned}$$

implying

$$|w \cap z| \leq r_n - 5. \tag{11}$$

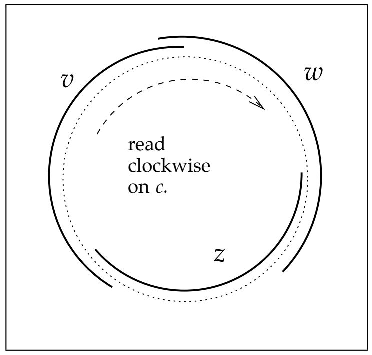

FIGURE 9. intervals v, w and z on c.

Subcase 2a: all $p \in F_c \setminus \{v, w\}$ have their starting point in z.

We divide z into the three intervals $w \cap z$, $z \setminus (v \cup w)$ and $v \cap z$ and determine how many elements of $F_c \setminus \{v, w\}$ can start in each interval in turn.

Clearly, if two elements of F_c have the same starting point on c, then they must be equal, since all posets in $\mathcal{F}$ have the same size. Thus $F_c \setminus \{v, w\}$ contains at most $|w \cap z|$ elements with starting points in $w \cap z$. Since w and p together do not cover c, and p starts at the end of w, we have $p \in A_c$. Since F_c is intersecting, p must intersect v, and this intersection must contain the starting point of v rather than the end point by (9). Moreover, by (10), $p \cap v$ must strictly contain $v \cap z$, and so p cannot start at the first $|v \cap z|$ points of $w \cap z$. In summary, we have at most $|w \cap z| - |v \cap z|$ elements $p \in F_c \setminus \{v, w\}$ which are distinct from z and have their starting point in $w \cap z$.

Next consider the case where $p \in F_c \setminus \{v, w\}$ has its starting point the gap between v and w. Then p does not intersect w at its end point, so $p \cap w$ must contain the starting point of w. But then $(p \cap w) \subset (v \cap w)$ which contradicts the definition of v and w. Thus there are no $p \in F_c$ starting in $z \setminus (v \cup w)$.

Finally, let p be an element of $F_c \setminus \{v, w\}$ with starting point in $v \cap z$. Then p cannot start at the starting point of v, so we have at most $|v \cap z| - 1$ such p.

Hence we conclude

$$\begin{aligned} |F_c \setminus \{v, w, z\}| &\leq (|w \cap z| - |v \cap z|) + (|v \cap z| - 1) \\ \implies |F_c| - 3 &\leq |w \cap z| - 1 \end{aligned}$$

which, together with (11), yields $|F_c| \leq r_n - 5 + 2 = r_n - 3$.

Subcase 2b: there exists $p \in F_c \setminus \{v, w\}$ whose starting point is not in z.

Denote the set of all such p by B_c:

$$B_c = \{\, p \in F_c \setminus \{v, w\} \mid \text{the starting point of } p \text{ does not lie in } z \,\}.$$

If $p \in B_c$ intersected v at the starting point of v then, since p does not start within z, we would have $|p \cap v| < |z \cap v|$, contradicting (10). Thus p must intersect v at the end point of v. In fact, since $p \neq w$ and $|v \cap w|$ is minimal, $(v \cap w)$ must be strictly contained in $(v \cap p)$. In other words, all $p \in B_c$ must start in $v \setminus w$. On the other hand, elements of B_c must intersect z so, since they do not start in z, they must end in z. In fact, this is true not only for z, but for all elements of

$$D_c := F_c \setminus \{v, w\} \setminus B_c = \{\, p \in F_c \setminus \{v, w\} \mid p \text{ starts in } z \,\}.$$

Thus the clockwise first element of B_c must cover

- the starting points of all other elements of B_c, which are all in $v \setminus w$,
- all of $w \setminus z$ including $v \cap w$,
- the starting points of all elements of D_c.

The length of this element is $r_n - 1$ and so $|B_c| + |w \setminus z| + |D_c| \leq r_n - 1$. Thus

$$|B_c| + |D_c| \leq r_n - 1 - (|w| - |w \cap z|) = |w \cap z|.$$

Since $|F_c| = |B_c| + |D_c| + 2$, this combines with (11) to give $|F_c| \leq r_n - 3$, concluding Subcase 2b.

