Groups whose proper quotients are virtually abelian

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Abstract

The just non-(virtually abelian) groups with non-trivial Fitting subgroup are classified. Particular attention is given to those which are virtually nilpotent and examples are given of the interesting phenomena that can occur.

1 Introduction

If $\mathscr{P}$ is a property of groups, a group G is said to be *just non-$\mathscr{P}$* if G does not have property $\mathscr{P}$ but all proper quotients of G do have property $\mathscr{P}$. Such groups are interesting to study for a number of reasons. On the one hand, a classic question in group theory is how the quotients of a group affect its structure. Just non-$\mathscr{P}$ groups naturally arise in this context. Another reason is that we can view just non-$\mathscr{P}$ groups as living on the boundary of the class of groups having property $\mathscr{P}$ and their study gives insight into the property $\mathscr{P}$ itself. In this work we consider the case when $\mathscr{P}$ is the property of being *virtually abelian* (that is, *abelian-by-finite*). We shall use the term *JNAF-group* to refer more briefly to just non-(abelian-by-finite) groups.

The first significant example of a study of just non-$\mathscr{P}$ groups was Newman's consideration of just non-abelian groups [10, 11]. Similar studies have been carried out for the cases when $\mathscr{P}$ is the property of being finite (McCarthy [8, 9]), of being polycyclic (Robinson and Wilson [13]), and of being finite-by-abelian (Robinson and Zhang [14]). A recent example is the work of De Falco [3] who considered just non-(nilpotent-by-finite) groups. There are many more studies of this kind and one source for further reading is the monograph of Kurdachenko, Otal and Subbotin [7]. In some sense, the study we present of JNAF-groups can be viewed as a simultaneous extension of the work of Newman and of McCarthy, as a natural analogue of that of

Robinson–Zhang, and as analysing an obvious gap introduced by the work of De Falco.

Of course, any infinite simple group is an example of a JNAF-group. To avoid such pathological examples we follow the model of all the above studies and consider only groups that possess a non-trivial normal abelian subgroup. Thus we shall classify JNAF-groups with non-trivial Fitting subgroup. It is worth noting, however, that progress can be made without making such an assumption. Wilson [15] made a study of just infinite groups possessing no non-trivial abelian subnormal subgroups and returned to it in a recent paper [16] where he also considered just infinite profinite groups. One important collection that arises in his classification lie within the class of *branch groups* (see, for example, Grigorchuk [4]). It is, in fact, known that every proper quotient of a branch group is virtually abelian and consequently they are examples of JNAF-groups. (This is [5, Proposition 1(b)], whose verification is embedded in the proof of [4, Theorem 4].) It is perhaps unsurprising then that Wilson's ideas can be extended to cover JNAF-groups that possess no non-trivial virtually abelian subnormal subgroups. This work was recently completed by Hardy [6], one of Wilson's research students. This gives further motivation for the study presented here since we are providing the "non-trivial Fitting subgroup" counterpart to the work of Hardy and Wilson.

Our description of JNAF-groups with non-trivial Fitting subgroup splits into several parts. Those which also happen to be not nilpotent-by-finite are necessarily just non-(nilpotent-by-finite) and hence fall among the groups examined by De Falco [3]. We shall similarly abbreviate this terminology to 'JNNF-group.' In Section 4 we exploit De Falco's results to describe most significant examples of JNAF-groups which are also JNNF-groups. In doing so we are obtaining analogues of [3, Theorems 2.6 and 2.13]. In fact, we observe that the JNNF-groups occurring in De Falco's theorems are in the most part actually JNAF-groups.

The most significant part of this paper is our description of JNAF-groups which are nilpotent-by-finite. We spend Section 3 examining these groups and in doing so we are in some sense describing the boundary between virtually abelian and virtually nilpotent groups. The Fitting subgroups of these groups are of finite index and have nilpotency class at least two. This is in contrast to the studies referred to above where the Fitting subgroup is usually abelian. We are therefore unable to apply the module theory that is often exploited. Furthermore it is not even the case that the Fitting subgroup needs to contain its centraliser as normally happens in such studies. Accordingly we require different methods to analyse the groups that occur. It turns out that the concepts of components (that is, quasisimple subnormal subgroups) and the generalised Fitting subgroup are key. These commonly arise in finite group theory but seem rarely to be exploited in infinite group theory. It is interesting to note their role here.

In Theorem 3.2 we characterise JNAF-groups which are nilpotent-by-finite and where the Fitting subgroup possesses torsion. In this case the Fitting subgroup can have arbitrarily large class and need not contain its centraliser. The group has a unique minimal normal subgroup and may possess components, each of which has non-trivial centre.

Theorem 3.7 contains the analogous characterisation of JNAF-groups which are nilpotent-by-finite and where the Fitting subgroup is torsion-free. Here the Fitting subgroup has class precisely two and actually does contain its centraliser, the group does not have a minimal normal subgroup, nor does it have any components.

2 Preliminaries

We begin first with some preliminaries which are used in Section 3. We shall use standard group theoretic terminology and conventions found in many textbooks. For example, $\mathrm{Z}(G)$ denotes the *centre* of the group G, G' denotes its *derived subgroup* and $\mathrm{F}(G)$ denotes its *Fitting subgroup*, this being the subgroup generated by all normal nilpotent subgroups of G. If H and K are normal subgroups of the group with $K \leqslant H$, we write $\mathrm{C}_G(H/K)$ for the *centraliser* of the quotient H/K in G; that is,

$$\mathrm{C}_G(H/K) = \{ x \in G \mid [H,x] \leqslant K \}.$$

Here $[x,y]$ denotes the *commutator* $x^{-1}y^{-1}xy$ and $[H,x]$ is the subgroup generated by all commutators $[h,x]$ as h ranges over the subgroup H.

The following elementary fact will be needed in the proofs of the main theorems. It holds, of course, since the class of abelian-by-finite groups is closed under taking subgroups and finite direct products.

