On Maximal Subgroups of Free Idempotent Generated Semigroups

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Abstract

We prove the following results: (1) Every group is a maximal subgroup of some free idempotent generated semigroup. (2) Every finitely presented group is a maximal subgroup of some free idempotent generated semigroup arising from a finite semigroup. (3) Every group is a maximal subgroup of some free regular idempotent generated semigroup. (4) Every finite group is a maximal subgroup of some free regular idempotent generated semigroup arising from a finite regular semigroup.

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1 Introduction and summary of results

Let S be a semigroup, and let $E = E(S)$ be the set of idempotents of S. The *free idempotent generated semigroup* on E is defined by the following presentation:

$$\mathsf{IG}(E) = \langle E \mid e \cdot f = ef \ (e, f \in E,\ \{e, f\} \cap \{ef, fe\} \neq \emptyset) \rangle. \tag{1}$$

(It is an easy exercise to show that if, say, $fe \in \{e, f\}$ then $ef \in E$. In the defining relation $e \cdot f = ef$ the left hand side is a word of length 2, and ef is the product of e and f in S, i.e. a word of length 1.) These semigroups arose in [12, 4], where abstract characterisations of the sets of idempotents of semigroups via structures called *biordered sets* was undertaken.

The semigroup $\mathsf{IG}(E)$ has the following properties:

(IG1) There exists a natural homomorphism ϕ from $\mathsf{IG}(E)$ into the subsemigroup S' of S generated by E.

(IG2) The restriction of ϕ to the set of idempotents of $\mathsf{IG}(E)$ is a bijection onto E (and an isomorphism of biordered sets). Thus we may identify those two sets.

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(IG3) ϕ maps the $\mathscr{R}$-class (respectively $\mathscr{L}$-class) of $e \in E$ onto the corresponding class of e in S'; this induces a bijection between the set of all $\mathscr{R}$-classes (resp. $\mathscr{L}$-classes) in the $\mathscr{D}$-class of e in $\mathsf{IG}(E)$ and the corresponding set in S'.

(IG4) The restriction of ϕ to the maximal subgroup of $\mathsf{IG}(E)$ containing $e \in E$ (i.e. to the $\mathscr{H}$-class of e in $\mathsf{IG}(E)$) is a homomorphism onto the maximal subgroup of S' containing e.

The assertion (IG1) is obvious; (IG2) is proved in [12] and [4]; (IG3) is a corollary of [6]; (IG4) follows from (IG2). For an introduction to biordered sets see [9]. For basics of semigroup theory, including Green's relations and their relationship with maximal subgroups see any standard monograph such as [9, 10]. The maximal subgroup of a semigroup S containing an idempotent e will be denoted by $\mathsf{H}(S,e)$.

If S is a regular semigroup, one also defines the *free regular idempotent generated semigroup* $\mathsf{RIG}(E)$ on E as follows. The *sandwich set* of a pair of idempotents $e, f \in E$ is defined as

$$S(e,f) = \{h \in E \,:\, ehf = ef,\ fhe = h\}.$$

The semigroup $\mathsf{RIG}(E)$ is then the homomorphic image of $\mathsf{IG}(E)$ obtained by adding the relations

$$ehf = ef\ (e,f \in E,\ h \in S(e,f))$$

to the presentation (1). This semigroup also satisfies the properties (IG1)–(IG4), and also:

(RIG1) $\mathsf{RIG}(E)$ is regular ([12]).

(RIG2) The natural homomorphism $\mathsf{IG}(E) \to \mathsf{RIG}(E)$ induces an isomorphism between the maximal subgroups of any $e \in E$ in $\mathsf{IG}(E)$ and $\mathsf{RIG}(E)$ ([1, Theorem 3.6]).

Maximal subgroups of free idempotent generated semigroups have been of interest for some time. Several papers [11, 13, 15] established various sufficient conditions guaranteeing that all the maximal subgroups are free. Indeed, it was conjectured in [11] that this was always the case. The first counterexample to this conjecture was given by Brittenham, Margolis and Meakin [1], where it was shown that the free abelian group $\mathbb{Z} \oplus \mathbb{Z}$ is the maximal subgroup of the free idempotent generated semigroup arising from a certain 72-element semigroup. The same authors report further counterexamples to appear in [2], where they show that the multiplicative group $\mathbb{F}^*$ of a field $\mathbb{F}$ arises as a maximal subgroup of $\mathsf{IG}(E(M_3(\mathbb{F})))$, where

$M_3(\mathbb{F})$ is the semigroup of all 3×3 matrices over $\mathbb{F}$. Related work concerning periodic elements in free idempotent generated semigroups may be found in [5].

