Combinatorics, Probability and Computing (2007) 00, 000–000. © 2007 Cambridge University Press
DOI: 10.1017/S0000000000000000 Printed in the United Kingdom

Growing at a Perfect Speed

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A collection of permutation classes is exhibited whose growth rates form a perfect set, thereby refuting some conjectures of Balogh, Bollobás and Morris.

1. Introduction

Let $\mathcal{P}$ be a collection of finite structures (also called a *property*), and for each non-negative integer n, let $\mathcal{P}_n$ denote those elements of $\mathcal{P}$ whose underlying set is $[n] = \{1, 2, \ldots, n\}$. The *speed* of $\mathcal{P}$ is just the function $n \mapsto |\mathcal{P}_n|$. If the speed of $\mathcal{P}$ is bounded above by some exponential function ($n \mapsto c^n$) then the *growth rate* of $\mathcal{P}$ is defined by $\mathrm{gr}(\mathcal{P}) = \limsup |\mathcal{P}_n|^{1/n}$.

In [3] Balogh, Bollobás and Morris generalized results of Kaiser and Klazar [7] for permutations and showed that, for hereditary properties of ordered graphs, there is a very limited set of available speeds (and corresponding growth rates) less than $n \mapsto 2^{n-1}$. They raised a number of questions, and posed a number of conjectures, which, roughly speaking, suggested that this sort of restrictive behaviour might continue for larger growth rates. We will show that Conjecture 3 (that growth rates have no accumulation points from above) and Conjecture 4 (that growth rates are integers or algebraic irrationals) of [3] are both false, by demonstrating the existence of a perfect set of growth rates in the more restrictive setting of permutations. The smallest element of this perfect set is approximately 2.47665, though this could almost certainly be improved. The available scope for improvement is limited by results of Vatter [9], which demonstrate that, if κ is the real root of $x^3 - 2x^2 - 1 = 0$ ($\kappa \approx 2.20557$) then, for permutation classes with growth rates below κ, results similar to those of [3] and [7] apply. In this paper we restrict ourself to consideration of permutation classes. However, it should be clear that the methods we use can be applied to many other sorts of relational structures.

2. Definitions and preliminary remarks

Before continuing to the results themselves, it is certainly time to define our terms a little more carefully. As usual, $\mathcal{S}_n$ will denote the permutations of $[n]$. If $\mathcal{C}$ is a set of permutations, then $\mathcal{C}_n$ denotes the set $\mathcal{C} \cap \mathcal{S}_n$, and the *generating function* of $\mathcal{C}$ is:

$$C(t) = \sum_{n=0}^{\infty} |\mathcal{C}_n| \, t^n.$$

Note that for algebraic convenience, we have implicitly allowed the empty permutation. Also note that we use calligraphic letters for sets of permutations, and an appropriate Roman letter for their generating functions throughout. If $C(t)$ is a generating function, then $\mathrm{rc}(C(t))$ will denote its radius of convergence.

A permutation $\pi \in \mathcal{S}_n$ *involves* a permutation $\sigma \in \mathcal{S}_k$ if there is a strictly increasing function $f : [k] \to [n]$ such that for all $i, j \in [k]$, $\sigma(i) < \sigma(j)$ if and only if $\pi(f(i)) < \pi(f(j))$. In this case we write $\sigma \preceq \pi$. If σ is not involved in π we say that π *avoids* σ. The range of such a function f is called an *occurrence of the pattern* σ. There are a number of more intuitive views of the involvement relation. The permutation σ is involved in π if:

- When π is written in standard one line notation as a sequence, there is a subsequence such that the relative ordering of the terms of the subsequence is the same as the relative ordering of the sequence representing σ.
- When the graph of π is drawn, some points can be erased so that what remains (after a rescaling of the axes) is the graph of σ.
- When π is thought of as a relational structure defined by two linear orders on $[n]$, it has a substructure isomorphic to σ (though of in the same way.)

