Covering and generating sporadic simple groups

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Abstract

For a non-cyclic finite group G that can be generated by two elements let $\sigma(G)$ be the least integer k so that G is the union of k of its proper subgroups, and let $\mu(G)$ be the largest integer m so that there exists a subset X of G of size m with the property that any two distinct elements of X generate G. Clearly $\mu(G) \leq \sigma(G)$. In this paper we prove that $\sigma(G) \leq 4 \cdot \mu(G)$ for a sporadic simple group G.

1 Introduction

Let G be a non-cyclic finite group. Cohn [6] defined the function $\sigma(G)$ to be the least integer k such that G is the union of k of its proper subgroups. Much is know about $\sigma(G)$ for various groups G. For example, for a finite solvable group G Tomkinson [16] showed that $\sigma(G) = q + 1$ where q is the size of the smallest chief factor group of G that has more than one complement. There are many papers on covering non-solvable groups by proper subgroups. See Lucido [13], Bryce, Fedri, Serena [5], Maróti [14], and Holmes [10]. For an interesting survey of the subject see Serena [15].

Let G be a non-cyclic finite group that can be generated by two elements. We define $\mu(G)$ to be the largest integer m so that there exists a subset X of G of size m with the property that any two distinct elements of X generate G. (Following Blackburn [1] we will say that a subset X of G generates G pairwise if any two distinct elements of X generates G. We will also say that X is a pairwise generating set for G.) A quick corollary to the solution of Dixon's conjecture, stated by Liebeck and Shalev in [12] (see Corollary 1.7), is that there exists a constant c so that $\mu(G) \geq c \cdot n$ for a finite simple group G where n is the minimal index of a proper subgroup in G.

An obvious relationship between the functions $\sigma(G)$ and $\mu(G)$ is $\mu(G) \leq \sigma(G)$. For certain symmetric, alternating, general linear, and special linear groups we can say more, namely that $\mu(G) = \sigma(G)$ (see [14], [1], [4]).

A group is said to have spread at least k if, for any non-identity $x_1, \ldots, x_k \in G$, there is some $y \in G$ such that $G =< x_i, y >$ whenever $1 \leq i \leq k$. The number $s(G)$ denotes the largest integer k so that G has spread at least k. There are several papers on the spread of sporadic groups, for example [3]. It is easy to see that for any finite group G that can be generated by two elements, the inequality $s(G) \leq \mu(G)$ holds.

In this paper we prove

Theorem 1.1. *Let G be a sporadic simple group. Then $\mu(G) \leq \sigma(G) \leq 4 \cdot \mu(G)$. We also have $\sigma(G) \leq 2 \cdot \mu(G)$ unless G is isomorphic to* J_2, Co_3, Suz *or* He.

In [10] the exact values of $\sigma(G)$ were already found for the groups M_{11}, M_{22}, M_{23}, Ly and O'N. Here we calculate the exact values of $\sigma(G)$ for the groups $G \cong \mathrm{Fi}_{22}$, HS, Ru, HN, He and M_{24}. The table in Section 1.1 gives upper and lower bounds for $\sigma(G)$ and $\mu(G)$ respectively for sporadic simple groups G.

1.1 Statement of results

We use $\sigma'(G)$ and $\mu'(G)$ to denote upper and lower bounds respectively for $\sigma(G)$ and $\mu(G)$. We prove the results given in the following table.

G	$\mu'(G)$	$\sigma'(G)$
M_{11}	23	23
M_{12}	131	210
M_{22}	732	771
M_{23}	41709	41079
M_{24}	2145	3336
J_1	4813	5415
J_2	380	1154
J_3	23648	44100
J_4	251012224202199409	251012689468122717
Fi_{22}	149276	221521
Fi_{23}	8768674848	8875272316
Fi_{24}'	3091639677952507908004	3091639678003799232170
Co_3	265413	832835
Co_2	4327363	4730457
Co_1	46490622576	58033615710
Suz	194928	540333
$\mathrm{M^cL}$	13245	24553
He	212937	464373
Ru	12970337	12992175
Th	103423277855	103614133000
HS	1247	1376
HN	162639021	229758831
O'N	20141165	36450855
Ly	112845655185250	112845655268156
B	*approx.* 3×10^{30}	*approx.* 3×10^{30}
$\mathbb{M}$	*approx.* 1.2×10^{49}	*approx* 1.5×10^{49}

We have $\sigma'(G) = \sigma(G)$ for the 11 sporadic simple groups G listed after the statement of Theorem 1.1.

2 Methods used

Unless specified otherwise, all information used in this section is given in the Atlas [7] and the Gap character table library [8]. All computations with group elements are performed using Magma [2]. Many calculations of bounds for $\sigma(G)$ mention auxiliary subgroups. These are defined in [10].

We use a formula and König's theorem to compute bounds for $\mu(G)$.

2.1 A formula

Proposition 2.1. *Let G be a finite group. Let $H_1, \ldots, H_n$ be representatives of the conjugacy classes of maximal subgroups of G. Let C be a conjugacy class of cyclic subgroups. Suppose $c \in C$ is contained in exactly k_i conjugates of H_i, $1 \leq i \leq n$. Then a pairwise generating set always exists which consists of generators of members of C and has size the integer part of*

$$\frac{|C|}{1 + \sum k_i(\frac{|C|k_i}{(G:H_i)} - 1)}.$$

Proof. Let S be a set that generates G pairwise. When S is empty, we are free to choose any of the $|C|$ members of C. Let c be the one that we choose. There are k_1 conjugates of H_1 that contain c, and each of these conjugates contain $|C|k_1/(G{:}H_1)$ members of C, including c itself. So if we add a generator of c to our pairwise generating set then there are at most

$$1 + \sum k_i(\frac{|C|k_i}{(G{:}H_i)} - 1)$$

members of C that generate a proper subgroup of G with c. We can always choose a generator from a member of C to add to S when

$$|S|(1 + \sum k_i(\frac{|C|k_i}{(G{:}H_i)} - 1)) < |C|.$$

This gives the result. □

Corollary 2.1. *Suppose that a pairwise generating set S contains elements from m_i conjugates of H_i, $1 \leq i \leq n$. Then the number of generators of members of C that can be added to S is at least the integer part of*

$$\frac{|C| - \sum m_i \frac{|C|k_i}{(G:H_i)}}{1 + \sum k_i(\frac{|C|k_i}{(G:H_i)} - 1)}.$$

We use this corollary to define functions $f(C)$ and $f(G)$. Let $f(C)$ be the maximum value that can be shown by Proposition 2.1 to be the size of a pairwise generating set consisting of generators of members of C. We define $f'(G)$ to be the size of a pairwise generating set obtained in the following manner.