Lemma 2.1 *Let G be a JNAF-group. Then G is subdirectly indecomposable; that is, if N_1 and N_2 are non-trivial normal subgroups of G, then $N_1 \cap N_2 \neq \mathbf{1}$.* □

As mentioned in the Introduction, we need the concept of a component of a group. Since these are more familiar in finite group theory, we recall here the relevant definitions and basic properties. These can be extracted, for example, from Section 31 of Aschbacher's monograph [1]. We omit most of the proofs as they can at worst be adapted from that source. The one exception to this is Proposition 2.7 where the most adaptation is needed.

Definition 2.2 A group L is called *quasisimple* if $L = L'$ and $L/\mathrm{Z}(L)$ is simple. A *component* of a group G is a quasisimple subnormal subgroup of G. We write $\mathrm{Comp}(G)$ for the set of component of G and $\mathrm{E}(G)$ for the subgroup generated by these components. The *generalised Fitting subgroup* of G is defined to be $\mathrm{F}^*(G) = \mathrm{F}(G)\mathrm{E}(G)$.

Lemma 2.3 *Let L be a group such that $L/\mathrm{Z}(L)$ is a non-abelian simple group. Then $L = L'\mathrm{Z}(L)$ and L' is quasisimple.*

Lemma 2.4 *Let L be a component of a group G and H be a subnormal subgroup of G. Then either L is a component of H or $[L, H] = \mathbf{1}$.*

This last fact is (31.4) in [1]. The proof given there exploits a minimal counterexample, so belongs to the world of finite group theory. It is not too hard, however, to convert the proof to one by induction on the defects of the subnormal subgroups H and L. (The *defect* of H is the length of a shortest series $G = H_0 > H_1 > \cdots > H_n = H$ where $H_i \trianglelefteqslant H_{i-1}$ for each i.)

Corollary 2.5 (i) *Distinct components of a group G commute.*

(ii) $\mathrm{Z}(\mathrm{E}(G)) = \langle\, \mathrm{Z}(L) \mid L \in \mathrm{Comp}(G)\,\rangle$.

(iii) $\mathrm{E}(G)/\mathrm{Z}(\mathrm{E}(G))$ *is the direct product of the central quotients of the components of G.*

Corollary 2.6 (i) *If L is a component of a group G and H is a soluble normal subgroup of G, then L and H commute.*

(ii) $\mathrm{F}^*(G)$ *is the central product of* $\mathrm{F}(G)$ *and* $\mathrm{E}(G)$.

(Recall a group G is the *central product* of subgroups $G_1, G_2, \ldots, G_n$ if $G = \langle G_1, G_2, \ldots, G_n\rangle$, $[G_i, G_j] = \mathbf{1}$ for $i \neq j$, and $G_i \cap \prod_{j\neq i} G_j \leqslant \mathrm{Z}(G_i)$ for all i.)

Proposition 2.7 *Suppose the Fitting subgroup of the group G has finite index in G. Then $\mathrm{C}_G(\mathrm{F}^*(G)) \leqslant \mathrm{F}^*(G)$.*

Proof: Let $C = \mathrm{C}_G(\mathrm{F}^*(G))$, $F = \mathrm{F}(G)$ and $Z = \mathrm{Z}(F)$. By definition we have $C \cap F \leqslant Z$. On the other hand, Corollary 2.6(i) tells us that Z commutes with $\mathrm{E}(G)$, which implies $Z \leqslant C$. Hence $C \cap F = Z$.

Suppose that $C \neq Z$. We have $C/Z = C/(C \cap F) \cong CF/F$, so C/Z is finite. Hence C/Z possesses a minimal normal subgroup M/Z, which is a direct product of finitely many isomorphic simple groups. If S/Z denotes one of these simple direct factors, then S/Z is a non-abelian simple group since $S \nleqslant F$. This forces $\mathrm{Z}(S) = Z$ and Lemma 2.3 tells us that $S = S'Z$ with S' is quasisimple. As $S \trianglelefteqslant M \trianglelefteqslant C \trianglelefteqslant G$, we see that S' is a component of G. Hence $S = S'Z \leqslant \mathrm{F}^*(G)$ and so $S \leqslant \mathrm{F}^*(G) \cap C = \mathrm{Z}(\mathrm{F}^*(G))$, which contradicts S being non-abelian.

Therefore $C = Z \leqslant F \leqslant \mathrm{F}^*(G)$, as required. □

The above result is sufficient for the application in Section 3. The conclusion also holds under a weaker hypothesis, namely it is enough to assume that the quotient $G/\mathrm{F}(G)$ satisfies the minimal condition on subnormal subgroups for then only minor changes are required in the above proof.

3 Nilpotent-by-finite JNAF-groups

In this section we characterise the JNAF-groups G which are nilpotent-by-finite. The section splits into two halves depending upon whether the Fitting subgroup possesses torsion or is torsion-free. In both cases after stating the theorems containing the conditions that characterise the particular class of JNAF-groups, we first verify that the conditions are sufficient and then complete the proof that the conditions necessarily hold. The advantage of proceeding in this order is that it enables us to continue our discussion immediately following the theorem and establish more structural properties of the JNAF-groups. At the end of each half we present some examples to illustrate the behaviour of these groups.

Torsion case

We begin by establishing a source of groups which are not abelian-by-finite.

Lemma 3.1 *Let K be a nilpotent group of class two such that K' is a non-trivial elementary abelian p-group of finite rank for some prime p and such that $K/\mathrm{Z}(K)$ is not finitely generated. Then K is not abelian-by-finite.*

Proof: Since K' is central, we calculate $[x^p, y] = [x, y]^p = 1$ for all $x, y \in K$. It follows that $K/\mathrm{Z}(K)$ is also an elementary abelian p-group. For the rest of the proof of this lemma, let us write $K/\mathrm{Z}(K)$ and K' additively and so view them as vector spaces over the field $\mathbb{F}_p$ of p elements. Then the commutator map $(x, y) \mapsto [x, y]$ induces a bilinear map $\beta\colon K/\mathrm{Z}(K) \times K/\mathrm{Z}(K) \to K'$.

