Maximal complements in finite groups

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Abstract

Let G be a finite group with a non-abelian minimal normal subgroup N which is a direct product of the simple group X. The maximal subgroups of G which complement N and their conjugacy classes are parametrised in terms of certain homomorphisms taking values in $\operatorname{Aut} X$ and satisfying particular conditions.

1 Introduction

In [8], Parker and the author classified maximal subgroups of a wreath product which complement its base group. This was subsequently applied to establish that certain iterated wreath products of non-abelian finite simple groups can be generated by two randomly chosen elements with high probability (see [9] for a precise statement). It is the author's hope that such probabilistic results can be extended to far wider classes of groups than just iterated wreath products. Accordingly we seek to generalise the main theorem of [8] so as to parametrise the complements of a normal subgroup in an arbitrary finite group.

Before we summarise the methods and results of the paper, let us set the scene for the situation we shall examine. Let G be a group and let N be a non-trivial normal subgroup of G. We are interested in understanding the possibilities for subgroups which both complement N and which are maximal in G. The following observation is straightforward.

Lemma 1.1 *Let G be a group with a non-trivial normal subgroup N. If G has a maximal subgroup which is a complement to N, then N is a minimal normal subgroup of G and so is characteristically simple.* $\square$

A characteristically simple subgroup of a finite group is necessarily a direct product of isomorphic simple groups. We therefore have a dichotomy:

either N is an elementary abelian p-group or it is a direct product of isomorphic non-abelian simple groups. In the former case, we obtain a representation of G/N over the field of p elements via the conjugation action on N. If a complement to N were to exist then it acts on N in the same way as G/N and we deduce that if one complement is maximal, then this representation is irreducible and hence *all* complements are maximal. Thus in this case one is required to determine whether the representation of G/N on N is irreducible and to determine all complements to N in G. The former takes us into the realm of representation theory while the latter leads to extensions with abelian kernels. We refer to Chapter 11 of Robinson [10] for a potential starting point for this type of extension theory.

In this paper we concentrate on the other possibility for N, namely the case when N is a direct product of isomorphic non-abelian simple groups. These direct factors are the minimal normal subgroups of N and they are permuted transitively under conjugation by elements of G.

In Section 2 below we discuss complements to a minimal normal subgroup N in G of this form. Standard arguments from the world of extension theory show that such complements correspond to homomorphisms $\zeta\colon G/N \to \operatorname{Aut} N$ satisfying a particular 'compatibility' condition. Since N is a direct product of isomorphic copies of some non-abelian finite simple group X, the automorphism group $\operatorname{Aut} N$ is isomorphic to a wreath product:

$$\operatorname{Aut} N \cong (\operatorname{Aut} X) \operatorname{wr}_\Omega \operatorname{Sym}(\Omega),$$

where the direct factors of N are indexed by the set Ω. We are therefore considering homomorphisms into a wreath product and accordingly the techniques of [8], in particular those in its second section, come into play. The principal role of our Section 2 below is to adapt these techniques to this more general situation and, as we do so, to set up the necessary notation.

We will have cause to consider a number of wreath products, all of which will have $\operatorname{Sym}(\Omega)$ as the top group. Let us recall the definition of a wreath product and also set up a few abbreviations to simplify our notation later on. Fix the set Ω and let L be any group. Throughout we shall write L^Ω for the direct product of copies of L indexed by Ω:

$$L^\Omega = \prod_{\alpha \in \Omega} L_\alpha.$$

The symmetric group $\operatorname{Sym}(\Omega)$ acts on L^Ω by permuting the direct factors:

$$L_\alpha^\pi = L_{\alpha\pi} \qquad \text{for all } \alpha \in \Omega \text{ and } \pi \in \operatorname{Sym}(\Omega).$$

Then the *wreath product* $L \operatorname{wr}_\Omega \operatorname{Sym}(\Omega)$ is the semidirect product of L^Ω by $\operatorname{Sym}(\Omega)$ via this action. We call L^Ω the *base group* of the wreath product and we shall also always use $W(L)$ to denote the wreath product

$L \operatorname{wr}_{\Omega} \operatorname{Sym}(\Omega)$. We shall also follow the convention from [8] of writing (x_α) for the element $(x_\alpha)_{\alpha \in \Omega}$ of L^{Ω}. We shall always index the entries of such elements by the variable α which will range over the set Ω. Given an element $w = (x_\alpha)\pi$ in $W(L)$, we refer to x_α as being the α-component of w.

We shall define $Q = G/N$ and give a method to induce a homomorphism $\Phi\colon Q \to \operatorname{Aut} N$ from $\phi\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$ (where $\operatorname{Stab}_Q(\omega)$ denotes the stabiliser of a point ω under an action of Q on Ω which will be determined by the group G). Theorem 2.8 and its corollary describes a parametrisation of the complements to N in G in terms of such homomorphisms $\phi\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$ satisfying another compatibility condition and here the complements are constructed via the induced maps. We then note (in Proposition 2.10) that the complements corresponding to homomorphisms ϕ_1 and ϕ_2 are conjugate in G if and only if $\phi_1 = \phi_2\sigma$ for some inner automorphism σ of $\operatorname{Inn} X$.

In Section 3 we discuss conditions for when the complement corresponding to an appropriate homomorphism $\phi\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$ is a maximal subgroup of our group G. Our main theorem is Theorem 3.4 where we observe that the complement is maximal in G if and only if the (compatible) homomorphism ϕ satisfies the following two conditions:

(a) the image of ϕ contains $\operatorname{Inn} X$; and

(b) ϕ is not the restriction of some homomorphism $\hat{\phi}\colon J \to \operatorname{Aut} X$ where $\operatorname{Stab}_Q(\omega) < J \leqslant Q$.

These are the natural generalisations of the conditions (c) and (d) that appear in [8, Theorem A]. We can then bring our condition for two complements to be conjugate to bear on this situation and obtain a parametrisation for the conjugacy classes of complements to N which are maximal subgroups in terms of equivalence classes of certain types of homomorphism.

To draw this introductory section to a close, we note (as we did in [8]) that studying maximal subgroups of a finite group has a considerable history. Two of the most significant examples of this are the work of Aschbacher and Scott [1] and that of Kovács [6]. The latter is probably closest to the study presented here and the third collection of maximal subgroups in [6, Theorem 4.3] are those we are considering in this paper. Kovács's techniques, however, appear very much to be from extension theory and he relies on two technical conditions for subgroups, which he calls 'high' and 'full,' respectively, and which are discussed at length in [5].

On the other hand, in [8] our arguments are more group-theoretical and revolve around whether it is possible to normalise certain subgroups of the base group of the wreath product that was considered there. Equally the goal with the work presented below has been to maintain this flavour as much as possible. Indeed, one of the points we address in the final section of this paper is the link between the two conditions (a) and (b) that appear in Theorem 3.4 below and subgroups of the normal subgroup N which are normalised by the constructed complement. In the end, one of the goals of this paper is to demonstrate that the methods of [8], which on the face of it appear to concern the specific case of a wreath product, can actually be extended to apply to complements in an arbitrary finite group (provided the minimal normal subgroup is non-abelian).

Around the time that the first draft of this paper was being completed, work by Cannon and Holt [4] was published. They returned to Kovács's work and described how to implement a practical algorithm for computing representatives for the conjugacy classes of maximal subgroups in a finite group. Our principal construction (see Definition 2.3 below) also appears in [4, Section 2.2]. However, we are using it to consider complements whereas Cannon and Holt use it to construct what they call 'diagonal-type subgroups' which are then used in the construction of the first collection of maximal subgroups from [6, Theorem 4.3].

The maximal subgroups that complement the normal subgroup are instead discussed in [4, Section 3.6]. The methods described there appear to be different from those given below in the current work. The induced homomorphism does not appear and, although condition (b) from our Theorem 3.4 is present, an alternative condition appears instead of our condition (a), namely that the complement corresponds to a maximal complement in a quotient. This latter condition is not used in our work below.