We have shown that in Case 2, $|F_c| \leq r_n - 3$ which is strictly less than (8). Therefore

$$|\{\, (p, c) \mid p \in \mathcal{F}, c \in \mathcal{C}_n, p \prec c \,\}| \leq |\mathcal{C}_n| \cdot (r_n - 1)$$

and combining this with (7) gives

$$|\mathcal{F}| \leq \frac{(n-2)(n+1)(n-1)!}{4}.$$

Equality in this bound implies that Case 1 (and not Case 2) holds. We must then have equality in (8) which is only possible if, for some $(a, b) \in \mathrm{Comp}_n$, we have $(v \cap w) = \{(a, b)\}$, which in turn implies

$$F_c = \{\, p \in \mathcal{P}_n \mid p \prec c,\ a <_p b \,\}.$$

Moreover, if $|\mathcal{F}|$ attains the bound of the theorem, then this must hold for *all* $c \in \mathcal{C}_n$, as required. □

Next we show that one way of obtaining an intersecting family of maximal size is by fixing a comparison:

Proposition 4.12. *For some fixed $x_1, x_2 \in [n]$, let*

$$F(n) = \{\, p \in \mathcal{M}_n \mid x_1 <_p x_2 \,\}.$$

Then $F(n)$ is an intersecting subset of $\mathcal{M}_n$ of maximal size.

Proof. Clearly, $F(n)$ is intersecting. To show that this family has maximal size, it suffices to show that $F(n)$ attains the bound in Theorem 4.11.

For fixed k, $Y_{k,n}$ contains $(n-2)!$ posets with $x_1 || x_2$, and exactly half of the remaining elements of $Y_{k,n}$ have $x_1 < x_2$. Thus

$$|\{\, p \in Y_{k,n} \mid x_1 <_p x_2 \,\}| = \frac{|Y_{k,n}| - (n-2)!}{2} = \frac{(n-2)! \cdot (n-2)(n+1)}{4}$$

since $|Y_{k,n}| = n!/2$. Combining this with Lemma 4.1 gives

$$|F(n)| = \sum_{i=1}^{n-1} |\{ p \in Y_{k,n} \mid x_1 <_p x_2 \}| = \frac{(n-1)! \cdot (n-2)(n+1)}{4}$$

as required. □

From Theorem 4.11 one might expect that the converse of Proposition 4.12 is true, i.e. that any intersecting subset of $\mathcal{M}_n$ of maximal size must fix a comparison. However, this is not the case:

Proposition 4.13. *Let $v_n \in \mathcal{P}_n$ be the poset $v_n = \{ (i, n) \mid i = 1, \ldots, n-1 \}$.*

- *If n is even, set $H(n) = \{ p \in \mathcal{M}_n \mid |p \cap v_n| \geq n/2 \}$.*
- *If $n = 2h + 1$, set*

$$H(n) = \{ p \in \mathcal{M}_n \mid \textit{either } |p \cap v_n| \geq h+1 \textit{ or } |p \cap v_n| = h,\ 1 <_p n \}.$$

Then $H(n)$ is an intersecting subset of $\mathcal{M}_n$ of maximal size.

Proof. Since $\mathcal{M}_n = \bigcup_{k=1}^{n-1} Y_{k,n}$, it follows that $H(n) = \bigcup_{k=1}^{n-1} G(k, n)$ where $G(k, n)$ is defined in Propositions 3.7 and 3.6. Moreover, this union is disjoint since $Y_{i,n} \cap Y_{j,n} = \emptyset$ for $i \neq j$, and so

$$\begin{aligned} |H(n)| &= \sum_{k=1}^{n-1} |G(k, n)| \\ &= \sum_{k=1}^{n-1} \begin{cases} (n-2) \cdot (n-1)!/4 & n \text{ even, } k = n/2 \\ (n-1) \cdot (n-1)!/4 & n = 2h+1,\ k \in \{h, h+1\} \\ n!/4 & \text{otherwise} \end{cases} \end{aligned}$$

by Propositions 3.7 and 3.6. When n is even, this gives

$$|H(n)| = (n-2) \cdot \frac{n!}{4} + \frac{(n-2) \cdot (n-1)!}{4} = \frac{(n-2)(n+1)(n-1)!}{4},$$

and when n is odd,

$$|H(n)| = (n-3) \cdot \frac{n!}{4} + 2 \cdot \frac{(n-1) \cdot (n-1)!}{4} = \frac{(n-2)(n+1)(n-1)!}{4}.$$

Hence $|H(n)|$ attains the bound in Theorem 4.11.

Finally, the definitions of $G(k, n)$ for n even and odd do not depend on k, and so the arguments used in Propositions 3.7 and 3.6 to show that $G(k, n)$ is intersecting also imply that $H(n)$ is intersecting. □

To summarise, in Section 3 we saw that fixing is not optimal in $Y_{k,n}$ (Remark 3.5), whereas saturation is optimal apart from a couple of exceptional cases (Propositions 3.6 and 3.7). By way of contrast, we saw in Section 4 that both fixing and saturation yield optimal intersecting families in $\mathcal{M}_n$ (Propositions 4.12 and 4.13).

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