A *permutation class*, or simply *class*, is a set of permutations closed downwards under involvement. The latter two interpretations of the involvement relation show that this is the correct notion of "hereditary property" for finite permutations, and in particular that permutation classes are actually special cases of the hereditary properties of ordered graphs considered in [3].

If $\mathcal{C}$ is a permutation class, then the set of $\preceq$-minimal elements not belonging to $\mathcal{C}$ is called its *basis.* If $\mathcal{X}$ is any set of permutations, then the set of permutations avoiding all the elements of $\mathcal{X}$, $\mathrm{Av}(\mathcal{X})$ is a class. Further, $\mathcal{X}$ is the basis of $\mathrm{Av}(\mathcal{X})$ if and only if $\mathcal{X}$ is an antichain for the involvement order. Marcus and Tardos proved in [8] that, if $\mathcal{C}$ is a proper permutation class (that is, $\mathcal{C} \neq \mathcal{S}$), then the speed of $\mathcal{C}$ is bounded by some exponential function, and so $\mathcal{C}$ has a growth rate. Of course this growth rate is simply the reciprocal of the radius of convergence of the generating function $C(t)$.

If $\alpha \in \mathcal{S}_n$ and $\beta \in \mathcal{S}_k$ are permutations, then the *sum* $\alpha \oplus \beta$ is the element of $\mathcal{S}_{n+k}$ defined by:

$$(\alpha \oplus \beta)(i) = \begin{cases} \alpha(i) & \text{if } i \leq n \\ n + \beta(i-n) & \text{otherwise.} \end{cases}$$

Again, there is a much more natural view of this operation: the graph of $\alpha \oplus \beta$ is obtained by stacking the graph of β above and to the right of that of α. A class, $\mathcal{C}$, is called *sum-closed* if for all $\alpha, \beta \in \mathcal{C}$ it is also the case that $\alpha \oplus \beta \in \mathcal{C}$. A permutation $\pi \in \mathcal{S}_n$ for $n \geq 1$ is called *sum-indecomposable*, or just *indecomposable*, if π cannot be written as a sum of strictly shorter permutations. Every permutation has a unique representation as a sum of indecomposable permutations, called its components. Furthermore, if σ, a plus indecomposable permutation, is involved in π, then it must be the case that σ is involved in at least one of π's components. This establishes:

Fact 2.1. *A class is sum-closed if and only if its basis consists of indecomposable permutations.*

If $\mathcal{C}$ is a permutation class then $\mathcal{C}^+$ denotes the set of indecomposable elements of $\mathcal{C}$. The representation referred to above implies that if $\mathcal{C}$ is sum-closed then:

$$C(t) = \frac{1}{1 - C^+(t)}.$$

Observation 2.2. *If $\mathcal{C}$ is a sum-closed class, and for some $0 < s < \mathrm{rc}(C^+(t))$ it is the case that $C^+(s) > 1$, then the radius of convergence of $C(t)$ is the least positive solution of $C^+(t) = 1$.*

This behaviour is by no means universal. For instance, the class $\mathcal{C} = \mathrm{Av}(312)$ is enumerated by the Catalan numbers and has radius of convergence $1/4$. However, both $C(t)$ and $C^+(t)$ have algebraic singularities at $1/4$.

We can now reveal the plan of our argument. We will construct an antichain of indecomposable permutations $\mathcal{A}_0 \cup \mathcal{A}$, where $\mathcal{A}$ is infinite such that all of the classes $\mathrm{Av}(\mathcal{A}_0 \cup \mathcal{Y})$ for $\mathcal{Y} \subseteq \mathcal{A}$ satisfy the conditions of Observation 2.2. It will then be relatively easy to establish that their growth rates form a perfect set.

We ask the reader to keep in mind that larger bases (in the sense of containment) produce smaller classes, and hence larger (or at least not smaller) radii of convergence for the corresponding generating functions.