There are also links between our work here and that of Baddeley in his fundamental paper [2]. Lafuente [7] observes that if N is a minimal normal subgroup of a group G, N is the direct product of copies of the non-abelian simple group X, and there is a complement H to N in G, then G is isomorphic to a twisted wreath product $X \operatorname{twr} H$ (see also Bercov [3]). Section 3 of [2] discusses conditions for the top group to be a maximal subgroup of the twisted wreath product and perhaps unsurprisingly our conditions (a) and (b) can also be found in the discussion there. In view of this, it is interesting to consider what the link is between our work and twisted wreath products and this is discussed in Section 4 below.

In both Kovács's and Baddeley's work, the possibility of parametrising the complements which are maximal seems to be implicit. The goal of the work presented here is to make such a parametrisation explicit and concrete noting both the need for the conditions (a) and (b) and also the compatibility condition required for a complement to exist. In doing so we bring the work begun in [8] to what appears to be its logical conclusion.

Finally we note that the proof of Lemma 3.2 (and hence the proof of the necessarity of condition (a) in Theorem 3.4) depends on the Classification of Finite Simple Groups. We did not need this in [8] since the corresponding homomorphism ϕ took values in the simple group X, whereas here we have the possibility that the image of $\phi\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$ is non-trivial but has trivial intersection with $\operatorname{Inn} X$. The corresponding point in Baddeley's work [2, Lemma 3.2] relies on the Schreier Conjecture and hence also on the Classification.

2 Complements and induced homomorphisms

In this section we consider basic properties of complements to a normal subgroup in a finite group. In particular, we shall see that if N is a normal subgroup of a finite group G and N is a direct product of k copies of a non-abelian finite simple group X then a complement to N corresponds to a certain type of homomorphism from $Q = G/N$ to the automorphism group of N. In this situation, this automorphism group is isomorphic to the wreath product $W(\operatorname{Aut} X) = (\operatorname{Aut} X) \operatorname{wr}_\Omega \operatorname{Sym}(\Omega)$ where $|\Omega| = k$. We then demonstrate how such homomorphisms may be induced from homomorphisms $\phi\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$ and obtain a parametrisation of the complements to N in terms of certain such homomorphisms. The notation we develop in this section will be maintained throughout the paper.

So let N be a normal subgroup of the finite group G and let $Q = G/N$. An element of Q is then a right coset Ng of N and we may define a map $\chi\colon Q \to \operatorname{Out} N$ by $Ng \mapsto (\operatorname{Inn} N)\sigma_g$, where σ_g denotes the automorphism of N induced by conjugation by the element g. This is a homomorphism and in the study of extension theory is known as a *coupling*. If x is an element of Q, then the image of x under χ is a coset of $\operatorname{Inn} N$ so it makes sense to define

$$E = \{\, (x, \sigma) \in Q \times \operatorname{Aut} N \mid \sigma \in x\chi \,\}.$$

There is a homomorphism from G to $Q \times \operatorname{Aut} N$ given by $g \mapsto (Ng, \sigma_g)$. The image of this homomorphism is E while the kernel is the centre $\mathrm{Z}(N)$ of N. Hence if this centre is trivial, this homomorphism is actually an isomorphism from G to E.

Henceforth we shall assume that $\mathrm{Z}(N) = \mathbf{1}$. Under the above isomorphism, the normal subgroup N then corresponds to the normal subgroup $\mathbf{1} \times \operatorname{Inn} N$ of E. A complement H to N in G corresponds to a subgroup $\tilde{H}$ of E such that for each $x \in Q$ there exists a unique automorphism σ of N with $(x, \sigma) \in \tilde{H}$. (For if both (x, σ_1) and (x, σ_2) lie in $\tilde{H}$, then $\tilde{H}$ contains the element $(1, \sigma_1^{-1}\sigma_2)$ and here $\sigma_1^{-1}\sigma_2 \in 1\chi = \operatorname{Inn} N$. Since H is a complement to N this forces $\sigma_1 = \sigma_2$.) It follows that H determines a homomorphism $\zeta\colon Q \to \operatorname{Aut} N$ via $(x, x\zeta) \in \tilde{H}$ for each $x \in Q$.

This homomorphism has a further property which we shall express in terms of the following notation.

Definition 2.1 If $\eta\colon Q \to \operatorname{Aut} Y$ is a homomorphism taking values in the automorphism group of some group Y, let $\operatorname{Out} Y$ denote the outer automorphism group of Y and write $\bar{\eta}\colon Q \to \operatorname{Out} Y$ for the homomorphism obtained by composing η with the natural map $\operatorname{Aut} Y \to \operatorname{Out} Y$.

Using this notation, we note that if H is a complement to N then the homomorphism $\zeta\colon Q \to \operatorname{Aut} N$ determined by $\tilde{H}$ must satisfy $\bar{\zeta} = \chi$ (since $\tilde{H}$ is contained in E). Conversely, given a homomorphism $\zeta\colon Q \to \operatorname{Aut} N$ such that $\bar{\zeta} = \chi$, we determine a complement $\{\, (x, x\zeta) \mid x \in Q \,\}$ to $\mathbf{1} \times \operatorname{Inn} N$ in E and hence a complement to N in G. Thus, complements to N in G are in one-one correspondence with homomorphisms $\zeta\colon Q \to \operatorname{Aut} N$ with the property that $\bar{\zeta} = \chi$. We refer to this condition $\bar{\zeta} = \chi$ as the *compatibility condition* for complements.

We shall now further restrict to the situation we are most interested in, namely that where N is a direct product of k copies of a non-abelian finite simple group X, say $N = X^{\Omega}$ where $|\Omega| = k$. Then

$$\operatorname{Aut} N \cong W(\operatorname{Aut} X) = (\operatorname{Aut} X) \operatorname{wr}_{\Omega} \operatorname{Sym}(\Omega)$$

and

$$\operatorname{Out} N \cong W(\operatorname{Out} X) = (\operatorname{Out} X) \operatorname{wr}_{\Omega} \operatorname{Sym}(\Omega).$$

The notation of Definition 2.1 will then take a homomorphism $\eta\colon Q \to W(\operatorname{Aut} X)$ and produce $\bar{\eta}\colon Q \to W(\operatorname{Out} X)$ (here we are taking $Y = N$ in the definition).

In this context, homomorphisms taking values in $\operatorname{Aut} N$ will also give rise to permutation representations on Ω. To exploit this, we shall make a similar notational definition to the previous one.

Definition 2.2 If $\eta\colon Q \to W(L)$ is a homomorphism taking values in the wreath product $W(L)$ where L is any group, write $\eta^{*}\colon Q \to \operatorname{Sym}(\Omega)$ for the homomorphism obtained by composing η with the natural map $W(L) \to \operatorname{Sym}(\Omega)$.

Comparing Definitions 2.1 and 2.2, we note that if $\eta\colon Q \to W(\operatorname{Aut} X)$ is a homomorphism, then $\eta^{*} = (\bar{\eta})^{*}$.

The coupling χ associated to the extension G of N by Q is a homomorphism $\chi\colon Q \to W(\operatorname{Out} X)$ from which we obtain the permutation representation $\chi^{*}\colon Q \to \operatorname{Sym}(\Omega)$ of Q on the set Ω. Since this permutation representation is determined by G and is fixed throughout our discussion, we shall write $\rho = \chi^{*}$ to have a permanent record of this map. Of course, this gives us an action of Q on Ω and we shall simplify notation by writing αx for the image of α (from Ω) under the action of the permutation $x\rho$.

The action of G on N induces an action of G on the set Ω corresponding to the permutation of the direct factors of N. It is straightforward to check that the corresponding permutation representation $G \to \operatorname{Sym}(\Omega)$ equals the composition of the natural map $G \to Q$ with ρ.