1. Take a pairwise generating set S consisting of members of C, where $f(C) \geq f(C')$ for any C'.
2. Let D be a conjugacy class of cyclic subgroups with $D \cap S = \emptyset$, and $f(D) \geq f(C')$ for all C' such that $C' \cap S = \emptyset$. Include in S the members of D given by Corollary 2.1.
3. Repeat Step 2 until all conjugacy classes have been used. Let $f'(G)$ be the size of S.

Note that $f'(G)$ is not well defined if $f(C) = f(D)$ but $C \neq D$ for any two conjugacy classes C and D. We define $f(G)$ to be the maximum value of all possible values of $f'(G)$.

2.2 König's Theorem

We will need König's theorem [11] on bipartite graphs.

A bipartite graph $\Gamma = (X, Y, E)$ is a graph with vertex set $X \cup Y$, edge set E with the property that every edge connects a vertex of X with a vertex of Y. A matching is a set of edges such that no pair of edges meet at a common vertex. A maximum matching is a matching of largest possible size. A covering K is a set of vertices of Γ so that every edge in Γ has an endpoint in K. A minimum covering is a covering of least possible size.

Theorem 2.1 (König, [11]). *Suppose that* $\Gamma = (X, Y, E)$ *is a bipartite graph. Then the number of edges in a maximum matching equals the number of vertices in a minimum covering.*

3 Using the formula

For all the groups in this subsection, $\mu'(G)$ was calculated using the formula $f(G)$ defined in Section 2.1. The values obtained are displayed in the table at the start of this section.

The values of $\sigma'(G)$ for G isomorphic to Ly, M_{22}, $\mathrm{M^cL}$, J_1 and O'N are given in [10].

Proposition 3.1. *Any minimal covering of* Co_2 *contains all conjugates of the maximal subgroups* $\mathrm{U}_6(2).2$, $2^{1+8}{:}\mathrm{S}_6(2)$, $\mathrm{M^cL}$ *and* M_{23}, *and at least some conjugates of* HS:2. *We have* $4443555 \leq \sigma(\mathrm{Co}_2) \leq 4730457$.

Proof. An element of class $23A$ is contained in one of the 4147200 conjugates of M_{23} and nothing else. This means that the covering must contain all conjugates of M_{23}.

If an element is in class $28A$ then it is contained in one of the 56925 conjugates of $2^{1+8}{:}\mathrm{S}_6(2)$ and two of the 1619200 conjugates of $\mathrm{U}_4(3).\mathrm{D}_8$. If we use the latter then we would be able to cover the classes $20B$ and $24B$ as well as $28A$, but it is cheaper to use all conjugates of $2^{1+8}{:}\mathrm{S}_6(2)$, $\mathrm{U}_6(2).2$ and $2^{10}{:}\mathrm{M}_{22}{:}2$. So we put all conjugates of $2^{1+8}{:}\mathrm{S}_6(2)$ into the covering.

When we look at $30C$ we see that the optimal way to cover the class is to use all 47104 conjugates of $\mathrm{M^cL}$. This does not give us any of the other remaining classes $20A$, $20B$ and $24B$, but we can complete the covering using 526332 subgroups, which is less than the number of conjugates of any subgroup other than $\mathrm{M^cL}$ needed to cover $30C$. The 526332 subgroups are all conjugates of HS:2, $\mathrm{M^cL}$ and $\mathrm{U}_6(2).2$.

Using the 2300 conjugates of $\mathrm{U}_6(2).2$ is optimal for $24B$, as any other method would use more than 526332 subgroups. So the cover includes all these subgroups. Each element of class $20A$ is in three of the 476928 conjugates of HS:2 and one conjugate of both the $2B$ and $5A$ normalisers. A $20B$ element is in one conjugate of HS:2, one of the 46575 conjugates of $2^{10}{:}\mathrm{M}_{22}{:}2$, and some other higher-index subgroups. So the best way to complete the covering is to use enough conjugates of HS:2 needed to cover $20A$, with possibly some auxiliary copies of $N_{\mathrm{Co}_2}(2B)$, and to top these up with enough conjugates of $2^{10}{:}\mathrm{M}_{22}{:}2$ and any auxiliary subgroups to complete the covering of $20B$. $\square$

Proposition 3.2. *We have* $505288 \leq \sigma(\mathrm{Co}_3) \leq 832835$*. All minimal coverings of* Co_3 *contain all conjugates of the maximal subgroups* M_{23} *and* $3^5{:}(\mathrm{M}_{11} \times 2)$*, between 327888 and 655149 of the 655776 conjugates of the maximal subgroup* $\mathrm{U}_3(5)\mathrm{S}_3$ *and a subset of the 276 conjugates of* $\mathrm{M^cL}{:}2$.

Proof. First we note that an element of class $23A$ is contained in exactly one maximal subgroup, namely a conjugate of M_{23}. So we must include all conjugates of M_{23} in any covering of Co_3.

Next we consider the conjugacy class $18A$. An element of this class is in one of the 128800 conjugates of $3^5{:}(\mathrm{M}_{11} \times 2)$, one of the 170775 conjugates of $2^{\cdot}\mathrm{S}_6(2)$, one of the 708400 conjugates of $3^{1+4}{:}4^{\cdot}\mathrm{S}_6$, and three of the 54648000 conjugates of $\mathrm{S}_3 \times \mathrm{L}_2(8){:}3$.

Suppose we use $3^5{:}(\mathrm{M}_{11} \times 2)$ to cover $18A$. After including these subgroups in the cover, the conjugacy classes remaining are $20B$, $21A$, $24A$ and $30A$. If we had used $2^{\cdot}\mathrm{S}_6(2)$ or $3^{1+4}{:}4^{\cdot}\mathrm{S}_6$ to cover $18A$ then we would also have covered $20B$, $24A$ and $30A$. This would have added 170775 or 708400 subgroups to the covering. As all three conjugacy classes $20B$, $24A$ and $30A$ are contained in the 276 conjugates of $\mathrm{M^cL}{:}2$, a cheaper way to cover them and $18A$ is to use the subgroups $3^5{:}(\mathrm{M}_{11} \times 2)$ and $\mathrm{M^cL}{:}2$. The only other alternative for covering $18A$ is to use a subset of the conjugates of $\mathrm{S}_3 \times \mathrm{L}_2(8){:}3$. If we were to use all conjugates of this subgroup then we would cover $21A$, but this can be done more efficiently by using $3^5{:}(\mathrm{M}_{11} \times 2)$ and $\mathrm{U}_3(5).\mathrm{S}_3$. We conclude that the covering should contain all conjugates of $3^5{:}(\mathrm{M}_{11} \times 2)$.