In this paper we prove:

Theorem 1. *Every group is a maximal subgroup of some free idempotent generated semigroup.*

Theorem 2. *Every finitely presented group is a maximal subgroup of some free idempotent generated semigroup arising from a finite semigroup.*

Theorem 3. *Every group is a maximal subgroup of some free regular idempotent generated semigroup.*

Theorem 4. *Every finite group is a maximal subgroup of some free regular idempotent generated semigroup arising from a finite regular semigroup.*

We remark that Theorem 2 provides a complete characterisation of groups appearing as maximal subgroups of free idempotent generated semigroups arising from finite semigroups. (And of course, trivially, Theorems 1 and 3 provide such characterisations with the finiteness assumption removed.) Indeed, every maximal subgroup in a free idempotent generated semigroup arising from a finite semigroup must be finitely presented. To see this, observe that in this case the presentation (1) is finite, and also that the set of $\mathcal{L}$-classes in the $\mathcal{D}$-class of any $e \in E$ is finite by (IG3). The assertion then follows from [19, Corollary 2.11]. By way of contrast, Theorem 4 leaves us with the following unresolved question: *Is every finitely presented group a maximal subgroup of some free regular idempotent generated semigroup arising from a finite regular semigroup?*

Theorems 1–4 will be proved by means of two explicit constructions described in Sections 3 and 4. Preceding this, in Section 2 we introduce the notation and tools common to both constructions.

2 Preliminaries and outline of the method

The environemnt semigroup $B_{I,J}$

There are two ways to compose mappings from a set X into itself: from left to right and from right to left; we denote the two resulting semigroups by $T_X^{(r)}$ and $T_X^{(l)}$ respectively (or $T_n^{(r)}$ and $T_n^{(l)}$ if $X = \{1, \ldots, n\}$). All the semigroups we are going to construct will be subsemigroups of some

$$B_{I,J} = T_I^{(l)} \times T_J^{(r)}.$$

A typical element of $\beta \in B_{I,J}$ has the form $\beta = (\beta^{(l)}, \beta^{(r)})$. To aid remembering the different orders in which compositions are formed we will write

$\beta^{(l)}$ to the left of its argument, and $\beta^{(r)}$ to the right. The semigroup $B_{I,J}$ has a unique minimal ideal

$$R_{I,J} = \{\rho_{ij} = (\rho_i, \rho_j) \, : \, i \in I, \, j \in J\},$$

where

$$\rho_i : I \to I, \, x \mapsto i,$$
$$\rho_j : J \to J, \, x \mapsto j,$$

are the constant maps. The multiplication in $R_{I,J}$ works as follows:

$$\rho_{ij}\rho_{kl} = \rho_{il},$$

i.e. $R_{I,J}$ is an $I \times J$ rectangular band. The semigroup $B_{I,J}$ is in fact the *translational hull* of $R_{I,J}$ (see [16]), an important background fact for our discussion, even though it will not be explicitly used in any of the arguments.

A presentation for the maximal subgroup

In each of the two constructions to be described in Sections 3, 4 we work with a semigroup S satisfying:

(S1) $R_{I,J} \leq S \leq B_{I,J}$.

For such a semigroup S we will want to determine the maximal subgroup H of $\mathsf{IG}(E(S))$ containing ρ_{11}. The crucial step in doing this is a certain 'canonical' presentation for H which we now describe. A method for obtaining presentations for subgroups of semigroups, akin to the classical Reidemeister–Schreier method for groups [18, Statement 6.1.8], is described in [19]. According to this method, a presentation for H is obtain by applying a certain rewriting process to the generators and relations of the defining presentation (1) for $\mathsf{IG}(E)$. The fundamental ingredient in this rewriting process is the action of the generators of $\mathsf{IG}(E)$ (i.e. of the elements of E) on the $\mathscr{L}$-classes in the $\mathscr{D}$-class of ρ_{11} in $\mathsf{IG}(E)$. By virtue of (IG3) (and (IG1)) this action is the same as the action of the elements of E on $R_{I,J}$.

If it were the case that $S = R_{I,J}$, this rewriting process would yield the presentation with generators

$$f_{ij} \; (i \in I, \, j \in J), \tag{2}$$

and relations

$$f_{1j} = f_{i1} = 1 \; (i \in I, \, j \in J), \tag{3}$$

i.e. H would be free of rank $(|I|-1)(|J|-1)$. This was originally proved in [14] using different methods. Notice the obvious correspondence between the generators (2) and the elements of $R_{I,J}$.

In the general case where $R_{I,J} < S$, on the face of it the rewriting process yields both new generators and relations. However, when rewriting the relations of the form $e \cdot f = ef$ coming from (1) with precisely one of e or f not in $R_{I,J}$, one sees that all the new generators are in fact redundant, but that eliminating them yields new defining relations

$$f_{ij}^{-1} f_{il} = f_{kj}^{-1} f_{kl} \ ((i,k;j,l) \in \Sigma), \tag{4}$$

where

$$\begin{aligned} \Sigma &= \Sigma_{LR} \cup \Sigma_{UD} \\ \Sigma_{LR} &= \{(i,k;j,l) \in I \times I \times J \times J \ : \\ &\qquad (\exists \beta \in E(S))(\beta^{(l)}(i) = i \ \& \ \beta^{(l)}(k) = k \ \& \ j\beta^{(r)} = l\beta^{(r)} = j)\} \\ \Sigma_{UD} &= \{(i,k;j,l) \in I \times I \times J \times J \ : \\ &\qquad (\exists \beta \in E(S))(\beta^{(l)}(i) = \beta^{(l)}(k) = i \ \& \ j\beta^{(r)} = j \ \& \ l\beta^{(r)} = l)\}. \end{aligned} \tag{5}$$

Finally, the relations obtained by rewriting the relations $e \cdot f = ef$ where $e, f, ef \notin R_{I,J}$ can be shown to be consequences of the above relations. Thus:

(H1) The group $H = \mathsf{H}(\mathsf{IG}(E(S)), \rho_{11})$ is defined by generators (2) and relations (3) and (4).