We are interested in the case when G permutes the direct factors of N transitively. Consequently let us assume the action of Q on Ω determined by ρ is transitive. Fix ω in Ω and write $\mathrm{Stab}_Q(\omega)$ for the stabiliser of ω in Q. Let $T = \{\, t_\alpha \mid \alpha \in \Omega \,\}$ be a transversal to $\mathrm{Stab}_Q(\omega)$ in Q such that $\omega t_\alpha = \alpha$ for all $\alpha \in \Omega$. We shall assume that $t_\omega = 1$. For $x \in Q$, define elements $h_{\alpha,x}$ in $\mathrm{Stab}_Q(\omega)$ by

$$t_\alpha x = h_{\alpha,x} t_{\alpha x}.$$

Definition 2.3 Let L be any group and let $\phi\colon \mathrm{Stab}_Q(\omega) \to L$ be a homomorphism. Define a map $\Phi\colon Q \to W(L)$ by

$$x\Phi = (h_{\alpha,x}\phi)(x\rho).$$

We shall say that this Φ is *induced* from ϕ.

Recall here that we are using an abbreviated notation for elements in the wreath product $W(L)$. Thus the α-component of the element $x\Phi$ is the image of $h_{\alpha,x}$ under the homomorphism ϕ. The terminology 'induced' is appropriate as there are similarities to induced representations and similar constructions. (Cannon and Holt term the similar construction appearing in [4, Definition 2.2] the *wreathed monomorphism*.)

Lemma 2.4 *The induced map Φ is a homomorphism.*

PROOF: If $x, y \in Q$, then $t_\alpha xy = h_{\alpha,x} t_{\alpha x} y = h_{\alpha,x} h_{\alpha x,y} t_{\alpha xy}$, so $h_{\alpha,xy} = h_{\alpha,x} h_{\alpha x,y}$. Thus

$$\begin{aligned}(xy)\Phi &= (h_{\alpha,x}\phi)(h_{\alpha x,y}\phi)(xy)\rho \\ &= (h_{\alpha,x}\phi)(x\rho) \cdot (h_{\alpha x,y}\phi)^{x\rho}(y\rho) \\ &= (h_{\alpha,x}\phi)(x\rho) \cdot (h_{\alpha,y}\phi)(y\rho) = (x\Phi)(y\Phi),\end{aligned}$$

as required. □

Therefore if we start with a homomorphism $\phi\colon \mathrm{Stab}_Q(\omega) \to \mathrm{Aut}\,X$ we can construct a homomorphism from Q to $\mathrm{Aut}\,N$. Of course, these induced homomorphisms will not typically determine a complement to N in G. It turns out that we will need to make particular choices for ϕ and then compose the induced map Φ with an inner automorphism of the wreath product to obtain a homomorphism ζ satisfying $\bar{\zeta} = \chi$. Note that a homomorphism satisfying $\bar{\zeta} = \chi$ necessarily satisfies $\zeta^* = \chi^* = \rho$, while an induced homomorphism Φ satisfies $\Phi^* = \rho$ by construction.

Also observe that given $\phi\colon \mathrm{Stab}_Q(\omega) \to \mathrm{Aut}\,X$ we may on the one hand induce $\Phi\colon Q \to W(\mathrm{Aut}\,X)$. On the other hand, we can compose ϕ with the natural projection to produce $\bar{\phi}\colon \mathrm{Stab}_Q(\omega) \to \mathrm{Out}\,X$ and then induce this to a homomorphism $Q \to W(\mathrm{Out}\,X)$. It is easy to see that this latter induced homomorphism coincides with $\bar{\Phi}$, obtained from Φ by the recipe described in Definition 2.1.

Although we shall be mostly considering homomorphisms which are induced with respect to the fixed transversal T, there will be points where we need to consider a different choice of transversal. The following lemma shows that this simply corresponds to conjugation of our image.

Lemma 2.5 *For each $\alpha \in \Omega$, let b_α be an element of $\mathrm{Stab}_Q(\omega)$ and $t'_\alpha = b_\alpha t_\alpha$. Let $\Phi' \colon Q \to W(L)$ be the homomorphism induced from ϕ with respect to the transversal $T' = \{\, t'_\alpha \mid \alpha \in \Omega \,\}$. Then $\Phi' = \Phi\beta$ where β denotes conjugation by the element $(b_\alpha \phi^{-1})$ of L^Ω.*

Proof: We define elements $h'_{\alpha,x}$ in $\mathrm{Stab}_Q(\omega)$ by the formula $t'_\alpha x = h'_{\alpha,x} t'_{\alpha x}$. We calculate

$$t'_\alpha x = b_\alpha t_\alpha x = b_\alpha h_{\alpha,x} t_{\alpha x} = b_\alpha h_{\alpha,x} b_{\alpha x}^{-1} t'_{\alpha x}.$$

Hence $h'_{\alpha,x} = b_\alpha h_{\alpha,x} b_{\alpha x}^{-1}$ for all $\alpha \in \Omega$ and $x \in Q$, so

$$\begin{aligned} x\Phi' &= \big((b_\alpha\phi)(h_{\alpha,x}\phi)(b_{\alpha x}^{-1}\phi) \big)(x\rho) \\ &= (b_\alpha\phi) \cdot (h_{\alpha,x}\phi)(x\rho) \cdot (b_{\alpha x}^{-1}\phi)^{x\rho} = (x\Phi)^{(b_\alpha\phi)^{-1}}. \end{aligned}$$

This establishes the lemma. □

Now consider any homomorphism $\eta \colon Q \to W(L)$ with the property that the map η^* coincides with the permutation representation ρ. If we consider the restriction of η to the stabiliser $\mathrm{Stab}_Q(\omega)$, then we note that its image $\mathrm{Stab}_Q(\omega)\eta$ must normalise the direct factor L_ω of the base group of the wreath product and hence

$$\mathrm{Stab}_Q(\omega)\eta \leqslant L_\omega \times \big(L \,\mathrm{wr}_{\Omega\setminus\{\omega\}} \mathrm{Sym}(\Omega \setminus \{\omega\}) \big).$$

It therefore makes sense to make the following definition.

Definition 2.6 If $\eta \colon Q \to W(L)$ is a homomorphism satisfying $\eta^* = \rho$, write $\theta_\eta \colon \mathrm{Stab}_Q(\omega) \to L$ for the homomorphism obtained by composing the restriction of η to the stabiliser $\mathrm{Stab}_Q(\omega)$ with the projection

$$L_\omega \times \big(L \,\mathrm{wr}_{\Omega\setminus\{\omega\}} \mathrm{Sym}(\Omega \setminus \{\omega\}) \big) \to L$$

onto the direct factor of the base group indexed by ω.

Thus, by definition, $x\theta_\eta$ equals the ω-component of the element $x\eta$ for each x in $\mathrm{Stab}_Q(\omega)$.

In particular, if we apply Definition 2.6 when η is an induced homomorphism $\Phi \colon Q \to W(L)$ as given in Definition 2.3, then θ_Φ is the map

$x \mapsto h_{\omega,x}\phi$ (for $x \in \operatorname{Stab}_Q(\omega)$). However, since we chose $t_\omega = 1$ we see that $h_{\omega,x} = x$ for all $x \in \operatorname{Stab}_Q(\omega)$. Hence $\theta_\Phi = \phi$ and so the recipe of Definition 2.6 simply recovers the homomorphism ϕ used to construct Φ.

We could equally well apply the recipe of Definition 2.6 to the coupling $\chi\colon Q \to W(\operatorname{Out} X)$ to yield $\theta_\chi\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Out} X$. This homomorphism will be important in the context of our compatibility condition $\bar{\zeta} = \chi$, so we will record θ_χ as $\psi\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Out} X$.