Suppose $|K'| = p^k$ and fix a k-element set which is a basis for K'. We then have k projection maps $\pi_i\colon K' \to \mathbb{F}_p$ associated to this basis (projecting an element of K' onto the coefficient of the ith basis vector). Then $\beta\pi_i$ is an alternating bilinear form on the vector space $K/\mathrm{Z}(K)$. Let R_i denote the radical of the form $\beta\pi_i$; that is,

$$R_i = \{\, x \in K/\mathrm{Z}(K) \mid (x, y)\beta\pi_i = 0 \text{ for all } y \in K/\mathrm{Z}(K) \,\}.$$

Then $\bigcap_{i=1}^{k} R_i = \mathbf{0}$, since elements in this intersection correspond to elements of K that belong to its centre.

Since $K/\mathrm{Z}(K)$ is infinite, one of the R_i must have infinite codimension in $K/\mathrm{Z}(K)$. Let V be the quotient of $K/\mathrm{Z}(K)$ by this R_i and denote by $\alpha\colon V \times V \to \mathbb{F}_p$ the alternating bilinear form induced by $\beta\pi_i$. Since we have taken the quotient by the radical of $\beta\pi_i$, this new bilinear form is non-degenerate.

Now let A be any abelian subgroup of K. Then β takes the value 0 on the image of $A \times A$ and consequently the image of A in V is a totally isotropic subspace (with respect to the form α). It is well-known that a totally isotropic subspace of an n-dimensional vector space possessing a non-degenerate (alternating) from has codimension at least $n/2$. It follows that the image of A cannot have finite codimension in the infinite dimensional space V. Consequently A cannot have finite index in K and thus K is not abelian-by-finite. □

Our theorem characterising nilpotent-by-finite JNAF-groups G where $\mathrm{F}(G)$ possesses torsion is now as follows:

Theorem 3.2 *Let G be a group and Z denote the centre of its Fitting subgroup. Then G is a nilpotent-by-finite JNAF-group such that Z possesses torsion if and only if the following three conditions hold:*

(i) *Z is a p-primary abelian group for some prime p;*

(ii) *there is a nilpotent normal subgroup K of class precisely two and of finite index in G such that $K/\mathrm{Z}(K)$ is not finitely generated and such that K' is the unique minimal $G/\mathrm{F}(G)$-submodule of Z;*

(iii) *every component of G has non-trivial centre.*

Note that if Z is a p-primary abelian group (for some prime p), then its socle is simply the set of elements x in Z satisfying $x^p = 1$. A consequence of Conditions (i) and (ii) above is then that K' is an elementary abelian p-group.

Proof of Theorem 3.2: First suppose that G satisfies the three given conditions. Condition (ii) tells us that G is nilpotent-by-finite (with K as a nilpotent subgroup of finite index). The centre Z of $\mathrm{F}(G)$ possesses torsion by the assumption of Condition (i). We must show that G is a JNAF-group.

The fact that $\mathrm{F}(G)$ has finite index in G then forces the minimal submodule K' to be a finitely generated torsion abelian group and hence finite. Thus K satisfies the hypotheses of Lemma 3.1 and so is not abelian-by-finite. Therefore G is not abelian-by-finite. On the other hand, let N be a non-trivial normal subgroup of G. Applying Proposition 2.7, we deduce that $N \cap \mathrm{F}^*(G) \neq \mathbf{1}$. We claim, in fact, that $N \cap \mathrm{F}(G) \neq \mathbf{1}$.

Suppose that $N \cap \mathrm{F}(G) = \mathbf{1}$. Let $F = \mathrm{F}(G)$, $E = \mathrm{E}(G)$ and $F^* = \mathrm{F}^*(G)$. Write $\bar{G}$ for the quotient G/Z and use the same bar notation for the images of subgroups of G under the canonical homomorphism $G \to \bar{G}$.

Since $N \cap F = \mathbf{1}$, we have $NZ \cap F = (N \cap F)Z = Z$ and so $\bar{N} \cap \bar{F} = \mathbf{1}$. Now by Corollary 2.6(ii), $\bar{F^*}$ is the direct product $\bar{F} \times \bar{E}$. Furthermore

$$\bar{E} = EZ/Z \cong E/(E \cap Z) = E/\mathrm{Z}(E)$$

(at the last step we use Corollary 2.6(i)). The latter is, by Corollary 2.5, the direct product of the central quotients of the components of G. Thus

$$\bar{F} \times \bar{E} = \bar{F} \times S_1 \times S_2 \times \cdots \times S_k$$

for some collection $S_1, S_2, \ldots, S_k$ of non-abelian finite simple groups. Since $\bar{N} \cap \bar{F} = \mathbf{1}$, there must exist some index i such that $\bar{N} \cap \bar{F^*}$ has non-trivial projection into S_i. As $\bar{N}$ is normal in $\bar{G}$, standard arguments then show that $S_i \leqslant \bar{N}$. Therefore there exists some component L of G such that $L \leqslant NZ$. As $N \cap F = \mathbf{1}$, elements of N commute with those of Z, so

$$L = L' \leqslant (NZ)' = N' \leqslant N.$$

We deduce that the centre of L is contained in $N \cap F$ (as $\mathrm{Z}(L) \leqslant \mathrm{Z}(E) \leqslant F$, using Corollary 2.5 and the definition of F). Now Condition (iii) yields a contradiction.

Thus it is indeed the case that our normal subgroup N meets the Fitting subgroup $\mathrm{F}(G)$ non-trivially and standard theory of nilpotent groups yields $N \cap Z \neq 1$. The fact that K' is the unique minimal normal subgroup of G contained in Z (Condition (ii)) forces $K' \leqslant N$. It follows that the image of K in G/N is abelian and thus G/N is abelian-by-finite.

This establishes the sufficiency of Conditions (i)–(iii) to ensure our G is a JNAF-group.

We now turn to the necessity of our conditions. Let G be a JNAF-group such that the Fitting subgroup $\mathrm{F}(G)$ has finite index and such that $Z = \mathrm{Z}(\mathrm{F}(G))$ possesses torsion. Let T be the torsion subgroup of Z. This abelian group is the direct product of its p-primary components, each of which is a normal subgroup of G and at least one of which is non-trivial. Lemma 2.1 then shows that T has a unique primary component and thus T is a p-primary abelian group for some prime p.