An alternative proof of this fact can be obtained by modifying the topological approach from [1].

Following [1] we will call a tuple belonging to Σ a *singular square* (singularised by β); those belonging to Σ_{LR} will be referred to as the *left-right* singular squares, and those belonging to Σ_{UD} as the *up-down* singular squares.

So, every singular square yields a defining relation. In order to aid understanding of the forthcoming considerations, let us highlight the relations produced by certain distinguished types of singular squares. For example, a square $(1,i;1,j) \in \Sigma$ yields the relation $f_{11}^{-1} f_{1j} = f_{i1}^{-1} f_{ij}$, which, keeping in mind (3), is equivalent to $f_{ij} = 1$. A singular square of the form $(1,i;j,l)$ yields the relation $f_{ij} = f_{il}$, while the square $(i,k;1,j)$ yields $f_{ij} = f_{kj}$. Let us call these three types the *corner*, *flush top* and *flush left* squares, respectively.

One further type of square will be utilised: Suppose that $(i,k;j,l) \in \Sigma$ and that we already know that $f_{ij} = 1$ (e.g. by virtue of a corner square involving i and j). Then the relation (4) becomes $f_{kj} f_{il} = f_{kl}$. Let us call this a $3/4$ square. All four different types of singular squares are illustrated in Figure 1.

The method

Let us outline the features and reasoning common to both forthcoming constructions. We will start with a group G, which ultimately we want to realise as a maximal subgroup of the free idempotent generated semigroup

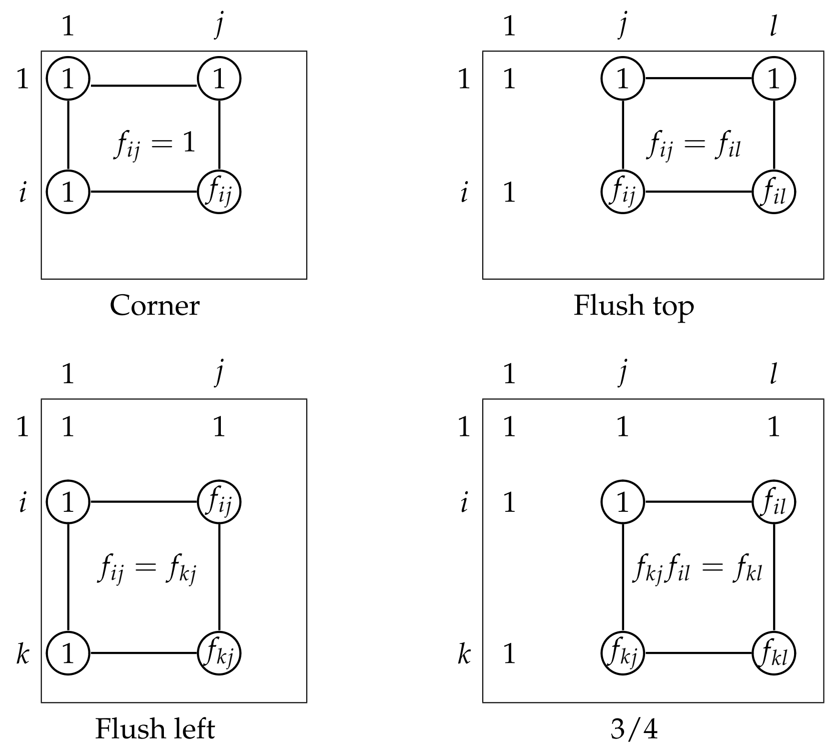

Figure 1: The four distinguished types of singular squares, and the relations they yield.

arising from some semigroup S. In each instance we will introduce an *auxiliary matrix*

$$Y = (y_{ij})_{I\times J}, \tag{6}$$

with entries from a generating set for G. Aided by this auxiliary matrix, we define a collection of idempotents from $B_{I,J}$, which, together with $R_{I,J}$, generate S. We then examine the presentation given in (H1) and prove that it indeed defines our initial group G. This we do by performing the following steps. First we consider certain idempotents from $S \setminus R_{I,J}$ such that the (corner and flush) squares singularised by them allow us to prove:

(Rel1) $f_{ij} = f_{kl}$ whenever $y_{ij} = y_{kl}$.

At this stage we may identify each letter f_{ij} with the corresponding entry y_{ij} (considered also as a letter, and not as a group element). Then we consider certain further idempotents such that:

(Rel2) The (3/4) squares singularised by these idempotents yield (after the identification above) a set of relations which define G.