Returning to our general homomorphism $\eta\colon Q \to W(L)$ satisfying $\eta^* = \rho$, we have our recipe to construct $\theta = \theta_\eta\colon \operatorname{Stab}_Q(\omega) \to L$ and, via Definition 2.3, we induce $\Theta\colon Q \to W(L)$. For each $\gamma \in \Omega$, let $a_\alpha^{(\gamma)}$ be the α-component of $t_\gamma\eta$, so that $t_\gamma\eta = (a_\alpha^{(\gamma)})(t_\gamma\rho)$. Let $x \in Q$ and consider $h_{\gamma,x}\theta$ for $\gamma \in \Omega$. Writing $x\eta = (x_\alpha)(x\rho)$, we have

$$\begin{aligned} h_{\gamma,x}\eta &= (t_\gamma x t_{\gamma x}^{-1})\eta \\ &= (a_\alpha^{(\gamma)})(t_\gamma\rho) \cdot (x_\alpha)(x\rho) \cdot (t_{\gamma x}^{-1}\rho)(a_\alpha^{(\gamma x)})^{-1} \\ &= \big(a_\alpha^{(\gamma)} x_{\alpha t_\gamma} (a_{\alpha h_{\gamma,x}}^{(\gamma x)})^{-1}\big)(h_{\gamma,x}\rho). \end{aligned}$$

We obtain $h_{\gamma,x}\theta$ by projecting onto the ω-component of the above element, so

$$h_{\gamma,x}\theta = a_\omega^{(\gamma)} x_{\omega t_\gamma} (a_{\omega h_{\gamma,x}}^{(\gamma x)})^{-1} = a_\omega^{(\gamma)} x_\gamma (a_\omega^{(\gamma x)})^{-1}.$$

Therefore

$$\begin{aligned} x\Theta &= \big(a_\omega^{(\alpha)} x_\alpha (a_\omega^{(\alpha x)})^{-1}\big)(x\rho) \\ &= (a_\omega^{(\alpha)}) \cdot (x_\alpha)(x\rho) \cdot (a_\omega^{(\alpha)})^{-1} \end{aligned}$$

for all $x \in Q$ and we deduce

$$x\eta = (x\Theta)^a$$

where $a = (a_\omega^{(\alpha)})_{\alpha\in\Omega}$, an element of the base group L^Ω of $W(L)$. Thus

$$\eta = \Theta\tau_a$$

where τ_a is the inner automorphism of $W(L)$ obtained by conjugating by the element a. Note that the ω-component of a is $a_\omega^{(\omega)}$, which is the identity element since $t_\omega = 1$ according to our original choice.

We shall use this observation repeatedly, so we record it as a lemma.

Lemma 2.7 *Let $\eta\colon Q \to W(L)$ be a homomorphism satisfying $\eta^* = \rho$. Define a_γ to be the ω-component of the element $t_\gamma\eta$. Then*

$$\eta = \Theta_\eta\tau_a$$

where Θ_η *is the homomorphism induced from* $\theta_\eta \colon \mathrm{Stab}_Q(\omega) \to L$ *(the latter as given by Definition 2.6) and* τ_a *is the inner automorphism of the wreath product obtained by conjugating by the element* $a = (a_\alpha)$ *from the base group* L^Ω *of* $W(L)$*.* □

Now suppose that $\zeta \colon Q \to W(\mathrm{Aut}\, X)$ is a homomorphism corresponding to a complement to N in G, which means, of course, that $\bar{\zeta} = \chi$. By Lemma 2.7, there exists a such that

$$\zeta = \Theta\tau_a$$

where Θ is induced from $\theta = \theta_\zeta \colon \mathrm{Stab}_Q(\omega) \to \mathrm{Aut}\, X$. Note that the homomorphism $\bar{\theta} \colon \mathrm{Stab}_Q(\omega) \to \mathrm{Out}\, X$ coincides with $\psi = \theta_\chi$ since $\bar{\zeta} = \chi$.

Conversely, suppose $\phi \colon \mathrm{Stab}_Q(\omega) \to \mathrm{Aut}\, X$ is a homomorphism satisfying $\bar{\phi} = \psi$. Then the induced map $\Phi \colon Q \to W(\mathrm{Aut}\, X)$ satisfies $\bar{\Phi} = \Psi$, where Ψ is the map induced from ψ. Applying Lemma 2.7 to χ produces a in the base group $(\mathrm{Aut}\, X)^\Omega$ of $W(\mathrm{Aut}\, X)$ such that

$$\chi = \Psi\tau_{\bar{a}}$$

where $\bar{a}$ is the image of a in $(\mathrm{Out}\, X)^\Omega$. Therefore $\bar{\zeta} = \chi$ where $\zeta = \Phi\tau_a$ and hence this ζ corresponds to a complement to N in G.

This leads us to the following:

Theorem 2.8 *Let* G *be a finite group with a normal subgroup* N *which is a direct product of* k *copies of a non-abelian finite simple group* X*. Let* $Q = G/N$ *and* $\chi \colon Q \to W(\mathrm{Out}\, X)$ *be the coupling associated to this extension. Define* $\psi = \theta_\chi \colon \mathrm{Stab}_Q(\omega) \to \mathrm{Out}\, X$ *to be as constructed by Definition 2.6.*

(i) *Every complement to* N *in* G *corresponds to a homomorphism* $\zeta \colon Q \to W(\mathrm{Aut}\, X)$ *of the form* $\zeta = \Phi\tau_a$*, where* Φ *is induced from a homomorphism* $\phi \colon \mathrm{Stab}_Q(\omega) \to \mathrm{Aut}\, X$ *satisfying* $\bar{\phi} = \psi$ *and* $a = (a_\alpha)$ *is an element in the base group* $(\mathrm{Aut}\, X)^\Omega$ *of* $W(\mathrm{Aut}\, X)$ *satisfying* $a_\omega = 1$*.*

(ii) *Given a homomorphism* $\phi \colon \mathrm{Stab}_Q(\omega) \to \mathrm{Aut}\, X$ *satisfying* $\bar{\phi} = \psi$*, there exists* $a = (a_\alpha)$ *in* $(\mathrm{Aut}\, X)^\Omega$ *with* $a_\omega = 1$ *such that* $\zeta = \Phi\tau_a$ *satisfies the compatibility condition* $\bar{\zeta} = \chi$ *and therefore corresponds to a complement to* N *in* G*.*

Furthermore, the element a *in both parts can be chosen so that its image in* $\mathrm{Out}\, N$ *(that is, its image in* $(\mathrm{Out}\, X)^\Omega$*) is uniquely determined independently of the complement in (i) and of the homomorphism* ϕ *in (ii).*

Proof: It just remains to check the uniqueness claim at the end of the statement. In our discussion prior to the statement of Lemma 2.7 we considered an arbitrary homomorphism $\eta \colon Q \to W(L)$ and noted that the element a could be determined by the values of $t_\gamma\eta$. When we compare this to how we established part (i) of this theorem, we note first that given a homomorphism $\zeta\colon Q \to W(\operatorname{Aut} X)$ such that $\bar{\zeta} = \chi$, the required a is determined by the $t_\gamma\zeta$ and hence the image of a in $\operatorname{Out} N$ is determined by $\bar{\zeta} = \chi$. Thus this image of a is indeed determined by χ and is independent of the particular complement.

Equally in establishing (ii), it was necessary to pick a such that $\chi = \Psi\tau_{\bar{a}}$ where $\bar{a}$ is the image of a in $\operatorname{Out} N$. Then as noted above, a can then be chosen so that $\bar{a}$ is determined by the values of $t_\gamma\chi$. This image is therefore independent of the homomorphism ϕ. □

We refer to the condition $\bar{\phi} = \psi$ occurring in the theorem as the *compatibility condition* for homomorphisms $\operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$.

Corollary 2.9 *There is a one-one correspondence between complements to N in G and homomorphisms $\phi\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$ satisfying the compatibility condition $\bar{\phi} = \psi$.*

Proof: Part (ii) of Theorem 2.8 provides us with a map $\phi \mapsto \zeta = \Phi\tau_a$ from the set of homomorphisms satisfying the compatibility condition to the set of homomorphisms $\zeta\colon Q \to \operatorname{Aut} N$ that correspond to a complement to N in G. We shall check that this map is a bijection.

We first note that part (i) of Theorem 2.8 tells us that the map is surjective. Secondly, if $\zeta = \Phi\tau_a$ where $a = (a_\alpha)$ satisfies $a_\omega = 1$, then applying the construction of Definition 2.6 yields

$$\theta_\zeta = \theta_\Phi = \phi.$$

This ensures our map $\phi \mapsto \zeta$ is injective. □

Finally for this section, we consider how conjugation relates to our construction.