Suppose that $Z \neq T$. Then there exists some element a in Z of infinite order. Now $\mathrm{F}(G)$ centralises a, so a has finitely many conjugates in G. Hence the normal closure A of the subgroup generated by a is a finitely generated abelian group, so is the direct product of a finite abelian group and a free abelian group. Hence there exists m such that A^m is free abelian and then $A^m \cap T = \mathbf{1}$, contrary to Lemma 2.1. Thus $Z = T$ and we have established Condition (i) in the statement of the theorem.

Let S be the socle of Z; that is, $S = \{\, x \in Z \mid x^p = 1 \,\}$. This is a non-trivial elementary p-subgroup which is normal in G and centralised by $\mathrm{F}(G)$.

We may therefore view S as a module over $\mathbb{F}_p(G/\mathrm{F}(G))$. Since this group algebra is finite, S has finite dimensional submodules and hence has at least one minimal non-trivial submodule. This corresponds to a minimal normal subgroup X of G contained in Z (and actually in S). Lemma 2.1 tells us that X is the unique minimal normal subgroup of G and that it is contained in all non-trivial normal subgroups of G.

By assumption the quotient G/X is abelian-by-finite. Hence there is a normal subgroup K of finite index in G which contains X and is contained in $\mathrm{F}(G)$ with K/X abelian. Now $K' \leqslant X$ and since G is not abelian-by-finite we have $K' = \mathbf{1}$. The minimality of X then forces $K' = X$. It follows that when S is viewed as a $G/\mathrm{F}(G)$-module, it has K' as its unique minimal submodule.

As $K \leqslant \mathrm{F}(G)$ and $K' = X \leqslant \mathrm{Z}(\mathrm{F}(G))$, we deduce that K is nilpotent of class two. Note that $[x^p, y] = [x, y]^p = 1$ for all $x, y \in K$, so that $K/\mathrm{Z}(K)$ is an elementary abelian p-group. Now as G is not abelian-by-finite, $\mathrm{Z}(K)$ cannot have finite index in G and so $K/\mathrm{Z}(K)$ must be infinite. Thus $K/\mathrm{Z}(K)$ is not finitely generated and we have now established that Condition (ii) holds.

Finally let

$$N = \langle\, L \in \mathrm{Comp}(G) \mid \mathrm{Z}(L) = \mathbf{1} \,\rangle.$$

Since distinct components of G commute (Corollary 2.5(i)), we see that N is actually the direct product of these centreless components. In particular N is a direct product of non-abelian simple groups, so $N \cap K = \mathbf{1}$ as K is nilpotent. Lemma 2.1 then forces $N = \mathbf{1}$ and we deduce that every component of G has non-trivial centre. Thus Condition (iii) holds and we have completed the proof of the theorem. □

Recall that a group is called *monolithic* if it possesses a unique minimal normal subgroup and this minimal normal subgroup is referred to as its *monolith*. In the course of the above proof we have observed that our JNAF-group G described in Theorem 3.2 is monolithic with K' as its monolith.

Corollary 3.3 *Let G be a nilpotent-by-finite JNAF-group such that the centre of its Fitting subgroup possesses torsion. Then G is a torsion group.*

Proof: The group G satisfies the conditions of Theorem 3.2. We shall maintain the notation of the last half of the proof of the theorem.

If a is an element of infinite order in the centre of the normal subgroup K then a has finitely many conjugates in G. Hence the normal closure A of the subgroup generated by a is a finitely generated abelian group. Therefore A is the direct product of a finite abelian group and a free abelian group of finite rank. There then exists a positive integer such that A^m is a non-trivial free abelian group and a normal subgroup of G. Then $K' \cap A^m = \mathbf{1}$, which is impossible by Lemma 2.1.

We deduce that $\mathrm{Z}(K)$ is a torsion group. Since G is a finite extension of K and $K/\mathrm{Z}(K)$ is an elementary abelian p-group, we conclude that G is a torsion group. $\square$

We shall complete our discussion of the torsion case by presenting three examples that illustrate the phenomena that can occur within the class of JNAF-groups which we have just classified.

Example 3.4 Let p be a prime and H be the group given by the presentation with generators

$$z, x_1, x_2, \dots, y_1, y_2, \dots$$

subject to the relations

$$\begin{aligned} z^p = x_i^p = y_i^p = 1 \quad \text{and} \quad [x_i, y_i] = z & \qquad \text{for } i \in \mathbb{N}, \\ [x_i, x_j] = [x_i, y_j] = [y_i, y_j] = 1 & \qquad \text{for } i \neq j, \\ [x_i, z] = [y_i, z] = 1 & \qquad \text{for } i \in \mathbb{N}. \end{aligned}$$

Then H is a nilpotent group of class two whose centre is cyclic of order p generated by z. In particular H is an example of a torsion nilpotent JNAF-group since it clearly satisfies Conditions (i)–(iii) of Theorem 3.2.

Furthermore suppose that P is a finite p-group of class $c \geqslant 2$ whose centre is cyclic generated by an element a of order p^k, say, and that Q is a Prüfer quasicyclic p-group with $b \in Q$ an element of order p^k. Define G to be the quotient of the direct product $H \times P \times Q$ by the central subgroup generated by $z^{-1}a^{p^{k-1}}$ and $a^{-1}b$. We identify H, P and Q with their images in G and so assume that $z = a^{p^{k-1}}$ and $a = b$. Then G is a nilpotent group of class c whose centre equals Q. This group G again possesses no components and the subgroup $K = HQ$ satisfies the requirements of Condition (ii) of Theorem 3.2. Hence G is a torsion nilpotent JNAF-group. This illustrates that the nilpotency class of the Fitting subgroup of a torsion nilpotent-by-finite JNAF-group can be arbitrarily large and also that the exponent of the centre of the Fitting subgroup need not be finite.