3. The construction

The set $\mathcal{A}_0$ will be taken to be:

$$\{4123, 4132, 4213, 4231, 4312, 4321\}$$

that is, all of the permutations in $\mathcal{S}_4$ which begin with 4. The class $\mathrm{Av}(\mathcal{A}_0)$ is called the class of 3-bounded permutations. This is because it can be described as the class of permutations $\pi = \pi_1 \pi_2 \cdots \pi_n$ having the property that for each j, π_j is amng the three smallest elements of the sequence $\pi_j \pi_{j+1} \cdots \pi_n$. It is convenient therefore to encode the elements of this class using the symbols 1, 2 and 3 where the code for π_j reflects its rank among the remaining elements. For instance, the encoding 3123311 represents the permutation 3146725, while the encoding of 245163 is 233121.

For $n \geq 0$, let α_n be the permutation whose encoding is $3(31)^n321$.

Proposition 3.1. *The set $\{\alpha_n : n \geq 0\}$ is an antichain with respect to $\preceq$.*

Proof. As the length of α_n is $2n + 4$, $\alpha_i \not\preceq \alpha_j$ when $i > j$. Since α_0, which in standard one line notation is 3421 has two occurrences of a 321 pattern, while α_k for $k > 0$ has only one, $\alpha_0 \not\preceq \alpha_k$ for any $k > 0$. Now consider α_n and α_k for some $0 < n < k$. As illustrated in Figure 1 for the case $n = 5$ there is a sequence of elements in α_n, beginning from the unique occurrence of a 321 pattern, whose values or position are uniquely determined by the previous elements of the sequence. Therefore, were it the case that $\alpha_n \preceq \alpha_k$ the pattern representing α_n inside α_k would need to use the corresponding sequence of elements from α_k. However, at the end of the sequence, two elements would be required to fill a certain gap by value, while in α_k only a single element would be available. So, $\alpha_n \not\preceq \alpha_k$. □

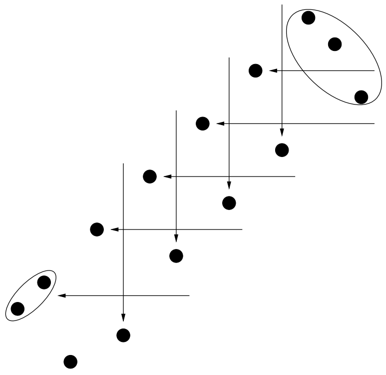

Figure 1. The graph of the permutation α_5. The upper ellipse represents the only occurrence of a 321 pattern. From top to bottom, all but the lowest arrow indicate elements that uniquely interpose, alternately by position or value, between previous elements which are similarly determined. The lowest arrow, and the lower ellipse show that in this gap by value, two elements are present. In the corresponding gap of α_k for any $k > 5$ there is only a single element.

We can extend this antichain by a few more 3-bounded permutations, specifically we define the antichain $\mathcal{A}$ to be the set of permutations whose encoding belongs to:

$$\{3221, 22221, 23311, 31221, 33311\} \cup \{3(31)^n321 : n \geq 0\}.$$

From now on we will be concerned only with the classes $\mathrm{Av}(\mathcal{A}_0 \cup \mathcal{X})$ for subsets $\mathcal{X}$ of $\mathcal{A}$, so we adjust our notational conventions slightly and use $X(t)$ for the generating function of such a class, and $X^+(t)$ for that of its indecomposable elements.

Fact 3.2. *The antichain $\mathcal{A}$ is maximal in the set of 3-bounded indecomposable permutations. The generating function of the class* $\mathrm{Av}(\mathcal{A}_0 \cup \mathcal{A})$ *is:*

$$A(t) = \frac{1-t^2}{1-t-2t^2-2t^3-4t^4-3t^5}$$

and its radius of convergence is $a \approx 0.403771$. The generating function for the indecomposable elements of this class is:

$$A^+(t) = t + t^2 + 3t^3 + \frac{5+6t}{1-t^2}.$$

These facts were discovered and verified using GAP [5] and the GAP automata package [4] together with routines written by the authors to implement the methods described in [1] for manipulating the automata representing regular classes and their bases.