Proposition 2.10 *Let H_1 and H_2 be complements to N in G (where N is a direct product of k copies of the non-abelian finite simple group X). For $i = 1, 2$, let H_i correspond to the homomorphism $\zeta_i\colon Q \to W(\operatorname{Aut} X)$ where $\zeta_i = \Phi_i\tau_{a_i}$, Φ_i is induced from $\phi_i\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$ and $a_i = (a_\alpha^{(i)})$ is an element of the base group $(\operatorname{Aut} X)^\Omega$ with $a_\omega^{(i)} = 1$. Then H_1 and H_2 are conjugate in G if and only if $\phi_1 = \phi_2\sigma$ for some inner automorphism σ of $\operatorname{Inn} X$.*

Proof: The subgroup H_i corresponds to $\tilde{H}_i = \{\, (x, x\zeta_i) \mid x \in Q \,\}$ and we shall assume that $\tilde{H}_1$ and $\tilde{H}_2$ are conjugate in E. Since they complement the subgroup $\mathbf{1} \times \operatorname{Inn} N$, we see this conjugation may be achieved by an element of the latter normal subgroup. Hence

$$\zeta_1 = \zeta_2\tau_b$$

for some $b = (b_\alpha)$ in $\operatorname{Inn} N = (\operatorname{Inn} X)^\Omega$. Since $a^{(1)}_\omega = a^{(2)}_\omega = 1$, we calculate that

$$\theta_{\zeta_1} = \theta_{\Phi_1 \tau_{a_1}} = \theta_{\Phi_1} = \phi_1$$

and that

$$\theta_{\zeta_2 \tau_b} = b_\omega^{-1} \theta_{\Phi_2} b_\omega = b_\omega^{-1} \phi_2 b_\omega.$$

Hence $\phi_1 = \phi_2 \sigma$ where σ denotes conjugation by the element b_ω from $\operatorname{Inn} X$.

Conversely suppose that $\phi_1 = \phi_2 \sigma$ where σ denotes conjugation by some element b_0 from $\operatorname{Inn} X$. Then, for $x \in Q$,

$$\begin{aligned} x\Phi_1 &= \big((h_{\alpha,x}\phi)^{b_0} \big)(x\rho) \\ &= b^{-1} \cdot (h_{\alpha,x}\phi)(x\rho) \cdot b = x\Phi_2 \tau_b \end{aligned}$$

where $b = (b_0)_{\alpha \in \Omega} \in \operatorname{Inn} N$. Therefore

$$x\zeta_1 = x\Phi_2 \tau_b \tau_{a_1} = x\zeta_2 \tau_{a_2^{-1} b a_1}.$$

It follows that $\tilde{H}_1 = \tilde{H}_2^{(1, a_2^{-1} b a_1)}$. The uniqueness part of Theorem 2.8 tells us that a_1 and a_2 are congruent modulo $\operatorname{Inn} N$. Hence $a_2^{-1} b a_1$ is an element of $\operatorname{Inn} N$ and we see that $\tilde{H}_1$ and $\tilde{H}_2$ are conjugate in E. Therefore H_1 and H_2 are conjugate in G and the proof is complete. $\square$

3 Conditions for maximality

In the context we are interested in, Theorem 2.8 and its corollary provides us with a parametrisation for the complements of our normal subgroup. In this section, we link this parametrisation to criteria for the maximality of the particular complement.

Let G be a finite group with a normal subgroup N which is a direct product of k copies of a non-abelian finite simple group X. We maintain the notation of the previous section. Specifically, let H be a complement to N in G and $\tilde{H} = \{ (x, x\zeta) \mid x \in Q \}$, where the homomorphism $\zeta \colon Q \to W(\operatorname{Aut} X)$ satisfies $\bar{\zeta} = \chi$. By Theorem 2.8(i), we may write $\zeta = \Phi \tau_a$, where Φ is induced from $\phi \colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$ and $a \in (\operatorname{Aut} X)^\Omega$ has its image in $(\operatorname{Out} X)^\Omega$ uniquely determined by G (that is, it is independent of H).

It is straightforward to see that H is a maximal subgroup of G if and only if it does not normalise any non-trivial proper subgroup of N. Passing to E (as in Section 2) and its subgroup $\tilde{H}$, we note that $\tilde{H}$ normalises a subgroup $\mathbf{1} \times M$ of $\mathbf{1} \times \operatorname{Inn} N$ if and only if the image of ζ normalises M and, since $\zeta = \Phi \tau_a$, this is in turn equivalent to the image of Φ normalising the subgroup $M^{a^{-1}}$ of $\operatorname{Inn} N$. We record this observation as a lemma.

Lemma 3.1 *The complement H corresponding to $\zeta = \Phi\tau_a$ is maximal in G if and only if the image of Φ normalises no non-trivial proper subgroup of* $\operatorname{Inn} N$. ☐

This lemma means that when determining whether the complement corresponding to a homomorphism $\phi\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$ is maximal or not, we need only consider the induced homomorphism Φ and we can ignore the conjugation by the element of $(\operatorname{Aut} X)^\Omega$ used to finish constructing the complement.

Let us first assume that our complement is a maximal subgroup of G; that is, that the image of Φ in $W(\operatorname{Aut} X)$ normalises no non-trivial proper subgroup of $\operatorname{Inn} N$. We begin by showing that the conditions appearing in the main theorem hold in this situation.

Lemma 3.2 *If the image of Φ normalises no non-trivial proper subgroup of* $\operatorname{Inn} N$*, then the image of ϕ in* $\operatorname{Aut} X$ *contains* $\operatorname{Inn} X$.

Proof: First assume that the image of ϕ normalises a non-trivial proper subgroup R of $\operatorname{Inn} X$. It follows that the image of the stabiliser $\operatorname{Stab}_Q(\omega)$ under Φ normalises the subgroup

$$R_\omega = \{\, (x_\alpha) \in (\operatorname{Aut} X)^\Omega \mid x_\omega \in R, \quad x_\alpha = 1 \text{ for } \alpha \neq \omega \,\},$$

of $\operatorname{Inn} N$. The distinct conjugates of R_ω under the action of $Q\Phi$ lie in distinct factors of $\operatorname{Inn} N$. Therefore the subgroup S generated by these conjugates has the form

$$S = \prod_{\alpha \in \Omega} R_\alpha,$$

a direct product of copies R_α of R. This subgroup S is then a non-trivial proper subgroup of $\operatorname{Inn} N$ which is normalised by the image of Φ, contrary to assumption. Hence the image of ϕ normalises no non-trivial proper subgroup of $\operatorname{Inn} X$.

Now consider $L = \operatorname{im}\phi \cap \operatorname{Inn} X$. If $L \neq \mathbf{1}$, then L is a normal subgroup of $\operatorname{im}\phi$ and taking $R = L$ in the above argument shows that $L = \operatorname{Inn} X$; that is, $\operatorname{im}\phi$ contains $\operatorname{Inn} X$ and the lemma would be proved.

The only remaining possibility is that $L = \mathbf{1}$. Then an argument of Wilson (see the proof of [11, Lemma 1]) shows that the centraliser in $\operatorname{Inn} X$ of a minimal normal subgroup M of $\operatorname{im}\phi$ is non-trivial. On the other hand, this centraliser is certainly a proper subgroup of $\operatorname{Inn} X$ and it is normalised by $\operatorname{im}\phi$, so we could take $R = \mathrm{C}_{\operatorname{Inn} X}(M)$ in the first paragraph. Thus, it is not possible that $L = \mathbf{1}$ and the proof of the lemma is complete. ☐

Lemma 3.3 *If the image of Φ normalises no non-trivial proper subgroup of* $\operatorname{Inn} N$ *and ϕ equals the restriction of some homomorphism $\hat{\phi}\colon H \to \operatorname{Aut} X$ where H is a subgroup of Q containing* $\operatorname{Stab}_Q(\omega)$*, then* $H = \operatorname{Stab}_Q(\omega)$.