Example 3.5 Let S be a non-abelian finite simple group whose Schur multiplier is non-trivial. Fix a prime p dividing the order of this Schur multiplier and let L be a central extension of a cyclic group of order p by S. (Thus L is the quotient of the universal central extension — see, for example, [1, Section 33] — of S by a sufficiently large subgroup of its centre.) Let H be the same group as defined in Example 3.4 (with respect to the prime p we have just fixed). Now define G to be the quotient of $H \times L$ by a diagonal subgroup of its centre. We identify H and L with their images in G, so that z generates the common centre of H and L. Our group G has L as its only component and, of course, $\mathrm{Z}(L) \neq \mathbf{1}$. Thus G satisfies the conditions of Theorem 3.2 (with $K = H$) and is a non-nilpotent example of a torsion nilpotent-by-finite JNAF-group. Note that the centraliser of the Fitting subgroup equals the component L. In particular we have an example where $\mathrm{C}_G(\mathrm{F}(G)) \nleqslant \mathrm{F}(G)$. (Of course, in this example, $\mathrm{E}(G) = L$ and $\mathrm{F}^*(G) = G$.)

Given a particular non-abelian finite simple group, there are, of course, considerable restrictions on the possible choices for the prime p. The data in the Atlas of Finite Groups [2] documents such restrictions.

Finally we give an example to illustrate that the centre of our Fitting subgroup need not be cyclic or quasicyclic.

Example 3.6 Let L be a finite group and, for some prime p, let V be an irreducible $\mathbb{F}_pL$-module on which L acts faithfully. Let k be the dimension of V over $\mathbb{F}_p$. Let H be the p-group constructed in Example 3.4 and let $D = H^{(k)}$ be the direct product of k copies of H. Denote by z_j, x_{ij}, y_{ij} $(1 \leqslant j \leqslant k)$ the copies of z, x_i, y_i in the jth direct factor of D. Note that $\langle z_1, z_2, \ldots, z_k\rangle$, $\langle x_{i1}, x_{i2}, \ldots, x_{ik}\rangle$ and $\langle y_{i1}, y_{i2}, \ldots, y_{ik}\rangle$ (for $i \in \mathbb{N}$) are elementary abelian p-groups and so can be viewed as vector spaces of dimension k over $\mathbb{F}_p$. Since these are isomorphic to V as vector spaces, we can define an action of L on each space which preserves these isomorphisms with V and also preserves the relations inherited from H. Thus we have an action of A on D and so we can form the semidirect product $G = D \rtimes L$ via this action. Note that $Z = \langle z_1, z_2, \ldots, z_k\rangle$ is a normal subgroup of G. Moreover, since L acts irreducibly on Z, we deduce that Z is a minimal normal subgroup of G.

Now D is a nilpotent normal subgroup of finite index in G. Hence $F = \mathrm{F}(G)$ is nilpotent and contains D. As $Z \trianglelefteqslant G$, we deduce $Z \cap \mathrm{Z}(F) \neq 1$. Minimality of Z then shows $Z \leqslant \mathrm{Z}(F)$. If $F \neq D$, then F contains some non-identity element $g \in L$. This element g then centralises Z, which contradicts the assumption that L acts faithfully on V. Hence D is the Fitting subgroup of G, $Z = \mathrm{Z}(\mathrm{F}(G))$ and, by assumption, Z is an irreducible $G/\mathrm{F}(G)$-module.

A consequence of this argument is also that $\mathrm{C}_G(D) = Z$ and therefore, by Corollary 2.6(i), G has no components. It now follows that G satisfies the conditions of Theorem 3.2 with $K = D$ in Condition (ii). Here $K' = Z$ is an elementary abelian p-group of rank k.

Of course, we can combine elements of Examples 3.4–3.6 to produce more complicated examples of JNAF-groups.

Torsion-free case

We now turn to the case of a nilpotent-by-finite JNAF-group where the Fitting subgroup is torsion-free.

Theorem 3.7 *Let G be a group and Z be the centre of its Fitting subgroup. Then G is a nilpotent-by-finite JNAF-group such that Z is torsion-free if and only if the following three conditions hold:*

(i) *there is a nilpotent normal subgroup K of class precisely two and of finite index in G such that K' is a free abelian subgroup of Z and such that, for each positive integer m, the centraliser $\mathrm{C}_K(K/(K')^m)$ has finite index in K;*

(ii) *every non-trivial $G/\mathrm{F}(G)$-submodule of Z contains a submodule which is commensurable with K';*

(iii) *G possesses no components.*

Recall that two subgroups H and K of a group G are called *commensurable* if $H \cap K$ has finite index both in H and in K.

Proof: Suppose that G satisfies the three conditions. Condition (i) ensures that G is nilpotent-by-finite. If Z possesses a non-identity element a of finite order, then a has only finitely many images under the induced action of $G/\mathrm{F}(G)$, so it generates a finite $G/\mathrm{F}(G)$-submodule A of Z. Since K' is infinite, no submodule of A could be commensurable with K'. Hence Condition (ii) shows that Z is torsion-free. For the sufficiency part of the proof, it remains to show that G is a JNAF-group.

Suppose that G is abelian-by-finite, so that it possesses an abelian normal subgroup of index n, say. Let $x, y \in K$. Then x^n and y^n belong to the abelian normal subgroup, so $[x, y]^{n^2} = [x^n, y^n] = 1$. Since K' is torsion-free, this forces $[x, y] = 1$ and we deduce that K is abelian. This contradicts K being nilpotent of class precisely two. Hence G is not abelian-by-finite.

Now let N be a non-trivial normal subgroup of G. Then Proposition 2.7 together with Condition (iii) implies $N \cap \mathrm{F}(G) \neq \mathbf{1}$. Standard theory of nilpotent groups yields $N \cap Z \neq \mathbf{1}$ and this is a non-trivial $G/\mathrm{F}(G)$-submodule of Z. Hence $N \cap Z$ contains a submodule commensurable with K' and we deduce that there exists a positive integer m such that $(K')^m \leqslant N$.