The argument is completed by a 'bookkeeping step', which ensures that we do get the group G and not a proper homomorphic image:

(Rel3) Check (by considering every idempotent and every square singularised by it) that every other relation given in (H1) holds in G.

3 First construction

In this section we present our first construction, which proves Theorems 1 and 2. In fact, we will present the construction in the finitely presented case (Theorem 2), and then remark that obvious trivial changes adapt it to the general case (Theorem 1).

So let G be any finitely presented group. It is well known that G can be defined by a presentation of the form

$$G = \langle a_1, \ldots, a_p \mid b_1c_1 = d_1, \ldots, b_qc_q = d_q\rangle, \tag{7}$$

where $b_r, c_r, d_r \in \{a_1, \ldots, a_p\}$ for all $r = 1, \ldots, q$. This essentially follows from the fact that a 'long' relator $x_1x_2 \ldots x_s$ ($s > 3$) can be replaced by two shorter relators $x_1x_2y^{-1}$ and $yx_3 \ldots x_s$, at the expense of introducing a new, redundant generator y.

Let us set

$$m = 1 + 2q,\ n = 1 + p + 2q,\ I = \{1, \ldots, m\},\ J = \{1, \ldots, n\}. \tag{8}$$

The auxiliary matrix (see Section 2) is:

$$Y=(y_{ij})_{m\times n}=\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 & 1 & 1 & \dots & 1 & 1\\ 1 & a_1 & a_2 & \dots & a_p & 1 & c_1 & 1 & 1 & \dots & 1 & 1\\ 1 & a_1 & a_2 & \dots & a_p & b_1 & d_1 & 1 & 1 & \dots & 1 & 1\\ 1 & a_1 & a_2 & \dots & a_p & 1 & 1 & 1 & c_2 & \dots & 1 & 1\\ 1 & a_1 & a_2 & \dots & a_p & 1 & 1 & b_2 & d_2 & \dots & 1 & 1\\ & \vdots & & & & & \vdots & & & \ddots & \vdots & \\ 1 & a_1 & a_2 & \dots & a_p & 1 & 1 & 1 & 1 & \dots & 1 & c_q\\ 1 & a_1 & a_2 & \dots & a_p & 1 & 1 & 1 & 1 & \dots & b_q & d_q \end{pmatrix}. \tag{9}$$

Next we define a family of idempotents which, together with $R_{m,n}$, will be used to generate S:

$$\begin{aligned} \sigma_u &= (\sigma_u^{(l)}, \sigma_u^{(r)})\ (u=2,\dots,m),\\ \tau_u &= (\tau_u^{(l)}, \tau_u^{(r)})\ (u=1,\dots,q), \end{aligned} \tag{10}$$

where

$$\sigma_u^{(l)}(x) = \begin{cases} 1 & \text{if } x=1\\ u & \text{if } x\neq 1 \end{cases} \tag{11}$$

$$x\sigma_u^{(r)} = \begin{cases} x & \text{if } 1\leq x\leq p+1\\ r+1 & \text{if } x>p+1 \text{ and } y_{u,x}=a_r \end{cases} \tag{12}$$

(with the convention that $a_0=1$), and

$$\tau_u^{(l)}(x) = \begin{cases} 2u+1 & \text{if } x=2u+1\\ 2u & \text{if } x\neq 2u+1 \end{cases} \tag{13}$$

$$x\tau_u^{(r)} = \begin{cases} p+2u & \text{if } x=p+2u, p+2u+1\\ 1 & \text{otherwise.} \end{cases} \tag{14}$$

Let us interrupt our exposition at this point in order to illustrate the construction thus far by means of a concrete example. Let us take G to be the Klein bottle group

$$G=\langle a,b \mid a^{-1}ba=b^{-1}\rangle.$$

By introducing a redundant generator $c=ba$ we obtain the following presentation

$$G=\langle a,b,c \mid ba=c,\ cb=a\rangle,$$

which has the form (7). Thus here $p=3$, $q=2$, $m=5$, $n=8$, and the auxiliary matrix is

$$Y=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\ 1 & a & b & c & 1 & a & 1 & 1\\ 1 & a & b & c & b & c & 1 & 1\\ 1 & a & b & c & 1 & 1 & 1 & b\\ 1 & a & b & c & 1 & 1 & c & a \end{pmatrix}.$$

The only definition from (11)–(14) that is perhaps not entirely straightforward is (12). So let us compute $\sigma_3^{(r)}$. Its image is $\{1,2,3,4\}$, and it acts identically on it. Each of the remaining columns is mapped into that image column which has the same 3rd entry. The 3rd entries for columns 5, 6, 7, 8 are b, c, 1, 1 respectively, and so 5, 6, 7, 8 are mapped to 3, 4, 1, 1 respectively.