PROOF: The subgroup H yields a (possibly trivial or improper) block system

$$\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_s$$

for the action of Q on Ω. Assume that $\omega \in \Omega_1$, so that $\Omega_1 = \omega H$, the orbit of H containing ω. Since $\mathrm{Stab}_Q(\omega)$ also equals the stabiliser of ω in H, there exists a transversal $U = \{\, u_\alpha \mid \alpha \in \Omega_1 \,\}$ to $\mathrm{Stab}_Q(\omega)$ in H such that $\omega u_\alpha = \alpha$ for each $\alpha \in \Omega_1$ and we may assume that $u_\omega = 1$. Let $V = \{v_1, v_2, \dots, v_s\}$ be a transversal to H in Q such that $\Omega_1 v_i = \Omega_i$ for $i = 1, 2, \dots, s$ and assume that $v_1 = 1$. Let $T' = \{\, u_\alpha v_i \mid \alpha \in \Omega_1,\ 1 \leqslant i \leqslant s \,\}$, which is a transversal to $\mathrm{Stab}_Q(\omega)$ in Q. Write t'_α for the element of T' satisfying $\omega t'_\alpha = \alpha$. Then $t'_\alpha = u_\alpha$ for all $\alpha \in \Omega_1$ since $v_1 = 1$. Let $\Phi' \colon Q \to W(\mathrm{Aut}\,X)$ be the homomorphism induced from ϕ with respect to T'. Lemma 2.5 tells us that Φ and Φ' differ by conjugation by some element of $(\mathrm{Aut}\,X)^\Omega$.

For each $\alpha \in \Omega_1$, let $\psi_\alpha = u_\alpha \hat{\phi}$ and define

$$D = \{\, (x^{\psi_\alpha^{-1}})_{\alpha \in \Omega_1} \mid x \in \mathrm{Inn}\,X \,\},$$

a diagonal subgroup of $(\mathrm{Inn}\,X)^{\Omega_1}$. (Here $x^{\psi_\alpha^{-1}}$ denotes the conjugate $\psi_\alpha x \psi_\alpha^{-1}$ of x by ψ_α^{-1}.) We shall view D as being embedded in $\mathrm{Inn}\,N$ in the obvious way. Let $y \in H$. Then if $\alpha \in \Omega_1$, we have

$$h'_{\alpha,y} = t'_\alpha y (t'_{\alpha y})^{-1} = u_\alpha y u_{\alpha y}^{-1}$$

and from this

$$h'_{\alpha,y}\phi = h'_{\alpha,y}\hat{\phi} = \psi_\alpha (y\hat{\phi}) \psi_{\alpha y}^{-1}.$$

Therefore, upon conjugating an element of D by $y\Phi'$, we find

$$\begin{aligned} \Big[(x^{\psi_\alpha^{-1}})_{\alpha\in\Omega_1}\Big]^{y\Phi'} &= \Big[(x^{\psi_\alpha^{-1}})_{\alpha\in\Omega_1}\Big]^{(h'_{\alpha,y}\phi)_{\alpha\in\Omega}(y\rho)} \\ &= \Big[\Big((x^{\psi_\alpha^{-1}})^{h'_{\alpha,y}\phi}\Big)_{\alpha\in\Omega_1}\Big]^{y\rho} \\ &= \Big[(z^{\psi_{\alpha y}^{-1}})_{\alpha\in\Omega_1}\Big]^{y\rho} \\ &= (z^{\psi_\alpha^{-1}})_{\alpha\in\Omega_1}, \end{aligned}$$

where $z = x^{y\hat{\phi}} \in \mathrm{Inn}\,X$. In particular, D is normalised by $H\Phi'$. Conjugating by the appropriate element of $(\mathrm{Aut}\,X)^\Omega$, we deduce that $H\Phi$ normalises some diagonal subgroup E of $(\mathrm{Inn}\,X)^{\Omega_1}$.

Note that $\{v_1\Phi, v_2\Phi, \dots, v_s\Phi\}$ is a transversal to $H\Phi$ in $Q\Phi$ and that $E^{v_i\Phi} \leqslant (\mathrm{Inn}\,X)^{\Omega_i}$ for each i. Hence the conjugates $E^{v_i\Phi}$ generate a direct product

$$F = E^{v_1\Phi} \times E^{v_2\Phi} \times \cdots \times E^{v_s\Phi}$$

and this is normalised by $Q\Phi$. Our assumption that the image of Φ normalises no non-trivial proper subgroup of $\mathrm{Inn}\,N$ then forces $|\Omega_1| = 1$. Therefore $H = \mathrm{Stab}_Q(\omega)$, as required. □

The previous two lemmas tell us that if a complement is maximal then we have two conditions on the associated homomorphism $\phi\colon \mathrm{Stab}_Q(\omega) \to \mathrm{Aut}\,X$. Conversely consider a complement to N in G and assume that it corresponds to the homomorphism $\zeta = \Phi\tau_a$ where Φ is induced from $\phi\colon \mathrm{Stab}_Q(\omega) \to \mathrm{Aut}\,X$. We assume that the conditions established above hold; that is,

(a) the image of ϕ in $\mathrm{Aut}\,X$ contains $\mathrm{Inn}\,X$, and

(b) ϕ is not the restriction of some homomorphism $\hat{\phi}\colon J \to \mathrm{Aut}\,X$ where $\mathrm{Stab}_Q(\omega) < J \leqslant Q$.

We seek to show that our complement is maximal in G and, in view of Lemma 3.1, this is equivalent to establishing that the image of Φ does not normalise any non-trivial proper subgroup of $\mathrm{Inn}\,N$.

Let L be a subgroup of $\mathrm{Inn}\,N$ which is normalised by the image of Φ. Let L_α be the subgroup of $\mathrm{Inn}\,X$ which is the image of L under the projection of $\mathrm{Inn}\,N$ onto its direct factor indexed by α. Since the representation of ρ used to construct the induced map Φ is transitive and since $\mathrm{im}\,\Phi$ normalises L, it follows that the subgroups L_α are conjugate in $\mathrm{Aut}\,X$. We therefore have three possibilities:

(i) $L_\alpha = \mathrm{Inn}\,X$ for all $\alpha \in \Omega$,

(ii) $\mathbf{1} < L_\alpha < \mathrm{Inn}\,X$ for all $\alpha \in \Omega$, or

(iii) $L_\alpha = \mathbf{1}$ for all $\alpha \in \Omega$.

Of course, the third case implies $L = \mathbf{1}$, which is perfectly acceptable for a subgroup of $\mathrm{Inn}\,N$ normalised by $\mathrm{im}\,\Phi$. We shall show that our two assumptions (a) and (b) above will force $L = \mathrm{Inn}\,N$ in case (i) and will yield a contradiction in case (ii).

We first deal with case (ii). Note that L is also normalised by the image of $\mathrm{Stab}_Q(\omega)$ under Φ and hence L_ω is normalised by the image of $(\mathrm{Stab}_Q(\omega))\Phi$ under the projection onto the direct factor of $\mathrm{Inn}\,N$ indexed by ω. However, the image of $(\mathrm{Stab}_Q(\omega))\Phi$ under this projection is merely the image of ϕ and thus L_ω is normalised by the image of ϕ. Our first assumption (a) then shows that L_ω is a normal subgroup of $\mathrm{Inn}\,X$ and since X ($\cong \mathrm{Inn}\,X$) is simple we obtain the required contradiction.