We now consider the classes $\mathrm{Av}(\mathcal{A}_0 \cup \mathcal{X})$ for various subsets $\mathcal{X}$ of $\mathcal{A}$. Fact 3.2 already provides us with important information about one extreme of this range. We also need the corresponding fact at the other extreme:

Fact 3.3. *The generating function of the class* $\mathrm{Av}(\mathcal{A}_0)$ *is:*

$$Z(t) = 1 + t + \frac{2t^2}{1-3t}$$

and its radius of convergence is $1/3$. The generating function of its indecomposable elements is:

$$Z^+(t) = \frac{t-t^2}{1-2t-t^2}$$

with radius of convergence $\sqrt{2}-1$.

We also note that

$$A^+(\sqrt{2}-1) \approx 1.06497 > 1 \quad \text{and} \quad \left.\frac{dA^+}{dt}\right|_{t=1/3} = 383/96 > 3.$$

We then obtain:

Proposition 3.4. *Let $\mathcal{X}$ be any subset of $\mathcal{A}$, and let $X(t)$ and $X^+(t)$ be the generating functions of* $\mathrm{Av}(\mathcal{A}_0 \cup \mathcal{X})$ *and the plus indecomposable elements of that class respectively. The radius of convergence of $X(t)$ is equal to the least positive solution of $X^+(t) = 1$.*

Proof. Observation 2.2 applies to $\mathrm{Av}(\mathcal{A}_0 \cup \mathcal{X})$ because termwise $A^+(t) \leq X^+(t) \leq Z^+(t)$. Hence, the radius of convergence of $X^+(t)$ is at least $\sqrt{2}-1$, which together with the first part of the previous note implies that $X^+(s) > 1$ for some $s < \sqrt{2} - 1$ as required. □

All of the remaining arguments are based on the following lemma:

Lemma 3.5. *Let a positive real number $\epsilon > 0$ be given. There exists a positive integer n such that for any two sets $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{A}$, if the symmetric difference between $\mathcal{X}$ and $\mathcal{Y}$ contains no permutations of length less than or equal to n, then*

$$|\mathrm{rc}(X(t)) - \mathrm{rc}(Y(t))| < \epsilon.$$

Proof. First define $Z^+_{>n}(t)$ to be the generating function of the set of all indecomposable permutations in $\mathrm{Av}(A_0)$ of size greater than n. Fix $s < \sqrt{2} - 1$ such that $\mathcal{A}^+(s) > 1$. Choose n sufficiently large that $Z^+_{>n}(s)/3 < \epsilon$.
It suffices to prove the lemma in the case $\mathcal{Y} \subseteq \mathcal{X}$ (since the general case follows by considering the intersection and union of the two given sets respectively). Let $r < s$ be the radius of convergence of $X(t)$. Then,

$$1 < Y^+(r) \leq X^+(r) + Z^+_{>n}(r) = 1 + Z^+_{>n}(r)$$

Also, for all t in $[1/3, s]$, the derivative of Y^+ evaluated at t is larger than the derivative of A^+ evaluated at $1/3$ (since Y^+ dominates A^+ termwise, and both are generating functions.) This value is known to exceed 3. So:

$$Y^+(r - \epsilon) \leq Y^+(r) - 3\epsilon \leq Y^+(r) - Z^+_{>n}(s) \leq Y^+(r) - Z^+_{>n}(r) \leq 1$$

and thus the radius of convergence of Y lies between $r - \epsilon$ and r. □

This allows us to show the main result:

Theorem 3.6. *The set*

$$R = \{\mathrm{rc}(\mathrm{Av}(\mathcal{A}_0 \cup X)) \ : \ \mathcal{X} \subseteq \mathcal{A}\}$$

is perfect. Additionally:

1 *If $\mathcal{X} \subseteq Y \subseteq \mathcal{A}$ and $\mathcal{X} \neq \mathcal{Y}$, then* $\mathrm{rc}(X(t)) < \mathrm{rc}(Y(t))$.
2 *If $\mathcal{X}_1 \subseteq \mathcal{X}_2 \subseteq \cdots \subseteq \mathcal{A}$ and $\mathcal{X} = \cup_{i \geq 1} \mathcal{X}_i$ then* $\mathrm{rc}(X(t)) = \sup_i \mathrm{rc}(X_i(t))$.
3 *If $\mathcal{A} \supseteq \mathcal{Y}_1 \supseteq \mathcal{Y}_2 \supseteq \cdots$ and $\mathcal{Y} = \cap_{i \geq 1} \mathcal{Y}_i$ then* $\mathrm{rc}(Y(t)) = \inf_i \mathrm{rc}(Y_i(t))$