The remaining classes are $20B$, $21A$, $24A$ and $30A$. An element of class $21A$ is in exactly two conjugates of the maximal subgroup $\mathrm{U}_3(5){:}\mathrm{S}_3$ of index 655776 as well as some other maximal subgroups of much higher index. All the conjugacy classes $20B$, $21A$, $24A$ and $30A$ are contained in $\mathrm{U}_3(5){:}\mathrm{S}_3$, so it is clear that we should use $\mathrm{U}_3(5){:}\mathrm{S}_3$ to cover $21A$.

The question is whether to use all 655776 conjugates of $\mathrm{U}_3(5){:}\mathrm{S}_3$ to cover all these conjugacy classes, or whether it is better to use only those conjugates of $\mathrm{U}_3(5){:}\mathrm{S}_3$ that are needed for covering $21A$. By [10], the number of conjugates needed to cover $21A$ is the total number minus the maximal size of a subset S of the conjugates, where no two of the members of S intersect in an element of $21A$.

We use Magma [2] to find good candidates for S. We take a permutation representation of Co_3 from the online Atlas [7] and compute the action on the 655776 cosets of $\mathrm{U}_3(5){:}\mathrm{S}_3$. We initialize a set S' to be empty, a set T to be the set of points $1, \ldots, 655776$, a point p to be one fixed point of a random element of class $21A$, and a set λ of the 6000 points that are fixed by the elements of class $21A$ that also fix p. Each pass round a loop we pick a random element g of G. If the point p^g is in T then we add it to S' and delete the members of λ^g from T. We continue until T is empty, and then the size of S' is a lower bound for the size of S.

The above method tells us that we need at most $655776 - 627$ conjugates of $\mathrm{U}_3(5){:}\mathrm{S}_3$ to cover $21A$. From this we deduce that the cheapest way to cover $20B$, $21A$, $24A$ and $30A$ is to use $\mathrm{U}_3(5){:}\mathrm{S}_3$ to cover $21A$ (possibly with some auxiliary subgroups conjugate to $\mathrm{L}_3(4){:}\mathrm{D}_{12}$ and $\mathrm{S}_3 \times \mathrm{L}_2(8){:}3$) and to use conjugates of $\mathrm{M^cL}{:}2$ to cover $20B$, $24A$ and $30A$.

□

Proposition 3.3. *There is a minimal covering of* Fi_{22} *consisting of all conju-*

gates of the maximal subgroups $2\mathrm{U}_6(2)$*,* $\mathrm{O}_8^+(2).3.2$ *and* $2^{10}{:}\mathrm{M}_{22}$*, and one conjugacy class of maximal subgroups isomorphic to* $\mathrm{O}_7(3)$*. This gives* $\sigma(\mathrm{Fi}_{22}) = 221521$*.*

Proof. Each element of class $22B$ is in one conjugate of $2\mathrm{U}_6(2)$ and nothing else, so the covering must contain this conjugacy class of subgroups.

An element of class $21A$ is in one subgroup in each class of S_{10}, one of the 1647360 conjugates of $\mathrm{S}_3 \times \mathrm{U}_4(3).2$ and one of the 61776 conjugates of $\mathrm{O}_8^+(2).3.2$. Using one of the conjugacy classes of S_{10} would cover $21A$ and $9C$ in 17791488 subgroups. A cheaper way is to use all conjugates of $\mathrm{O}_8^+(2).3.2$ and one class of 14080 conjugates of $\mathrm{O}_7(3)$. Using $\mathrm{S}_3 \times \mathrm{U}_4(3).2$ would not give us any conjugacy classes not available in $\mathrm{O}_8^+(2).3.2$. So the covering contains all conjugates of $\mathrm{O}_8^+(2).3.2$.

The optimal way of covering $16B$ is to use all conjugates of $2^{10}{:}\mathrm{M}_{22}$. Another choice would be to use conjugates of the Tits group, as a $16B$ element is in four of these. This would also cover $13B$. But the cheapest way to cover these two conjugacy classes is to use all conjugates of $2^{10}{:}\mathrm{M}_{22}$ and one conjugacy class of subgroups isomorphic to $\mathrm{O}_7(3)$. This completes the covering. ☐

Proposition 3.4. *The bounds for* $\sigma(\mathrm{Fi}_{23})$ *are* $8804338591 \leq \sigma(\mathrm{Fi}_{23}) \leq 8946269383$*. Any minimal covering of* Fi_{23} *must contain all conjugates of* $\mathrm{O}_8^+(3).3.2$*,* $2^{11}\mathrm{M}_{23}$*,* $\mathrm{A}_{12}.2$*, and at least half of the conjugates of* $2^{2\cdot}\mathrm{U}_6(2).2$*,* $\mathrm{S}_8(2)$ *and* $2^{\cdot}\mathrm{Fi}_{22}$*.*

Proof. First note that we need all conjugates of $\mathrm{A}_{12}{:}2$ to cover the conjugacy class $35A$.

Next we look at $23A$. This is in one conjugate of $\mathrm{L}_2(23)$ and one of $2^{11}\mathrm{M}_{23}$. If we use $\mathrm{L}_2(23)$ then we would also get $12O$, but a cheaper way to get $23A$ and $12o$ is to use all conjugates of the 2^{11} normaliser and all conjugates of $2^{\cdot}\mathrm{Fi}_{22}$. So the covering must contain all conjugates of $2^{11}\mathrm{M}_{23}$.

The optimal way to cover $39A$ is to use all conjugates of $\mathrm{O}_+^8(3).3.2$. This is also in the maximal subgroups $\mathrm{S}_3 \times \mathrm{O}_7(3)$ and max 8 of index 6165913600, but neither of these subgroups would give us any conjugacy class not covered by all conjugates of $\mathrm{O}_+^8(3).3.2$.

The remaining classes are $17A$, $22A$ and $24B$. An element of class $17A$ is contained in one conjugate of $\mathrm{S}_4(4).4$ and two conjugates of $\mathrm{S}_8(2)$. Neither of these subgroups contain any elements of $22A$ or $24B$, so we can choose the covering of these independently. The best way is to use at least half the 86316516 conjugates of $\mathrm{S}_8(2)$ and possibly some auxiliary conjugates of $\mathrm{S}_4(4).4$.