The complete list of our idempotents is:

$$\sigma_2^{(l)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 2 & 2 & 2 \end{pmatrix} \qquad \sigma_2^{(r)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 1 & 2 & 1 & 1 \end{pmatrix}$$
$$\sigma_3^{(l)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 3 & 3 & 3 \end{pmatrix} \qquad \sigma_3^{(r)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 3 & 4 & 1 & 1 \end{pmatrix}$$
$$\sigma_4^{(l)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 4 & 4 & 4 \end{pmatrix} \qquad \sigma_4^{(r)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 1 & 1 & 1 & 3 \end{pmatrix}$$
$$\sigma_5^{(l)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 5 & 5 & 5 \end{pmatrix} \qquad \sigma_5^{(r)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 1 & 1 & 4 & 2 \end{pmatrix}$$
$$\tau_1^{(l)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 2 & 3 & 2 & 2 \end{pmatrix} \qquad \tau_1^{(r)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 1 & 5 & 5 & 1 & 1 \end{pmatrix}$$
$$\tau_2^{(l)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 4 & 4 & 4 & 5 \end{pmatrix} \qquad \tau_2^{(r)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 7 & 7 \end{pmatrix}.$$

Returning to our general argument, let S be the subsemigroup of $B_{m,n}$ generated by the idempotents σ_u $(u = 2,\dots,m)$, τ_u $(u = 1,\dots,q)$ and $R_{m,n}$. We start working towards accomplishing (Rel1) (see Section 2) by identifying the squares singularised by the idempotent σ_u $(u = 2,\dots,m)$ and the relations these squares yield. Beginning with the left-right squares, notice that $\mathrm{im}(\sigma_u^{(l)}) = \{1,u\}$, and so a typical square singularised has the form $(1,u;r+1,l)$ where $0 \le r \le p$, $l > p+1$ and $y_{u,l} = a_r$. This is a flush top square, and yields the relation

$$f_{ul} = f_{u,r+1} \; (2 \le u \le m,\ p+1 < l < n,\ 0 \le r \le p,\ y_{ul} = a_r). \tag{15}$$

The up-down squares singularised by σ_2 are of the form $(2,i;j,l)$ $(2 < i \le m,\ 1 \le j < l \le p+1)$. For $j = 1$ we obtain a flush left square $(2,i;1,l)$, which yields the relation

$$f_{il} = f_{2l} \; (2 < i \le m,\ 1 < l \le p+1). \tag{16}$$

A general up-down square singularised by any σ_u has the form $(u,i;j,l)$ $(2 \le i \le m,\ i \ne u,\ 1 \le j < l \le p+1)$ and the relation it yields is an easy consequence of (16):

$$f_{uj}^{-1} f_{ul} = f_{2j}^{-1} f_{2l} = f_{ij}^{-1} f_{il}.$$

We see that the squares singularised by σ_u ($2 \leq u \leq m$) allow us precisely to deduce (Rel1): A typical generator f_{ij} is equal to an appropriate f_{i,j_1} (with $1 \leq j_1 \leq p+1$) by (15), which in turn is equal to its 'canonical representative' f_{2,j_1} by (16). So from now on we identify each generator f_{ij} with y_{ij}.

Let us now examine the squares singularised by τ_u ($u = 1, \dots, q$). The most important such square is $(2u, 2u+1; p+2u, p+2u+1)$. It is a left-right 3/4 square, since under our identification $f_{2u,p+2u} = y_{2u,p+2u} = 1$, and yields the relation

$$b_u c_u = d_u \ (u = 1, \dots, q). \tag{17}$$

Every other left-right square for τ_u is flush left $(2u, 2u+1; 1, j)$ ($j \neq 1, 2u, 2u+1$), yielding either $a_{j-1} = a_{j-1}$ (for $j \leq p+1$) or $1 = 1$ (for $j > p+1$). The up-down squares have the form $(2u, i; 1, p+2u)$ ($i \neq 2u+1$), and yield the trivial relation $1 = 1$.

At this stage we have accomplished step (Rel2), and have precisely a presentation defining G. Furthermore we have examined all the squares singularised by $\sigma_2, \dots, \sigma_m, \tau_1, \dots, \tau_q$. So, to verify (Rel3), i.e. prove that no further relations are introduced, it is sufficient to prove that S contains no further idempotents:

$$E(S) = R_{m,n} \cup \{\sigma_2, \dots, \sigma_m, \tau_1, \dots, \tau_q\}. \tag{18}$$

In order to prove (18), let us examine products of σs and τs of length 2. Clearly, $\sigma_u^{(l)}\sigma_v^{(l)} = \sigma_u^{(l)}$ for any $u, v \in \{2, \dots, m\}$. Next note that $\mathrm{im}(\sigma_u^{(r)}) = \{1, \dots, p+1\}$, and that $\sigma_v^{(r)}$ acts identically on it. Hence $\sigma_u^{(r)}\sigma_v^{(r)} = \sigma_u^{(r)}$, and we conclude that

$$\sigma_u \sigma_v = \sigma_u. \tag{19}$$

Let us now examine the product $\tau_u \tau_v$ ($1 \leq u, v \leq q$). If $u = v$ then of course $\tau_u \tau_v = \tau_u$; so let us suppose $u \neq v$. Since $\mathrm{im}(\tau_v^{(l)}) = \{2v, 2v+1\}$, and both these points are mapped to $2u$ by $\tau_u^{(l)}$, it follows that $\tau_u^{(l)}\tau_v^{(l)} = \rho_{2u}$, the constant mapping with value $2u$. A similar argument shows that $\tau_u^{(r)}\tau_v^{(r)} = \rho_1$, the constant 1. Hence

$$\tau_u \tau_v = \begin{cases} \tau_u & \text{if } u = v \\ \rho_{2u,1} & \text{if } u \neq v. \end{cases} \tag{20}$$