We then turn to case (iii) where $L_\alpha = \mathrm{Inn}\,X$ for all $\alpha \in \Omega$. This says that L is a subdirect product in $\mathrm{Inn}\,N$. Since $\mathrm{Inn}\,X$ is isomorphic to the non-abelian simple group X, this subdirect product is a direct product of diagonal subgroups, say

$$L = D_1 \times D_2 \times \cdots \times D_s$$

where $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_s$ is a partition of Ω and each D_i is a diagonal subgroup of the direct product of copies of $\mathrm{Inn}\,X$ indexed by Ω_i. Therefore each D_i is isomorphic to X and they are the minimal normal subgroups of L. Conjugation by elements of the image of Φ permutes the D_i and noting that the action of Q on Ω is defined by $\rho = \Phi^*$, we see that $\{\Omega_1, \Omega_2, \dots, \Omega_s\}$ is a block system for the action of Q on Ω. Assume $\omega \in \Omega_1$ and let J be the subgroup of Q containing $\mathrm{Stab}_Q(\omega)$ that stabilises the block Ω_1 in the block system. Then $\Omega_1 = \omega J$, the orbit of ω under the action of J.

Now $J\Phi$ is a subgroup of $Q\Phi$ which normalises the subgroup $(\mathrm{Inn}\, X)^{\Omega_1}$ of $\mathrm{Inn}\, N$ and hence normalises $L \cap (\mathrm{Inn}\, X)^{\Omega_1} = D_1$. We may write

$$D_1 = \{ \, (\sigma_{x\phi_\alpha})_{\alpha \in \Omega_1} \mid x \in X \, \}$$

where each ϕ_α is an automorphism of X and σ_a denotes the inner automorphism of X obtained by conjugating by a. Without loss of generality we shall assume ϕ_ω is the identity map. Then

$$(\sigma_{x\phi_\alpha})_{\alpha \in \Omega_1} \mapsto x$$

is an isomorphism and is equal to the composition of the projection of elements of D_1 onto their ω-component followed by the inverse of the natural isomorphism $X \to \mathrm{Inn}\, X$. The action of $J\Phi$ on D_1 then induces an action of J on X and hence a homomorphism $\hat{\phi} \colon J \to \mathrm{Aut}\, X$.

Consider the restriction of $\hat{\phi}$ to $\mathrm{Stab}_Q(\omega)$. If $y \in \mathrm{Stab}_Q(\omega)$, then the ω-component of

$$[(\sigma_{x\phi_\alpha})_{\alpha \in \Omega_1}]^{y\Phi}$$

is

$$\sigma_x^{h_{\omega,y}\phi} = \sigma_x^{y\phi} = \sigma_{x(y\phi)}.$$

Hence $y\hat{\phi}$ is the map $x \mapsto x(y\phi)$; that is, $y\hat{\phi} = y\phi$. Since then ϕ is the restriction of $\hat{\phi}$ to $\mathrm{Stab}_Q(\omega)$ we may now use assumption (b) to deduce $J = \mathrm{Stab}_Q(\omega)$ and hence $|\Omega_i| = 1$ for all i. Therefore $L = D_1 \times D_2 \times \dots \times D_s$ is the direct product of copies of $\mathrm{Inn}\, X$ indexed by Ω. Thus in case (i) the subgroup normalised is $\mathrm{Inn}\, N$.

We now conclude that under our assumptions there is no non-trivial proper subgroup of $\mathrm{Inn}\, N$ normalised by $\mathrm{im}\, \Phi$ which tells us that our complement to N in G is a maximal subgroup. This completes the proof of the following theorem.

Theorem 3.4 *Let G be a finite group with a normal subgroup N which is a direct product of copies of a non-abelian finite simple group X indexed by the set Ω. Let $Q = G/N$ and let Q act on Ω via the coupling $\chi \colon Q \to W(\mathrm{Aut}\, X)$. A complement corresponding to a homomorphism $\phi \colon \mathrm{Stab}_Q(\omega) \to \mathrm{Aut}\, X$ (which necessarily satisfies the compatibility condition $\bar{\phi} = \psi$, where $\psi = \theta_\chi$) is maximal in G if and only if*

(a) *the image of ϕ in* $\mathrm{Aut}\, X$ *contains* $\mathrm{Inn}\, X$*, and*

(b) *ϕ is not the restriction of some homomorphism $\hat{\phi}\colon J \to \operatorname{Aut} X$ where $\operatorname{Stab}_Q(\omega) < J \leqslant Q$.* $\square$

The set of all complements is parametrised by homomorphisms ϕ satisfying the compatibility condition (see Corollary 2.9). This theorem tells us that those complements which are maximal are parametrised by homomorphisms satisfying the compatibility condition *and* the two conditions (a) and (b).

4 Further considerations: normalised subgroups and twisted wreath products

As has been the case throughout the majority of this paper, let G be a finite group with a normal subgroup N which is a direct product of copies of a non-abelian finite simple group X indexed by the set Ω. We retain the notation previously described. In particular, $Q = G/N$ and there is an action of Q on Ω determined by the coupling $\chi\colon Q \to W(\operatorname{Aut} X)$. Corollary 2.9 tells us that complements to N in G correspond to homomorphisms $\phi\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$ satisfying the given compatibility condition. Theorem 3.4 then says that a complement is maximal in G if and only if the associated homomorphism ϕ satisfies two further conditions. Since a complement to N is maximal if and only if it normalises no non-trivial proper subgroup of N, we would expect these two conditions to provide information about possible normalised subgroups of N. Accordingly we begin this final section by extracting from the work of Sections 2 and 3 the subgroups which are normalised.

Suppose first that the homomorphism $\phi\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$ satisfies the compatibility condition $\bar{\phi} = \psi$ but fails condition (a) of Theorem 3.4. The proof of Lemma 3.2 shows that the induced homomorphism $\Phi\colon Q \to W(\operatorname{Aut} X)$ has the property that its image normalises a subgroup S of $\operatorname{Inn} N$ of the form

$$S = \prod_{\alpha \in \Omega} R_\alpha$$

where each R_α is an isomorphic copy of a non-trivial proper subgroup R of $\operatorname{Inn} X$ normalised by the image of ϕ. The discussion prior to Lemma 3.1 then tells us that the image of the corresponding $\zeta\colon Q \to W(\operatorname{Aut} X)$ normalises

$$S^a = \prod_{\alpha \in \Omega} R'_\alpha$$

where R'_α is the conjugate of R_α by some element in $\operatorname{Aut} X$. (Note that Theorem 2.8 tells us what form the element a has.) It follows that $\tilde{H} = \{\, (x, x\zeta) \mid x \in Q \,\}$ normalises the subgroup $\mathbf{1} \times S^a$ of $\mathbf{1} \times \operatorname{Inn} N$. Finally recall that the isomorphism from G to $E = \{\, (x, \sigma) \in Q \times \operatorname{Aut} N \mid \sigma \in x\chi \,\}$ is given by

$$g \mapsto (Ng, \sigma_g).$$

Applying the inverse of this isomorphism, we deduce that the complement H corresponding to ϕ normalises the subgroup

$$M = \{\, g \in N \mid \sigma_g \in S^a \,\}$$

of N. Now the isomorphism $\operatorname{Inn} N \to N$ given by $\sigma_g \mapsto g$ (for $g \in N$) preserves the decomposition of S^a into a direct product of isomorphic copies of R. We have therefore established our first observation.