An element of $22A$ is in two of the 31671 conjugates of $2\mathrm{Fi}_{22}$ and one of the 55582605 conjugates of $2^{2\cdot}\mathrm{U}_6(2).2$. An element of class $24B$ is also contained in two conjugates of the latter group, and this is the lowest index subgroup containing elements of class $24B$. The other subgroups are the other 2^2 normaliser, the 2^6 normaliser, and the centraliser of an element of class $3B$. So any minimal covering would contain at least half the conjugates of $2^{2\cdot}\mathrm{U}_6(2).2$ and any auxiliary subgroups needed to cover $24B$, together with as many conjugates of $2\mathrm{Fi}_{22}$ and the 2^2 normaliser needed to cover $22A$.

We have $f(35A, 23B, 17A, 24B, 27A, 39B, 22A) = 8537488128 + 195747378 + 43156474 + 27086048 + 134637 + 2989 + 401 = 8803616055$. ☐

Proposition 3.5. *Any minimal covering of* Fi_{24}' *contains all conjugates of the* *maximal subgroups* Fi_{23}*,* $(3\times \mathrm{O}_8^+(3){:}3){:}2$*,* $\mathrm{O}_{10}^-(2)$*,* $3^7.\mathrm{O}_7(3)$ *and* $29{:}14$*. We have*

$$3091639677934679981816 \le \sigma(\mathrm{Fi}_{24}') \le 3091639677998938747142.$$

Proof. An element of class $29A$ is only in one conjugate of $29{:}14$, so we need all conjugates of this subgroup in the covering.

A $23B$-element is in one of the 306936 conjugates of Fi_{23} and one conjugate of $2^{11\cdot}\mathrm{M}_{24}$. If we use all the conjugates of $2^{11\cdot}\mathrm{M}_{24}$ then we also cover $24G$, $20B$, $21B$, $24C$ and $24D$. All of these are either in a conjugate of $N_{\mathrm{Fi}_{24}'}(3A)$ or $2^{\cdot}\mathrm{Fi}_{22}.2$, so we can use these subgroups together with Fi_{23} to cover those classes more cheaply. This puts all conjugates of Fi_{23} into the covering.

Next we look at $42C$. An element of this class is in one conjugate from each class of He:2, one conjugate of $C_{\mathrm{Fi}_{24}'}(2B)$, one conjugate of $3^7\mathrm{O}_7(3)$ and two conjugates of $N_{\mathrm{Fi}_{24}'}(7B)$. The optimal way is to use all conjugates of $3^7\mathrm{O}_7(3)$, because although using $C_{\mathrm{Fi}_{24}'}(2B)$ or He:2 would give $20B$, $21B$, $24C$ and $24D$, we can cover these better with all conjugates of $2\mathrm{Fi}_{22}.2$ and $N_{\mathrm{Fi}_{24}'}(3A)$.

The remaining classes are $20B$, $21B$, $24C$, $24D$, $33B$ and $39B$. An element of each of the conjugacy classes $21B$, $24C$, $24D$ and $39B$ is in two conjugates of $N_{\mathrm{Fi}_{24}'}(3A)$, and this is the best subgroup for each of them. The only subgroup containing both $33B$ and $20B$ is $\mathrm{O}_{10}^-(2)$. This subgroup is optimal for $33B$, and an element in this class is in two conjugates of the subgroup, but the best subgroup for $20B$ is $2^{\cdot}\mathrm{Fi}_{22}.2$. This shows that these six conjugacy classes can be covered by all conjugates of $N_{\mathrm{Fi}_{24}'}(3A)$ and $\mathrm{O}_{10}^-(2)$. The number of subgroups in this set is less than any other number of subgroups that could be used to cover $33B$ alone, so we see that the best covering is to use as many conjugates of $N_{\mathrm{Fi}_{24}'}(3A)$ as we need for $21B$, $24C$, $24D$ and $39B$, as many conjugates of $\mathrm{O}_{10}^-(2)$ as are needed for $33B$, and enough conjugates of $2\mathrm{Fi}_{22}.2$ and auxiliary subgroups to top up $20B$. □

Proposition 3.6. *Any minimal covering of* He *consists of all conjugates of the maximal subgroups* $\mathrm{S}_4(4){:}2$*,* $2^2.\mathrm{L}_3(4).\mathrm{S}_3$*,* $2^{1+6}\mathrm{L}_3(2)$ *and* $3\mathrm{S}_7$*, and we have* $\sigma(\mathrm{He}) = 464373$*.*

Proof. An element of class $17B$ is in one conjugate of $\mathrm{S}_44.2$ and no other subgroup. This implies that a minimal covering must contain all conjugates of this subgroup.

The remaining classes are $12B$, $14D$, $21B$, $21D$ and $28B$. An element of class $14D$ is in one of the 187425 conjugates of $2^{1+6}\mathrm{L}_3(2)$, one of the 625800 conjugates of $7^{1+2}{:}(\mathrm{S}_3\times 3)$ and one of the 244800 conjugates of $7^2{:}\mathrm{SL}_2(7)$. Using the involution centraliser would give us $12B$, but the 7-normaliser would give $21B$ and $21D$. The 7^2-normaliser does not give any of the other conjugacy classes. We note that using all conjugates of the involution centraliser, $3^{\cdot}\mathrm{S}_7$ and $2^2\mathrm{L}_3(3).\mathrm{S}_3$ completes the covering in 462315 subgroups. This is smaller than the index of the 7-normaliser, so it must be best to include all conjugates of the involution centraliser in the covering at this point.