Let $C = \mathrm{C}_K(K/(K')^m)$. Then C projects onto an abelian subgroup of $K/(K')^m$ (indeed, it is a central subgroup) and by Condition (i) this is of finite index. We conclude that $G/(K')^m$ is abelian-by-finite and therefore G/N is abelian-by-finite. Hence G is a JNAF-group, as claimed.

Conversely suppose that G is a JNAF-group such that the Fitting subgroup has finite index and such that $Z = \mathrm{Z}(\mathrm{F}(G))$ is torsion-free. We now derive the three conditions of the theorem.

Let x be any non-identity element in Z. Then the Fitting subgroup $\mathrm{F}(G)$ centralises x and so x has finitely many conjugates in G. The subgroup generated by these conjugates is then a non-trivial finitely generated normal subgroup of G contained in Z. Let X be such a non-trivial normal subgroup of G contained in Z such that the number of generators of X is as small as possible. Now, by hypothesis, G/X is abelian-by-finite. Hence there exists a normal subgroup K of finite index in G which contains X, is contained in $\mathrm{F}(G)$ and has K/X abelian. Now $K' \leqslant X$ and since G is not abelian-by-finite we have $K' \neq \mathbf{1}$. The minimal choice for X ensures that K' has the same number of generators as X (and consequently it has finite index in X). We shall replace X by K' for convenience.

Thus we have $K \leqslant \mathrm{F}(G)$ and $K' = X \leqslant \mathrm{Z}(\mathrm{F}(G))$. We deduce that K is nilpotent of class two. We also note that K' is a free abelian group of finite rank.

Now if m is a positive integer, then $(K')^m$ is a non-trivial normal subgroup of G and so the quotient $K/(K')^m$, as a subgroup of $G/(K')^m$, is abelian-by-finite. Choose a normal subgroup L of finite index in K containing $(K')^m$ such that $L/(K')^m$ is abelian. Now $K = \langle L, x_1, x_2, \ldots, x_d \rangle$ for some elements x_1, x_2, …, x_d in K. Write $\bar{K} = K/(K')^m$ and use this bar notation for images of subgroups and elements in this quotient. Now the derived subgroup of $\bar{K}$ is a finitely generated abelian group of exponent m, so is finite. Therefore each $\bar{x}_i$ has finitely many conjugates in $\bar{K}$ and we deduce that $\mathrm{C}_{\bar{K}}(\bar{x}_i)$ has finite index in $\bar{K}$. Let C be the subgroup of K containing $(K')^m$ such that

$$\bar{C} = \bar{L} \cap \left(\bigcap_{i=1}^{d} \mathrm{C}_{\bar{K}}(\bar{x}_i) \right).$$

Then C has finite index in K and, moreover, an element in C commutes modulo $(K')^m$ with L and each x_i. Hence $C \leqslant \mathrm{C}_K(K/(K')^m)$. We have now established all parts of Condition (i).

If M is a non-trivial $G/\mathrm{F}(G)$-submodule of Z, then M is a normal subgroup of G contained in Z. Lemma 2.1 implies $M \cap K' \neq \mathbf{1}$. The minimality of the number of generators of K' implies $M \cap K'$ has the same number of generators and hence has finite index in K'. Thus $M \cap K'$ is a submodule of Z which is commensurable with K', so Condition (ii) holds.

Finally since G is nilpotent-by-finite, it possesses at most finitely many components and they are all finite. Therefore, by Corollary 2.5, $\mathrm{E}(G)$ is finite. Thus $\mathrm{E}(G) \cap K' = \mathbf{1}$ and now Lemma 2.1 gives $\mathrm{E}(G) = \mathbf{1}$. Therefore Condition (iii) holds. This completes the proof of the theorem. □

The derived subgroup K' appearing in the theorem is not a minimal normal subgroup of the JNAF-group G just described. Indeed G is not monolithic. However, as observed, any non-trivial normal subgroup of G must contain $(K')^m$ for some m.

Corollary 3.8 *Let G be a nilpotent-by-finite JNAF-group such that the* *centre of its Fitting subgroup is torsion-free. Then* $\mathrm{F}(G)$ *is a torsion-free nilpotent group of class two and* $\mathrm{C}_G(\mathrm{F}(G)) = \mathrm{Z}(\mathrm{F}(G))$.

PROOF: Our group G satisfies the Conditions of Theorem 3.7. We maintain the notation of the last half of the proof of the theorem. Now the elements of finite order in the nilpotent subgroup $\mathrm{F}(G)$ form a subgroup T which is normal in G. Necessarily $T \cap K' = \mathbf{1}$ and we deduce that $T = \mathbf{1}$. Thus $\mathrm{F}(G)$ is torsion-free. Since K has finite index in $\mathrm{F}(G)$, they must have the same class (see, for example, [12, Exercise 5.2.13]). Finally $\mathrm{C}_G(\mathrm{F}(G)) \leqslant \mathrm{F}(G)$ by Proposition 2.7 and Condition (iii) of Theorem 3.7, and this establishes the final equality. □

In our characterisation of torsion nilpotent-by-finite JNAF-groups, Condition (ii) of Theorem 3.2 tells us that $K/\mathrm{Z}(K)$ must be at least of cardinality $\aleph_0$ for the subgroup K occurring. For the torsion-free case, the analogous condition is that $\mathrm{C}_K(K/(K')^m)$ has finite index in K for all $m \in \mathbb{N}$. This condition essentially corresponds to $K/\mathrm{Z}(K)$ not being too large.

Corollary 3.9 *Let G be a nilpotent-by-finite JNAF-group such that the centre of the Fitting subgroup is torsion-free and let K be the nilpotent normal subgroup of finite index in G provided by Condition (i) of Theorem 3.7. Then*

$$|K/\mathrm{Z}(K)| \leqslant 2^{\aleph_0}.$$

PROOF: For each $m \in \mathbb{N}$, let $C_m = \mathrm{C}_K(K/(K')^m)$. Define

$$\phi\colon K \to \prod_{m=1}^{\infty} K/C_m$$

to be the natural map. Then $\ker\phi = \bigcap_{m=1}^{\infty} C_m = \mathrm{Z}(K)$ since K' is a free abelian group of finite rank. Hence $K/\mathrm{Z}(K)$ embeds in the Cartesian product of a countable number of finite groups. □

We shall see in Example 3.13 that it is permissible for $K/\mathrm{Z}(K)$ to be uncountable. On the other hand, in the case that $K/\mathrm{Z}(K)$ is finitely generated (so countable) the condition that $\mathrm{C}_K(K/(K')^m)$ has finite index in K comes for free.