Proof. Lemma 3.5 immediately implies that every element of R is an accumulation point of R, since given a set $\mathcal{X}$ if we form $\mathcal{Y}$ by adding to or removing from $\mathcal{X}$ a single sufficiently long element of $\mathcal{A}$,then $\mathrm{rc}(Y(t)) \neq \mathrm{rc}(X(t))$, but their difference can be made arbitrarily small. So it remains to argue that R is closed. Suppose that r_i is a sequence in R, converging to r. Choose $\mathcal{X}_i$ so that $r_i = \mathrm{rc}(X_i(t))$. By passing to a subsequence if necessary, we may assume that, for each positive integer n, if $i, j \geq n$, then the elements

of $\mathcal{X}_i$ and of $\mathcal{X}_j$ of length at most n are the same. For each n, set $\mathcal{Y}_n$ to be that common set of elements. Then the $\mathcal{Y}_n$ form an increasing sequence of sets with union $\mathcal{Y}$ and $\mathrm{rc}(Y(t)) = r$.
The additional items all follow easily from Lemma 3.5. ☐

Since the reciprocals of the elements of R are the growth rates of the corresponding classes, we have shown the existence of a perfect set of growth rates between $1/a \approx 2.47665$ and 3, and including both those values.

4. Discussion and conclusions

Theorem 3.6 certainly indicates that the collection of real numbers that can arise as growth rates of permutation classes has rather more complex structure than had been observed previously. In particular it contains a perfect subset and hence is uncountable, refuting both Conjectures 3 and 4 of [3]. Undaunted, we would like to suggest:

Conjecture. The set of growth rates of permutation classes includes some interval (λ, ∞).

In fact, we suspect that the least such λ, if it exists, is not very large. However, the methods we have used here do not provide sufficient finesse to prove that result. The issue is that when we have a finitely based class then we cannot easily construct classes with slightly larger growth rates, and when we have a class whose basis is cofinite relative to our chosen antichain we have a similar problem with slightly smaller growth rates. However, our construction is very specific, and there is no *a priori* reason to restrict attention to classes whose bases are subsets of some fixed antichain.

It is our opinion that Conjecture 1 of [3] (that the limit superior used to define growth rates is in fact always a limit) is probably true for permutation classes. We certainly regard it as a significant open problem. Most of the following discussion assumes it to be true (or should be restricted to refer to growth rates of classes for which it is true.)

In [2] it was remarked, though this was probably folkloric, that the collection of growth rates of permutation classes is closed under addition. This is because, given two classes $\mathcal{A}$ and $\mathcal{B}$ we can form their juxtaposition – that set of permutations π which have a prefix α whose pattern comes from $\mathcal{A}$, and such that the corresponding suffix β has a pattern from $\mathcal{B}$. The simplest example of this type is the class of permutations having at most one descent, which is the juxtaposition of the increasing class with itself. A simple maximization exercise then shows that the growth rate of this new class is equal to the sum of the growth rates of $\mathcal{A}$ and of $\mathcal{B}$. In particular it follows that in order to prove our Conjecture 4 it would suffice to demonstrate the existence of any non-trivial interval all of whose elements were growth rates.

There are more complicated constructions related to juxtapositions – generalizations of the notion of grid classes from [6]. For example, we could allow juxtapositions of permutations of three types in an L shape. Now the constrained maximization problem that one must solve in order to determine the growth rate of such a class is rather more complex and yields interesting values. However, we know of no construction which can be used to add "just a little bit" to the growth rate of a class, nor any construction that can reliably be used to decrease the growth rate of a class in any consistent fashion.

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