A $28B$-element is in one of the 8330 conjugates of $2^2\mathrm{L}_3(3).\mathrm{S}_3$ and one conjugate each of $7{:}3\times \mathrm{L}_3(2)$ and $\mathrm{S}_4\times \mathrm{L}_3(2)$. Both of these subgroups have index greater than 462315, so we use the conjugates of $2^2\mathrm{L}_3(3).\mathrm{S}_3$. The only other class is $21B$. The best way to cover this class is to use all conjugates of $3^{\cdot}\mathrm{S}_7$. This completes the covering. □

Proposition 3.7. *There is a minimal covering of* HS *consisting of all conjugates of the maximal subgroups* M_{22} *and* S_8*, and one of the conjugacy classes of subgroups isomorphic to* $\mathrm{U}_3(5).2$*. This gives* $\sigma(\mathrm{HS}) = 1376$*.*

Proof. An element of class $15A$ is in one conjugate of each of the 1100 maximal subgroups conjugate to S_8 and one of the 5775 maximal subgroups conjugate to $5{:}4 \times \mathrm{A}_5$. If we use all conjugates of S_8 then we also cover the classes $6A$, $7A$, $8A$, $10B$ and $12A$, but if we were to use $5{:}4 \times \mathrm{A}_5$ then we could cover $20A$. Considering $20A$ and $15A$ alone, we see that this would not be the most efficient choice for covering those classes, as it could be done by using S_8 and the 176 conjugates of $\mathrm{U}_5(2)$. So we include all conjugates of S_8 in our covering.

Next we look at $11A$. This is in one conjugate of each class of M_{11} and one conjugate of M_{22}. Using any of those subgroups would also cover $5C$, but either choice of class of M_{11}'s would give a conjugacy class of elements of order 8. This would use 5600 subgroups to cover $11A$ and one class of elements of order 8, but a cheaper method would be to use all conjugates of M_{22} and all conjugates of $\mathrm{U}_3(5).2$. So the conjugates of M_{22} go into the covering.

This leaves $20A$, $8B$ and $8C$. Using either class of $\mathrm{U}_5(3).2$ is an optimal way to cover $20A$, and would complete the covering.

The formula at the beginning of this section gives a lower bound for $\mu(\mathrm{HS})$. □

Proposition 3.8. *Any minimal covering of* J_2 *contains all conjugates of* $3\mathrm{S}_6$ *and* $2^{1+4}{:}\mathrm{A}_5$ *and at least half the conjugates of* $\mathrm{U}_3(3)$ *and* $2^{2+4}(3 \times \mathrm{S}_3)$*. We have* $907 \le \sigma(\mathrm{J}_2) \le 1154$*.*

An element of class $10D$ is in one of the 315 conjugates of $2^{1+4}{:}\mathrm{A}_5$, one of the 1008 conjugates of $\mathrm{A}_5 \times \mathrm{D}_{10}$ and one of the 2016 conjugates of $5^2{:}\mathrm{D}_{12}$. If we use all conjugates of $2^{1+4}{:}\mathrm{A}_5$ then we also cover $8A$ and $12D$. Using $\mathrm{A}_5 \times \mathrm{D}_{10}$ would give $10B$ and $15A$, but elements from both of these are contained in the 280 conjugates of $3\mathrm{S}_6$, giving a better covering of $10B$, $15A$ and $10D$. It isn't optimal to use $5^2{:}\mathrm{D}_{12}$ either. This gives $6B$ and $12A$, but both of those can be covered using the 525 conjugates of $2^{2+4}(3 \times \mathrm{S}_3)$. So the covering must contain all conjugates of the involution centraliser.

The remaining elements are in classes $6B$, $7A$, $10B$ and $15A$. An element of class $7A$ is in one of the 1800 conjugates of $\mathrm{PGL}_2(7)$ and two of the 100 conjugates of $\mathrm{U}_3(3)$. We could also cover $6B$ with $\mathrm{PGL}_2(7)$ but a cheaper way is to use all conjugates of $2^{2+4}(3 \times \mathrm{S}_3)$. So the covering contains at least half the conjugates of $\mathrm{U}_3(3)$.

By the results of [10], we can calculate an upper bound for the number of conjugates of $\mathrm{U}_3(3)$ needed to cover $7A$ by subtracting the size of a set of conjugates of $\mathrm{U}_3(3)$ that do not intersect in an element of class $7A$ from 100. A random search using MAGMA [2] finds such a set of size 10.

A $6B$-element is in two conjugates of $2^{2+4}(3\times \mathrm{S}_3)$, two conjugates of $5^2{:}\mathrm{D}_{12}$, two conjugates of $\mathrm{PGL}_2(7)$ and one conjugate of $\mathrm{A}_4 \times \mathrm{A}_5$. We note that elements from $10B$ and $15A$ are both contained in $3{}^{\cdot}\mathrm{S}_6$ of index 280, so the covering can be completed in 805 subgroups. This shows that no subgroup can be used for $6B$ and also the other classes, so we can consider $6B$ separately. It is clear that the best method uses at least half the conjugates of $2^{2+4}(3 \times \mathrm{S}_3)$. Finally, a $15A$-element is only in one conjugate of $3{}^{\cdot}\mathrm{S}_6$ and some much smaller subgroups, so we complete the covering with all conjugates of this subgroup.

We use the same method as for $7A$ to bound the number of conjugates of $2^{2+4}(3 \times \mathrm{S}_3)$ needed to cover $6B$. We find that we can subtract 56 conjugates.

Proposition 3.9. *The bounds for* $\mu(\mathrm{J}_4)$ *and* $\sigma(\mathrm{J}_4)$ *are as given in the table.*

Proof. There is a covering of size 251012689468122717 of J_4 consisting of all conjugates of the maximal subgroups $29{:}28$, $43{:}14$, $2^{11}{:}\mathrm{M}_{24}$, $2^{1+12\cdot}3^{\cdot}\mathrm{M}_{22}{:}2$, $2^{3+12\cdot}(\mathrm{S}_5 \times \mathrm{L}_3(2))$, $2^{10}{:}\mathrm{L}_5(2)$ and $\mathrm{U}_3(11){:}2$. $\square$

Proposition 3.10. *Any minimal covering of* Ru *has size 12992175 and consists of all conjugates of the maximal subgroups* $\mathrm{L}_2(29)$, $(2^2 \times \mathrm{Sz}(8)){:}3$ *and* $2^{1+4+6}{:}\mathrm{S}_5$.

Proof. First we note that an element of class $29A$ is only in one conjugate of the maximal subgroup $\mathrm{L}_2(29)$, so all conjugates of this subgroup must be in a covering.

Next we consider $26C$. This is in one of the 417600 conjugates of $(2^2 \times \mathrm{Sz}(8)){:}3$ and two of the 4677120 conjugates of $\mathrm{L}_2(25).2^2$. The advantage of using the latter would be covering some elements of $24B$. But we would need at least half the conjugates of $\mathrm{L}_2(25).2^2$ to cover $26C$, so a better way to cover $26C$ and $24B$ would be to use all conjugates of $(2^2 \times \mathrm{Sz}(8)){:}3$ and all 593775 conjugates of $2^{1+4+6}{:}\mathrm{S}_5$. So we put all conjugates of $(2^2 \times \mathrm{Sz}(8)){:}3$ into the covering.