Lemma 3.10 *Let K be a nilpotent group of class two such that $K/\mathrm{Z}(K)$ is finitely generated. Then $\mathrm{C}_K(K/(K')^m)$ has finite index in K for all $m \in \mathbb{N}$.*

PROOF: Let $x_1, x_2, \dots, x_d \in K$ be representatives for the generators of the quotient $K/\mathrm{Z}(K)$. Then for $i = 1, 2, \dots, d$, we obtain a homomorphism $\phi_i\colon K \to K'$ given by $g \mapsto [g, x_i]$. The hypotheses that K is nilpotent of class two and $K/\mathrm{Z}(K)$ is finitely generated ensure that K' is finitely generated.

Thus $(K')^m$ has finite index in K' and L_i, its inverse image under ϕ_i, then also has finite index in K. Let $C = \bigcap_{i=1}^{d} L_i$, a subgroup of finite index in K. Then $[C, x_i] \leqslant (K')^m$ for $i = 1, 2, \ldots, d$, and we deduce that C centralises K modulo $(K')^m$ (since it certainly centralises $\mathrm{Z}(K)$). Hence $\mathrm{C}_K(K/(K')^m)$ has finite index in m. □

Finally we present some examples to illustrate the possible behaviours of nilpotent-by-finite JNAF-groups in the torsion-free case.

Example 3.11 Let H be the Heisenberg group, that is, the group with presentation

$$H = \langle\, x, y, z \mid [x, y] = z,\ [x, z] = [y, z] = 1 \,\rangle.$$

Then H is itself a nilpotent group satisfying the conditions of Theorem 3.7, so is a JNAF-group. If Q denotes a multiplicatively-written abelian group isomorphic to the additive group of the rational numbers and if $e \in Q$ is taken to $1 \in \mathbb{Q}$ under this isomorphism, then let G be the quotient of $H \times Q$ by the central subgroup generated by $z^{-1}e$. We identify H and Q with their images in G. Then $\mathrm{Z}(G) = Q$, $\mathrm{F}(G) = G$ and $G' = \langle z \rangle = \langle e \rangle$. Condition (ii) in Theorem 3.7 requires us then to examine a subgroup of Q and show it contains a subgroup of finite index in $\langle e \rangle$. Since every subgroup of $\mathbb{Q}$ contains a positive integer and hence contains a subgroup of finite index in $\mathbb{Z}$, this condition holds and it follows that G is a JNAF-group with $K = G$ as the required nilpotent normal subgroup for Condition (i).

In the previous example, the centre of the Fitting subgroup did not coincide with the derived subgroup of the nilpotent normal subgroup K. If it were the case, however, that $K' = Z = \mathrm{Z}(\mathrm{F}(G))$, then the condition that every $G/\mathrm{F}(G)$-submodule of Z contains a submodule commensurable with K' (Condition (ii)) becomes more transparent. If $K' = Z$, then this condition is simply that K' is a *just infinite* $G/\mathrm{F}(G)$*-module*; that is, every proper quotient module of K' is finite. Such modules occur in our next example.

Example 3.12 Let L be a finite group. Since a finitely generated L-module is also finitely generated as an abelian group, a minor variation on the standard argument for just infinite groups (see, for example, [4, Proposition 3]) shows that every infinite finitely generated L-module possesses a just infinite quotient module. In particular, there does exist a finitely generated L-module V which is just infinite. We shall assume that L acts faithfully on V. As an abelian group, V is torsion-free since the torsion subgroup is of infinite index and is an L-submodule. Suppose that, as an abelian group, V is isomorphic to a free abelian group of rank k.

Let K be the direct product of k copies of the Heisenberg group H (from Example 3.11). Let x_i, y_i, z_i denote, respectively, the copies of the generators x, y, z in the ith direct factor of K. Then each of $\langle x_1, x_2, \dots, x_k\rangle$, $\langle y_1, y_2, \dots, y_k\rangle$ and $\langle z_1, z_2, \dots, z_k\rangle$ are free abelian groups of rank k. We can then exploit their isomorphism with V and define an action of L on each of these abelian groups so that each is isomorphic as an L-module to V and so that L preserves the relations on K inherited from H. Thus we have an action of L on K and we can form the semidirect product $G = K \rtimes L$.

The same argument as in Example 3.6 shows that $\mathrm{F}(G) = K$ and $\mathrm{C}_G(K) = Z$. Therefore G has no components. Since K is finitely generated, Lemma 3.10 shows $\mathrm{C}_K(K/(K')^m)$ has finite index in K for all $m \in \mathbb{N}$. Finally as V is a just infinite L-module, any $G/\mathrm{F}(G)$-submodule of K' $(= Z)$ has finite index. Hence G satisfies the conditions of Theorem 3.2 and so is a JNAF-group.

Example 3.13 Let X be the direct produc of countably many infinite cyclic groups $\langle x_i\rangle$ (for $i \in \mathbb{N}$) and Y be the Cartesian product of countably many infinite cyclic groups $\langle y_i\rangle$ (for $i \in \mathbb{N}$). Let $\langle z\rangle$ be a further infinite cyclic group. Let p_1, p_2, … be an enumeration of the prime numbers and define

$$m_{i+1} = p_1^i p_2^i \dots p_i^i \qquad \text{for } i = 0,\ 1,\ 2,\ \dots.$$

Now define G to be the group generated by the abelian groups X and Y and the element z subject to $z \in Z(G)$ and

$$[x_1^{r_1} x_2^{r_2} \dots, y_1^{s_1} y_2^{s_2} \dots] = z^t,$$

where $t = \sum_{i=0}^{\infty} m_i r_i s_i$, for all appropriate sequences (r_i) and (s_i). Note that the value of t, and hence the above commutator, are well-defined since all but finitely many of the r_i are zero. This has the consequence that $[x_i, y_i] = z^{m_i}$ for all $i \in \mathbb{N}$. Then G is a nilpotent group of class precisely two whose centre Z is cyclic generated by z. Note $G' = Z$ and that certainly G' is just infinite.