A very similar argument shows that

$$\sigma_u \tau_v = \rho_{u,1}. \tag{21}$$

Finally, let us consider the product $\tau_u \sigma_v$. On the left we have

$$\tau_u^{(l)}\sigma_v^{(l)} = \begin{cases} \rho_{2u} & \text{if } v \neq 2u+1 \\ \mu & \text{if } v = 2u+1, \end{cases}$$

where

$$\mu(x) = \begin{cases} 2u & \text{if } x = 1 \\ 2u+1 & \text{if } x \neq 1. \end{cases}$$

Note that μ is not an idempotent, since

$$\mu(\mu(1)) = 2u + 1 \neq 2u = \mu(1).$$

On the right $\tau_u^{(r)}\sigma_v^{(r)}$ is either constant ρ_1, or else it maps $\{p+2u, p+2u+1\}$ onto some $r \in \{2, \dots, p+1\}$, and everything else, including r, to 1, in which case it is not an idempotent. So we conclude that

$$\tau_u\sigma_v = \rho_{2u,1} \in R_{m,n} \text{ or } \tau_u\sigma_v \notin E(S). \tag{22}$$

From (19)–(22) we see that the only products of length 2 of generators of S that are not in $R_{m,n} \cup \{\sigma_2, \dots, \sigma_m, \tau_1, \dots, \tau_q\}$ are some of $\tau_u\sigma_v$,and that these products are not idempotent. It is now straightforward to see that no product of length greater than 2 is going to produce any further new elements of S, and hence that (18) indeed holds. This completes the proof of (Rel3), and we can conclude that

$$\mathsf{H}(\mathsf{IG}(E(S)), \rho_{11}) \cong G,$$

proving Theorem 2.

Theorem 1 is proved in exactly the same way. We just omit the assumption that G be finitely presented, at the expense of allowing the presentation (7) to become infinite. This in turn makes the index sets I and J, the family of idempotents σ_u, τ_v, and ultimately the semigroup S, infinite. Alternatively, Theorem 1 can be deduced as a corollary of Theorem 3, which will be proved in the next section.

4 Second construction

The semigroup S constructed in Section 3 is not regular: the non-constant products $\tau_u\sigma_v$ are easily seen to be non-regular. Furthermore, there does not exist a regular semigroup S' with the same set of idempotents as S. Indeed, for τ_u and σ_v as above, the sandwich set $S(\tau_u, \sigma_v)$ is easily seen to be empty, implying that $E(S)$ is not a regular biordered set. So, in order to prove Theorems 3 and 4 we introduce a new construction.

This construction has two advantages over that described in the previous section: it is in a sense more compact, and it always yields a regular semigroup. Its main disadvantage is that the constructed semigroup is finite if and only if the input group is finite.

Let G be an arbitrary group, let N be its (possibly infinite) order, and let $n = N^2$. We will work inside the semigroup $B_{3,n} = T_3^{(l)} \times T_n^{(r)}$, which has the $3 \times n$ rectangular band $R_{3,n}$ as its minimal ideal. The corresponding generators for the maximal subgroup containing ρ_{11}, as introduced in Section 2, are

$$f_{ij} \ (i = 1, 2, 3; j = 1, \ldots, n) \tag{23}$$

with

$$f_{1j} = f_{i1} = 1 \ (i = 1, 2, 3; j = 1, \ldots, n). \tag{24}$$

The auxiliary matrix this time is

$$Y = (y_{ij})_{3\times n} = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & y_{22} & y_{23} & \ldots & y_{2n} \\ 1 & y_{32} & y_{33} & \ldots & y_{3n} \end{pmatrix}.$$

Its entries are the elements of G, arranged arbitrarily subject to the condition that every possible column appears (once and only once):

$$\{(1, y_{2j}, y_{3j}) \ : \ j = 1, \ldots, n\} = \{(1, g, h) \ : \ g, h \in G\}. \tag{25}$$

In fact, in what follows we will sometimes identify the index set $J = \{1, \ldots, n\}$ and the set $\{(1, g, h) \ : \ g, h \in G\}$ of all columns of Y. When we do want to distinguish between the two sets we will write Y_j for the jth column: $Y_j = (1, y_{2j}, y_{3j})$. The index set I is, of course, just $\{1, 2, 3\}$.