The most efficient way to cover $24B$ is to use all conjugates of $2^{1+4+6}{:}\mathrm{S}_5$, and this completes the covering. $\square$

Proposition 3.11. *We have* $338625 \leq \sigma(\mathrm{Suz}) \leq 540333$. *Any minimal covering of* Suz *contains all conjugates of the maximal subgroups* $\mathrm{G}_2(2)4$ *and* $2^{1+6}\mathrm{U}_4(2)$, *and at least some conjugates of* $\mathrm{U}_5(2)$ *and* $\mathrm{J}_2{:}2$.

Proof. An element of class $13A$ is in one of the 1782 conjugates of $\mathrm{G}_2(4)$, the 39916800 conjugates of both classes of $\mathrm{L}_3(3).2$, and three of the 57480192 conjugates of $\mathrm{L}_2(25)$. If we use all conjugates of $\mathrm{G}_2(4)$ then we would miss out on the elements in classes $8C$, $12D$ and $12E$ that could be covered by one of the higher-index subgroups. This is no loss, however, as we can cover those classes by using all conjugates of the subgroups $\mathrm{J}_2{:}2$, $3\mathrm{U}_4(3).2$, and $\mathrm{U}_5(2)$. This would use a total of 428078 subgroups (including the conjugates of $\mathrm{G}_2(4)$), which is far better than using any of the alternatives for $13A$. This puts all conjugates of $\mathrm{G}_2(4)$ into the covering.

Next we look at $20A$. This is in two of the 17297280 conjugates of the maximal subgroup $(3^2{:}4 \times \mathrm{A}_6).2$ and the 1216215 conjugates of the maximal subgroup $2^{2+8}(\mathrm{A}_5 \times \mathrm{S}_3)$. It is also in one of the 135135 conjugates of the maximal subgroups $2^{1+6}\mathrm{U}_4(2)$ and the 10378368 conjugates of the subgroup $(\mathrm{A}_6 \times \mathrm{A}_5).2$. If we use $(3^2{:}4 \times \mathrm{A}_6).2$ instead of the involution centralisers then we would cover the conjugacy class $12D$, but a cheaper way to cover $20A$ and $12D$ is to use all conjugates of the involution centraliser and the 370656 conjugates of $\mathrm{J}_2{:}2$. So we put all conjugates of $2^{1+6}\mathrm{U}_4(2)$ into the covering.

The remaining classes are $11A$, $12D$ and $14A$. The elements of class $11A$ is in two of the 32760 conjugates of $\mathrm{U}_5(2)$, two of the 232960 conjugates of $3^5{:}\mathrm{M}_{11}$ and one of the 2358720 conjugates of $\mathrm{M}_{12}{:}2$. Elements of class $12D$ are in three of the 370656 conjugates of $\mathrm{J}_2{:}2$ and several other, much higher indexed, subgroups. The class $14A$ elements are contained in two conjugates of $\mathrm{J}_2{:}2$ and one conjugate of $(\mathrm{A}_4 \times \mathrm{L}_3(4)){:}2$. Thus the smallest conceivable covering would use half the conjugates of $\mathrm{U}_5(2)$ and half the conjugates of $\mathrm{J}_2{:}2$, and using all conjugates of these groups is guaranteed to cover the group. ☐

4 Using König's Theorem

The results in this subsection are obtained using Theorem 4.3.

Proposition 4.1. *The bound for* $\mu(\mathrm{HN})$ *is as given in the table. The value of* $\sigma(\mathrm{HN})$ *is 229758831, and any minimal covering of* HN *must consist of all conjugates of the maximal subgroups* $\mathrm{U}_3(8).3$, $2\mathrm{HS}.2$, A_{12}, $2^{1+8}(\mathrm{A}_5 \times \mathrm{A}_5).2$ *and* $5^{1+4}{:}2^{1+4}5.4$.

Proof. The only maximal subgroup containing an element of class $19B$ is $\mathrm{U}_3(8).3$ There are 16500000 conjugates of this subgroup. The only one containing an element of class $22A$ is 2HS.2, and there are 1539000. There are two subgroups containing an element of class $35B$. These are $(\mathrm{D}_{10} \times \mathrm{U}_3(5)).2$ and A_{12}. There are 1140000 conjugates of the latter. An element of class $25B$ is in both of the 5 normalisers, so Theorem 4.3 says that we may use 1140000 elements of class $35B$ and 136515456 elements of class $25B$. This gives a pairwise generating set of size 16500000+1539000+1140000+136515456.

The above proves that any minimal covering must contain all conjugates of $\mathrm{U}_3(8).3$ and $2^{\cdot}\mathrm{HS}.2$. The only remaining conjugacy class contained in a conjugate of $(\mathrm{D}_{10} \times \mathrm{U}_3(5)).2$ is $35B$, so the best way to cover $35B$ is to use all conjugates of A_{12}. This only leaves $20E$, $25B$ and $30C$.

Look at $30C$. An element of this class is contained in $2^{1+8}(\mathrm{A}_5 \times \mathrm{A}_5).2$, $5^{2+1+2}4^{\cdot}\mathrm{A}_5$, and a 3-centraliser. If we use the involution centraliser then we also get $20E$, but if we use the 5-normaliser then we get $25B$. The 5-normaliser has index 364041216, and an element of class $20E$ is contained in three conjugates of it, so this would complete the covering with 364041216 groups. But this is not optimal, as using the involution centraliser means that we only need the 136515456 conjugates of $5^{1+4}{:}2^{1+4}5.4$ to complete the covering. ☐

Proposition 4.2. *The bounds for* $\mu(\mathrm{J}_3)$ *and* $\sigma(\mathrm{J}_3)$ *are as given in the table.*

Proof. An element of class $19A$ is in the 14688 conjugates of both classes of maximal subgroups isomorphic to $\mathrm{L}_2(19)$. By Theorem 4.3 we get a pairwise generating set of 14688 elements of 19A. The conjugacy classes not in either subgroup are $12A$, $15A$, $17A$ and $8A$. We find that $f(12A, 15A, 17A, 8A)$ is 8960. This gives $\mu'(\mathrm{J}_3) = 23648$. ☐

Proposition 4.3. *Any minimal covering of* M_{24} *is the union of the conjugacy classes of maximal subgroups* M_{23}, $\mathrm{M}_{12}{:}2$ *and* $\mathrm{L}_3(4).3.2$. *This gives* $\sigma(\mathrm{M}_{24}) =$ *3336. The bound for* $\mu(\mathrm{M}_{24})$ *is as given in the table.*

Proof. An element of class $23A$ is in one of the 24 conjugates of M_{23} and one of the 40320 conjugates of $\mathrm{L}_2(23)$. If we use all conjugates of M_{23} then we would also cover the classes $8A$, $11A$, $14A$ and $15A$. The only class that we could cover by using $\mathrm{L}_2(23)$ instead would be $12B$, but a cheaper way to cover the two classes $23A$ and $12B$ would be to use all conjugates of M_{23} and all 1288 conjugates of $\mathrm{M}_{12}{:}2$. So any minimal covering of M_{24} must include all 24 conjugates of M_{23}.