If m is any positive integer, write

$$m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$$

for some non-negative integers k, α_1, α_2, …, α_k. We then note that m divides m_i for all $i > r$, where $r = \max\{k, \alpha_1, \alpha_2, \dots, \alpha_k\}$. Consequently the centraliser $\mathrm{C}_G(G/(G')^m)$ contains all products

$$x_1^{m\beta_1} x_2^{m\beta_2} \cdots x_r^{m\beta_r} x_{r+1}^{\beta_{r+1}} \cdots x_s^{\beta_s} \qquad (\text{for } \beta_i \in \mathbb{Z} \text{ and } s \geqslant r)$$

and all products

$$y_1^{m\beta_1} y_2^{m\beta_2} \cdots y_r^{m\beta_r} y_{r+1}^{\beta_{r+1}} y_{r+2}^{\beta_{r+2}} \cdots \qquad (\text{for } \beta_i \in \mathbb{Z}).$$

Therefore $|G : \mathrm{C}_G(G/(G')^m)| \leqslant m^{2r}$. This shows that G satisfies the conditions of Theorem 3.7 and so is a nilpotent JNAF-group.

Note that $G/\mathrm{Z}(G)$ is uncountable and so the bound given in Corollary 3.9 is best possible.

4 Groups which are both JNAF and JNNF

In our final section we consider JNAF-groups which are not nilpotent-by-finite. These are therefore JNNF-groups all of whose proper quotients happen to be abelian-by-finite. Theorem 2.2 of [3] then tells us that the Fitting subgroup of such a JNAF-group is either torsion-free or is an elementary abelian p-group for some prime p. We shall characterise in Theorem 4.2 a JNAF-group which is not nilpotent-by-finite and which has non-trivial Fitting subgroup of finite Prüfer rank. Secondly we characterise a JNAF-group which is not nilpotent-by-finite, which is monolithic and where the Fitting subgroup is a non-trivial elementary abelian p-group. These correspond to Theorems 2.6 and 2.13 of [3]. (We choose not to state an analogue of [3, Theorem 2.8], though this could easily be done, since the groups described there also have Fitting subgroup of finite Prüfer rank.) These restrictions are also consistent both with previous work on just non-$\mathscr{P}$ groups (for various properties $\mathscr{P}$) and, to some extent, with Theorems 3.2 and 3.7 above. In Theorem 3.2, the JNAF-groups considered are monolithic and the monolith has finite rank, while in Theorem 3.7 the JNAF-group has a finite rank abelian normal subgroup, broadly corresponding to the assumptions we now make.

To perform our analysis we exploit [3, Section 2] together with the following lemma. This shows that many of the JNNF-groups occurring in the theorems of [3] are actually JNAF-groups.

Lemma 4.1 *Let G be a group possessing abelian subgroups A and B such that*

(i) *A is normal in G and is faithful and just infinite as a G/A-module;*

(ii) *$A \cap B = \mathbf{1}$;*

(iii) *the semidirect product $A \rtimes B$ has finite index in G.*

Then every proper quotient of G is abelian-by-finite.

PROOF: By assumption, $C_G(A) = A$. Therefore if N is a non-trivial normal subgroup of G, then $A \cap N \neq \mathbf{1}$. Now $A \cap N$ is a G/A-submodule of A, so $A \cap N$ has finite index in A. Now consider the induced action of B on the finite quotient $A/(A \cap N)$. The kernel of this action, $C = C_B(A/(A \cap N))$, has finite index in B.

Write $\bar{G} = G/(A \cap N)$ and use this bar notation for images of subgroups of G in the quotient $\bar{G}$. Here $\bar{A}\bar{C} = \bar{A} \times \bar{C}$ is an abelian subgroup of finite index in $\bar{A} \rtimes \bar{B}$. Hence $\bar{G}$ is abelian-by-finite and we deduce G/N is abelian-by-finite. □

Since the Fitting subgroup A of a JNNF-group is either torsion-free or of exponent p, and if furthermore $A \neq \mathbf{1}$, then (also by Theorem 2.2 of [3]) A is self-centralising. Certainly A cannot then be finite and so if A is of finite Prüfer rank it cannot be of finite exponent. Therefore the following theorem describes all JNAF-groups which are not nilpotent-by-finite and where the Fitting subgroup has finite Prüfer rank.

Theorem 4.2 *Let G be a group whose Fitting subgroup has finite Prüfer rank. Then G is a JNAF-group which is not nilpotent-by-finite if and only if it contains non-trivial torsion-free abelian subgroups A and X satisfying the following conditions:*

(i) *A is normal in G and is faithful and just infinite as a G/A-module;*

(ii) $A \cap X = \mathbf{1}$;

(iii) *the semidirect product $A \rtimes X$ has finite index in G.*

PROOF: This follows immediately from the comment before the statement, Theorem 2.6 of [3] and Lemma 4.1 above. □

Finally for the case when the Fitting subgroup has exponent p, we describe monolithic JNAF-groups which are not nilpotent-by-finite and where the Fitting subgroup is non-trivial of exponent p.

Theorem 4.3 *Let G be a monolithic group with monolith M. Then G is a JNAF-group with non-trivial Fitting subgroup of exponent p and infinite index if and only if there exists a subgroup X of G such that G is the semidirect product $M \rtimes X$ and*

(i) *M is an elementary abelian p-group with $\mathrm{C}_G(M) = M$;*

(ii) *X is an infinite abelian-by-finite group.*

PROOF: This follows from [3, Theorem 2.13] together with Lemma 4.1 and the observation that if $M \neq \mathbf{1}$ and $G = M \rtimes X$ is a JNAF-group then X must be abelian-by-finite. □

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