Lemma 4.1 *If the homomorphism $\phi\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$ satisfies the compatibility condition $\bar{\phi} = \psi$ but the image of ϕ does not contain $\operatorname{Inn} X$, then the complement to N in G associated to ϕ normalises a subgroup of N of the form $\prod_{\alpha \in \Omega} M_\alpha$ where the M_α are isomorphic copies of a non-trivial proper subgroup of X.* □

Now consider the case when $\phi\colon \operatorname{Stab}_Q(\omega) \to \operatorname{Aut} X$ satisfies the compatibility condition $\bar{\phi} = \psi$ but fails condition (b) of Theorem 3.4. Thus we are assuming that ϕ is the restriction of some homomorphism $\hat{\phi}\colon J \to \operatorname{Aut} X$ where $\operatorname{Stab}_Q(\omega) < J \leqslant Q$. The proof of Lemma 3.3 shows that the image of Φ normalises a subgroup of $\operatorname{Inn} N$ of the form

$$F = E^{v_1\Phi} \times E^{v_2\Phi} \times \cdots \times E^{v_s\Phi}$$

where $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_s$ is a block system for Q on Ω with $\Omega_1 = \omega J$, $\{v_1, v_2, \dots, v_s\}$ is a transversal to J in Q, $v_1 = 1$, and E is a diagonal subgroup of the direct product of copies of $\operatorname{Inn} X$ indexed by Ω_1. Thus F is a direct product of diagonal subgroups. Conjugating by the element a in the base group of $W(\operatorname{Aut} X)$ produced by Theorem 2.8 preserves this state of affairs and we deduce that the image of ζ normalises the subgroup

$$F^a = F_1 \times F_2 \times \cdots \times F_s$$

where each F_i is a diagonal subgroup of copies of $\operatorname{Inn} X$ indexed by Ω_i. Then $\tilde{H}$ normalises $\mathbf{1} \times F^a$ and, applying the inverse of the isomorphism $G \to E$, we deduce that the complement H associated to ϕ normalises the subgroup

$$D = \{\, g \in N \mid \sigma_g \in F^a \,\}$$

of N. Once again the isomorphism $\operatorname{Inn} N \to N$ preserves the decomposition into the direct product and we obtain our second observation about the conditions in the main theorem.

Lemma 4.2 *If the homomorphism* $\phi\colon \mathrm{Stab}_Q(\omega) \to \mathrm{Aut}\,X$ *satisfies the compatibility condition* $\bar{\phi} = \psi$ *but is the restriction of some homomorphism* $\hat{\phi}\colon J \to \mathrm{Aut}\,X$ *where* $\mathrm{Stab}_Q(\omega) < J \leqslant Q$*, then the complement to* N *in* G *associated to* ϕ *normalises a subgroup of* N *of the form*

$$D = D_1 \times D_2 \times \cdots \times D_s$$

where $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_s$ *is a block system for the action of* Q *on* Ω *with* $\Omega_1 = \omega J$ *and each* D_i *is a diagonal subgroup of the direct product of copies of* X *indexed by* Ω_i*.* $\square$

Finally let us discuss the link between our work and work relating to twisted wreath products (particularly that of Lafuente [7]). Before we define what is meant by a twisted wreath product we shall examine our earlier work in sufficient detail that the link to twisted wreath products can be elucidated with ease.

We continue to assume that G is a finite group with a normal subgroup N which is a direct product of copies of a non-abelian simple group X indexed by a set Ω. Suppose that H is a complement to N in G. Then the projection $G \to Q = G/N$ induces an isomorphism from H onto Q and therefore G is isomorphic to the semidirect product of N by Q. The isomorphism class of G is therefore determined by the induced action of Q on N.

We observed earlier that G is isomorphic to $E = \{\, (x,\sigma) \in Q \times \mathrm{Inn}\,N \mid \sigma \in x\chi \,\}$ and that under this isomorphism N corresponds to $\mathbf{1} \times \mathrm{Inn}\,N$ and H corresponds to $\tilde{H} = \{\, (x, x\zeta) \mid x \in Q \,\}$ where $\zeta = \Phi\tau_a$ as determined by Theorem 2.8. One isomorphism of N with $\mathbf{1} \times \mathrm{Inn}\,N$ is given by $g \mapsto \sigma_g^a = a^{-1}\sigma_g a$ and the conjugation action of $\tilde{H}$ on $\mathbf{1} \times \mathrm{Inn}\,N$ is then

$$(1, \sigma_g^a)^{(x,x\zeta)} = (1, \sigma_g^{a(x\Phi\tau_a)}) = (1, \sigma_g^{(x\Phi)a}) = (1, \sigma_{g(x\Phi)}^a).$$

It follows that the induced action of Q on N is given by the homomorphism $\Phi\colon Q \to \mathrm{Aut}\,N$.

Now let us recall the definition of a twisted wreath product. We shall express this in a similar form to our earlier description of the wreath product. We shall construct the twisted wreath product $X \,\mathrm{twr}\, Q$ with respect to the homomorphism $\phi\colon \mathrm{Stab}_Q(\omega) \to \mathrm{Aut}\,X$ and the previously chosen transversal $T = \{\, t_\alpha \mid \alpha \in \Omega \,\}$. Define elements $h_{\alpha,x}$ in $\mathrm{Stab}_Q(\omega)$ by the formula

$$t_\alpha x = h_{\alpha,x} t_{\alpha x}$$

for $\alpha \in \Omega$ and $x \in Q$. Let

$$N = \prod_{\alpha \in \Omega} X_\alpha,$$

the direct product of copies of X indexed by Ω, and define an action of Q on N by

$$(g_\alpha)^x = (g_{\alpha x^{-1}}^{(h_{\alpha,x^{-1}}^{-1})\phi})$$

for $g_\alpha \in X$ for all $\alpha \in \Omega$ and $x \in Q$. (Lafuente [7] uses the set of maps $T \to X$ instead of the direct product of copies of X. There is, of course, an obvious translation between the two and the previous formula is simply the translation of the formula he gives for multiplication of the elements of the twisted wreath product into this setting.) The definition of the elements $h_{\alpha,x}$ ensures that $h_{\alpha,x^{-1}}^{-1} = h_{\alpha x^{-1},x}$ and hence the action of Q on N is given by

$$(g_\alpha)^x = (g_{\alpha x^{-1}}^{(h_{\alpha x^{-1},x})\phi}). \tag{1}$$

The *twisted wreath product* $X \operatorname{twr}_\phi Q$ is then the semidirect product of N by Q with respect to this action.

Now, however, we note that Equation (1) says that the image of an element $g \in N$ under the action of $x \in Q$ is simply the image of g under the automorphism

$$x\Phi = (h_{\alpha,x}\phi)(x\rho)$$

from $\operatorname{Aut} N$. That is, $X \operatorname{twr}_\phi Q$ is the semidirect product of N by Q with respect to the homomorphism $\Phi\colon Q \to \operatorname{Aut} N$. Since we used precisely the same action to construct G as the semidirect product of N by Q, we deduce $G \cong X \operatorname{twr}_\phi Q$. Since Q is isomorphic to H, we deduce Lafuente's result as a consequence of our work.

Theorem 4.3 (Lafuente [7]) *If G is a finite group with a minimal normal subgroup N that is a direct product of copies of a non-abelian finite simple group X and H is a complement to N in G, then G is isomorphic to a twisted wreath product $X \operatorname{twr} H$.* □

The similarities between our conditions for maximality of a complement and those of Baddeley [2] for the maximality of the top group of a twisted wreath product are therefore unsurprising.

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References

[1] M. Aschbacher and L. Scott, Maximal subgroups of finite groups, *J. Algebra* 92 (1985), 44–80.

[2] R. W. Baddeley, Primitive permutation groups with a regular non-abelian normal subgroup, *Proc. London Math. Soc. (3)* 67 (1993), 547–595.

[3] R. Bercov, On groups without abelian composition factors, *J. Algebra* 5 (1967), 106–109.

[4] John Cannon and Derek F. Holt, Computing maximal subgroups of finite groups, *J. Symbolic Comput.* 37 (2004), 589–609.

[5] F. Gross and L. G. Kovács, On normal subgroups of groups which are direct products, *J. Algebra* 90 (1984), 133–168.

[6] L. G. Kovács, Maximal subgroups in composite finite groups, *J. Algebra* 99 (1986), 114–131.

[7] Julio Lafuente, On restricted twisted wreath products of groups, *Arch. Math. (Basel)* 43 (1984), 208–209.

[8] Christopher Parker and Martyn Quick, Maximal complements in wreath products, *J. Algebra* 266 (2003), 320–337.

[9] Martyn Quick, Probabilistic generation of wreath products of non-abelian finite simple groups, II, (2004, to appear).

[10] Derek J. S. Robinson, *A Course in the Theory of Groups (Second Edition)*, Graduate Texts in Mathematics 80 (Springer-Verlag, New York 1996).

[11] Robert A. Wilson, Maximal subgroups of automorphism groups of simple groups, *J. London Math. Soc. (2)* 32 (1985), 460–466.