The remaining classes are $10A$, $12A$, $12B$ and $21A$. An element of the class $21A$ is in one of the 2024 conjugates of $\mathrm{L}_3(4).3.2$ and one of the 3795 $2^6{:}(\mathrm{L}_3(2)) \times \mathrm{S}_3)$. If we use the latter conjugacy class of subgroups, then we would also cover $12A$ and $12B$. But fewer subgroups are needed if we choose all conjugates of $\mathrm{L}_3(4).3.2$ and all conjugates of $\mathrm{M}_{12}{:}2$. This proves that we must include all conjugates of $\mathrm{L}_3(4).3.2$ in our covering.

The remaining classes are $10A$, $12A$ and $12B$. The most efficient way to cover $12B$ is to use the 1288 conjugates of $\mathrm{M}_{12}{:}2$. This gives us all the remaining group elements, so this must be the best decision. This proves that a minimal covering is as stated in the lemma.

By Theorem 4.3 we get a pairwise generating set of 2024 elements of class 21A. The conjugacy classes not in either subgroup are $10A$, $23A$ and $11A$. We find that $f(10A, 23A, 11A)$ is 121. This gives $\mu'(\mathrm{M}_{24}) = 2145$. □

Proposition 4.4. *Any minimal covering of* Co_1 *must contain all conjugates of the maximal subgroups* $(\mathrm{A}_4 \times \mathrm{G}_2(4)){:}2$, $3^{\cdot}\mathrm{Suz}.2$, $2_+^{1+8}\mathrm{O}_8^+(2)$ *and* $(\mathrm{A}_5 \times \mathrm{J}_2){:}2$, *and at least half the conjugates of* $2^{11}{:}\mathrm{M}_{24}$. *This gives* $58029469522 \le \sigma(\mathrm{Co}_1) \le 58033615710$. *The bound for* $\mu(\mathrm{Co}_1)$ *is as given in the table.*

Proof. An element of the class $39B$ is in the 688564800 conjugates of $(\mathrm{A}_4 \times \mathrm{G}_2(4)){:}2$ and is in the 1545600 3Suz2. By Theorem 4.3 we get a pairwise generating set of 688564800 elements of $39B$. The conjugacy classes not in either subgroup are $35A$, $36A$, $21C$, $23B$, $20C$, $28A$, $24F$, $30D$, $20B$, $30E$ and $12I$. We find that

$$f(35A, 36A, 21C, 23B, 20C, 28A, 24F, 30D, 20B, 30E, 12I) = 45802057776.$$

This gives $\mu'(\mathrm{Co}_1) = 46490622576$.

¿From the above, we see that any minimal covering needs all conjugates of $(\mathrm{A}_4 \times \mathrm{G}_2(4)){:}2$. Then $33A$ is in $3^{\cdot}\mathrm{Suz}.2$, $3^6{:}2^{\cdot}\mathrm{M}_{12}$ and $\mathrm{U}_6(2).3.2$. Conjugacy classes not in $3^{\cdot}\mathrm{Suz}.2$ that can be covered using the other subgroups are $12I$, $20B$, $21C$, $24F$, $30D$ and $36A$. These classes can be covered by using all conjugates of Co_2, $2^{11}{:}\mathrm{M}_{24}$ and $2_+^{1+8}\mathrm{O}_8^+(2)$. This is cheaper than using either $\mathrm{U}_6(2).3.2$ or $3^6{:}2^{\cdot}\mathrm{M}_{12}$, so all conjugates of $3^{\cdot}\mathrm{Suz}.2$ are in a minimal covering.

An element of class $36A$ is in one conjugate of each of $2_+^{1+8}\mathrm{O}_8^+(2)$, $\mathrm{U}_6(2).3.2$ and the two smallest 3-local maximal subgroups. Neither 3-local subgroup gives any conjugacy classes not available in the involution centraliser, but $\mathrm{U}_6(2).3.2$ covers $21C$. This conjugacy class can be covered using conjugates of $2^{11}{:}\mathrm{M}_{24}$, so we use the involution centraliser to cover $36A$.

The remaining classes are $20B$, $21C$, $23B$ and $35A$. We note that we can complete the covering with 57296883735 more subgroups, by using all conjugates of $(\mathrm{A}_5 \times \mathrm{J}_2){:}2$ and $2^{11}{:}\mathrm{M}_{24}$. An element of class $35A$ is in one conjugate of $(\mathrm{A}_5 \times \mathrm{J}_2){:}2$, two conjugates of $(\mathrm{A}_6 \times \mathrm{U}_3(3)){:}2$, and one conjugate of $(\mathrm{A}_7 \times \mathrm{L}_2(7)){:}2$. Both the latter subgroups have index far higher than 57296883735, so we do not use them.

Then the optimal way to cover $21C$ is to use at least half the conjugates of $2^{11}{:}\mathrm{M}_{24}$. This would give us at least half the elements of $20B$ and $23B$ too. Elements from both those conjugacy classes are also contained in conjugates of the subgroup Co_2 of index a mere 98280, so this suggests that the best way to complete the covering is to use conjugates of Co_2. □

Proposition 4.5. *The bounds for* $\mu(\mathrm{Th})$ *and* $\sigma(\mathrm{Th})$ *are as given in the table.*

Proof. An element of the class $27A$ is in the 96049408000 conjugates of $N_{\mathrm{Th}}(3B)$ and $N_{\mathrm{Th}}(3^2)$. By Theorem 4.3 we get a pairwise generating set of 96049408000 elements of 27A. An element of class $39B$ is only in the subgroup $(3 \times \mathrm{G}_2(3)){:}2$, of which there are 3562272000 conjugates. Conjugacy classes not in either subgroup are $19A$, $20A$, $21A$, $28A$, $30B$, $31B$. We find that $f(19A, 20A, 30B, 28A, 31B, 21A)$ is 3811597855. This gives $\mu'(\mathrm{Th}) = 103423277855$. □

Proposition 4.6. *When* $G \cong \mathrm{M}_{23}$*, we have* $\mu(G) = \sigma(G) = 41079$

Proof. An element of the class $14B$ is in one of the 253 conjugates of $2^4\mathrm{A}_7$ and one of the 253 conjugates of $PSL(3,4).2$. By Theorem 4.3 we get a pairwise generating set of 253 elements of $14B$. Similarly we get a pairwise generating set of 506 elements of class $15B$, as an element of this class is contained in one of the 506 conjugates of A_8 and one of the 1771 conjugates of $2^4(3 \times \mathrm{A}_5)$. An element of class $23A$ is one conjugate of the maximal subgroup 23:11 and no other maximal subgroup, so we can adjoin 40320 elements of class $23A$ to our pairwise generating set. This gives $\mu(\mathrm{M}_{23}) = 41079$.

The exact value of $\sigma(\mathrm{M}_{23})$ is computed in [10]. □

5 The other groups

The first proposition is a slight extension of the relevant results of [10] and [14].

Proposition 5.1. *For the Mathieu group* M_{11} *we have* $\mu(\mathrm{M}_{11}) = \sigma(\mathrm{M}_{11}) = 23$.

Proof. ¿From the Atlas [7] we know that the set of all 12 conjugates of $PSL(2,11)$ and all 11 conjugates of M_{10} is a covering for the Mathieu group M_{11}. This gives $\sigma(\mathrm{M}_{11}) \leq 23$. Each element of order 11 is contained in exactly one copy of $PSL(2,11)$ and no other maximal subgroup of M_{11}. Similarly, each element of order 8 is in exactly one copy of M_{10} and no other maximal subgroup. Hence we may choose one element of order 11 from each copy of the maximal subgroups $PSL(2,11)$ and one element of order 8 from each copy of the subgroups M_{10} to obtain a pairwise generating set for M_{11} of size 23. This gives $\mu(\mathrm{M}_{11}) \geq 23$. Finally, the result follows from $23 \leq \mu(\mathrm{M}_{11}) \leq \sigma(\mathrm{M}_{11}) \leq 23$.

□

Proposition 5.2. *The bounds for* $\sigma'(\mathrm{M}_{12})$ *and* $\mu'(\mathrm{M}_{12})$ *are as given in the table.*

We use the computer package Magma [2] to perform a random search for a pairwise generating set. This finds a pairwise generating set of size 131. There is a covering of size between 150 and 222 consisting of one class each of maximal subgroups isomorphic to M_{11} and $\mathrm{A}_6.2^2$, and enough conjugates of $\mathrm{L}_2(11)$ to cover the conjugacy class $6A$.

Proposition 5.3. *We have* $\mu(\mathrm{B})/\sigma(\mathrm{B}) > 1 - 1/10^{13}$.

Proof. The power maps between conjugacy classes of B show that every maximal cyclic group contains an element of class $2A$, $2B$, $2C$, $2D$, $3A$, $3B$, $5A$, $5B$, $31A$, or $47A$.

The $2C$ centraliser is $(2^2 \times \mathrm{F}_4(2)){:}2$ where the 2^2 has two elements of class $2A$ and one of class $2C$. So any element that powers up to a $2C$ involution is in the subgroup $(2^2 \times \mathrm{F}_4(2))$ and hence can be found in a $2A$ centraliser, and we do not need to include any $2C$ centralisers in a covering if it includes all conjugates of the $2A$ centraliser.

The $2D$ centraliser $2^{[}26]^{\cdot}\mathrm{O}_8^+(2)$ is a subgroup of the maximal subgroup $2^9.2^16^{\cdot}\mathrm{S}_8(2)$ and the $31A$ centraliser is contained in Th. So there is a covering consisting all conjugates of the maximal subgroups $N_{\mathrm{B}}(2A)$, $N_{\mathrm{B}}(2B)$, $N_{\mathrm{B}}(3A)$, $N_{\mathrm{B}}(3B)$, $N_{\mathrm{B}}(5A)$, $N_{\mathrm{B}}(5B)$, $N_{\mathrm{B}}(47A)$, $2^9.2^16^{\cdot}\mathrm{S}_8(2)$ and Th. This has approximate size $\sigma' = 3 \times 10^{30}$.

Taking one element from each cyclic group of order 47 gives an independent set of approximate size $\mu' = 3 \times 10^{30}$. The ratio $\mu'/\sigma' \leq \mu(\mathrm{B})/\sigma(\mathrm{B})$ is more than $1 - 1/10^{13}$.

□

Proposition 5.4. *We have* $\mu(\mathrm{M})/\sigma(\mathrm{M}) > 4/5$.

The power maps between conjugacy classes of M show that every maximal cyclic group contains an element of class $2A$, $2B$, $3A$, $3B$, $3C$, $5A$, $5B$, $7A$, $7B$, $13A$, $13B$, $41A$, $59A$ or $71A$.

We know from [9] that the $59A$ centralisers are contained in the maximal subgroup $\mathrm{L}_2(59)$ and the $71A$ centralisers are contained in the maximal subgroup $\mathrm{L}_2(71)$. The $41A$ centralisers are contained in the maximal subgroups $N_{\mathrm{M}}(3^8)$. This gives a covering consisting of all conjugates of $N_{\mathrm{M}}(2A)$, $N_{\mathrm{M}}(2B)$, $N_{\mathrm{M}}(3A)$, $N_{\mathrm{M}}(3B)$, $N_{\mathrm{M}}(3C)$, $N_{\mathrm{M}}(5A)$, $N_{\mathrm{M}}(5B)$, $N_{\mathrm{M}}(7A)$, $N_{\mathrm{M}}(7B)$, $N_{\mathrm{M}}(13A)$, $N_{\mathrm{M}}(13B)$, $N_{\mathrm{M}}(3^8)$, $\mathrm{L}_2(59)$ and $\mathrm{L}_2(71)$. This covering has approximate size 1.5×10^{49}.

An independent set can be found by taking an element of order 71 from each conjugate of $\mathrm{L}_2(71)$ and an element of order 59 from each conjugate of $\mathrm{L}_2(59)$. This independent set has approximate size 1.2×10^{49}.

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