Now we proceed to define our extra idempotents. There are six of them

$$\sigma_u = (\sigma_u^{(l)}, \sigma_u^{(r)}) \ (u = 1, \ldots, 6), \tag{26}$$

and are given by

$$\begin{aligned}
\sigma_1^{(l)} &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix} & \sigma_1^{(r)} &: (1, g, h) \mapsto (1, g, g) \\
\sigma_2^{(l)} &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} & \sigma_2^{(r)} &: (1, g, h) \mapsto (1, g, 1) \\
\sigma_3^{(l)} &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix} & \sigma_3^{(r)} &: (1, g, h) \mapsto (1, 1, h) \\
\sigma_4^{(l)} &= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{pmatrix} & \sigma_4^{(r)} &: (1, g, h) \mapsto (1, h, h) \\
\sigma_5^{(l)} &= \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix} & \sigma_5^{(r)} &: (1, g, h) \mapsto (1, 1, hg^{-1}) \\
\sigma_6^{(l)} &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix} & \sigma_6^{(r)} &: (1, g, h) \mapsto (1, gh^{-1}, 1).
\end{aligned}$$

Note that only $\sigma_5^{(r)}$ and $\sigma_6^{(r)}$ actually depend on the group G.

Let us illustrate this with an example. We take

$$G = K_4 = \langle a, b \mid a^2 = b^2 = 1, \ ab = ba \rangle = \{1, a, b, c\},$$

the Klein four-group. The index sets are $I = \{1,2,3\}$ and $J = \{1,\ldots,16\}$, and the auxiliary matrix is

$$Y = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & a & a & a & a & b & b & b & b & c & c & c & c \\ 1 & a & b & c & 1 & a & b & c & 1 & a & b & c & 1 & a & b & c \end{pmatrix}.$$

If we want to compute, say, $(12)\sigma_6^{(r)}$ we proceed as follows. Take the 12th column of Y – that is $(1,b,c)$. Transforming it via $(1,g,h) \mapsto (1,gh^{-1},1)$ yields $(1,a,1)$, which is column 5. Thus $(12)\sigma_6^{(r)} = 5$.

The full list of all the mappings is:

$$\sigma_1^{(l)} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix} \quad \sigma_1^{(r)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 1 & 1 & 1 & 6 & 6 & 6 & 6 & 11 & 11 & 11 & 11 & 16 & 16 & 16 & 16 \end{pmatrix}$$

$$\sigma_2^{(l)} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} \quad \sigma_2^{(r)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 1 & 1 & 1 & 5 & 5 & 5 & 5 & 9 & 9 & 9 & 9 & 13 & 13 & 13 & 13 \end{pmatrix}$$

$$\sigma_3^{(l)} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix} \quad \sigma_3^{(r)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\sigma_4^{(l)} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{pmatrix} \quad \sigma_4^{(r)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 6 & 11 & 16 & 1 & 6 & 11 & 16 & 1 & 6 & 11 & 16 & 1 & 6 & 11 & 16 \end{pmatrix}$$

$$\sigma_5^{(l)} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix} \quad \sigma_5^{(r)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 2 & 3 & 4 & 2 & 1 & 4 & 3 & 3 & 4 & 1 & 2 & 4 & 3 & 2 & 1 \end{pmatrix}$$

$$\sigma_6^{(l)} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix} \quad \sigma_6^{(r)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 5 & 9 & 13 & 5 & 1 & 13 & 9 & 9 & 13 & 1 & 5 & 13 & 9 & 5 & 1 \end{pmatrix}.$$

If we were to change the group G, say to $G = C_4 = \langle a \mid a^4 = 1 \rangle = \{1, a, a^2, a^3\} = \{1, a, b, c\}$, the cyclic group of order 4, only the mappings $\sigma_5^{(r)}$ and $\sigma_6^{(r)}$ would change:

$$\sigma_5^{(r)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 2 & 3 & 4 & 4 & 1 & 2 & 3 & 3 & 4 & 1 & 2 & 2 & 3 & 4 & 1 \end{pmatrix}$$

$$\sigma_6^{(r)} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 13 & 9 & 5 & 5 & 1 & 13 & 9 & 9 & 5 & 1 & 13 & 13 & 9 & 5 & 1 \end{pmatrix}.$$

We now return to the main argument. A routine verification shows that the semigroup generated by $\{\sigma_u \ : \ u = 1,\ldots,6\}$ has 21 elements: σ_u $(u = 1,\ldots,6)$ themselves, three elements ρ_{11}, ρ_{21}, ρ_{31} belonging to the minimal ideal $R_{3,n}$, and twelve further elements:

$$
\begin{aligned}
\sigma_7 &= \sigma_1\sigma_3 &&= (\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}, (1,g,h) \mapsto (1,1,g)) \\
\sigma_8 &= \sigma_2\sigma_5 &&= (\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix}, (1,g,h) \mapsto (1,1,g^{-1})) \\
\sigma_9 &= \sigma_3\sigma_6 &&= (\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 3 \end{pmatrix}, (1,g,h) \mapsto (1,h^{-1},1)) \\
\sigma_{10} &= \sigma_4\sigma_2 &&= (\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \end{pmatrix}, (1,g,h) \mapsto (1,h,1)) \\
\sigma_{11} &= \sigma_5\sigma_4 &&= (\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}, (1,g,h) \mapsto (1,hg^{-1},hg^{-1})) \\
\sigma_{12} &= \sigma_6\sigma_1 &&= (\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \end{pmatrix}, (1,g,h) \mapsto (1,gh^{-1},gh^{-1})) \\
\sigma_{13} &= \sigma_1\sigma_3\sigma_6 &&= (\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix}, (1,g,h) \mapsto (1,g^{-1},1)) \\
\sigma_{14} &= \sigma_2\sigma_5\sigma_4 &&= (\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}, (1,g,h) \mapsto (1,g^{-1},g^{-1})) \\
\sigma_{15} &= \sigma_3\sigma_6\sigma_1 &&= (\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}, (1,g,h) \mapsto (1,h^{-1},h^{-1})) \\
\sigma_{16} &= \sigma_4\sigma_2\sigma_5 &&= (\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \end{pmatrix}, (1,g,h) \mapsto (1,1,h^{-1})) \\
\sigma_{17} &= \sigma_5\sigma_4\sigma_2 &&= (\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \end{pmatrix}, (1,g,h) \mapsto (1,hg^{-1},1)) \\
\sigma_{18} &= \sigma_6\sigma_1\sigma_3 &&= (\begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}, (1,g,h) \mapsto (1,1,gh^{-1})).
\end{aligned}
$$

The elements $\{\sigma_1, \ldots, \sigma_{18}\}$ form a single $\mathscr{D}$-class D illustrated in Figure 2.

It follows that the semigroup

$$S = \langle R_{3,n} \cup \{\sigma_1, \ldots, \sigma_6\} \rangle = R_{3,n} \cup D \leq B_{3,n}$$

has the following properties:

- S is regular;
- S has two $\mathscr{D}$-classes: $R_{3,n}$ and D;
- S has precisely six idempotents ($\sigma_1, \ldots, \sigma_6$) outside $R_{3,n}$;
- S is finite if and only if $R_{3,n}$ is finite, which is the case if and only if G is finite.

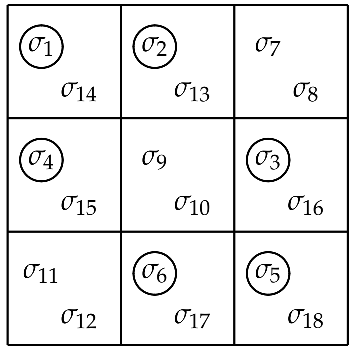

Figure 2: The $\mathscr{D}$-class D, with the idempotents circled.

We now follow the process outlined in Section 2 to prove that the maximal subgroup of $\mathsf{IG}(E(S))$ containing $\rho_{11} \in R_{3,n}$ is isomorphic to G. So far we have got the generators (23) and relations (24). Any further relations arise from the squares singularised by σ_u, $u = 1, \ldots, 6$.

Since $\mathrm{im}(\sigma_1^{(l)}) = \{1, 2\}$ it follows that the left-right squares singularised by σ_1 are of the form $(1, 2; j, l)$, where $Y_j = (1, g, g)$, $Y_l = (1, g, h)$ for some $g, h \in G$. These are flush top squares yielding the relations

$$f_{2j} = f_{2l} \text{ (whenever } y_{2j} = y_{2l}). \tag{27}$$

The left-right squares singularised by σ_2 yield exactly the same relations, while those singularised by σ_3 and σ_4 yield

$$f_{3j} = f_{3l} \text{ (whenever } y_{3j} = y_{3l}). \tag{28}$$

Next note that $\mathrm{im}(\sigma_1^{(r)}) = \{(1, g, g) \,:\, g \in G\}$; so the up-down squares singularised by σ_1 have the form $(2, 3; j, l)$, where $Y_j = (1, g, g)$, $Y_l = (1, h, h)$. For $j = 1$ we obtain the flush left square $(2, 3; 1, l)$, yielding the relation

$$f_{2l} = f_{3l} \text{ (whenever } y_{2l} = y_{3l}). \tag{29}$$

The relation $f_{2j}^{-1} f_{2l} = f_{3j}^{-1} f_{3l}$ produced from a general square $(2, 3; j, l)$ is an easy consequence of (29).

Combining (27), (28), (29) together gives us precisely the relations

$$f_{ij} = f_{kl} \text{ (whenever } y_{ij} = y_{kl}), \tag{30}$$

i.e. we have completed step (Rel1), and can identify each generator f_{ij} with the entry y_{ij} of the auxiliary matrix (considered as a formal symbol).

The up-down squares singularised by σ_2, σ_3, σ_4 do not yield any relations over and above (30). So there remains to analyse the relations yielded by σ_5 and σ_6. From $\mathrm{im}(\sigma_5^{(l)}) = \{2,3\}$ it follows that the left-right squares singularised by σ_5 are of the form $(2,3;j,l)$, where $Y_j = (1,1,hg^{-1})$ and $Y_l = (1,g,h)$. This is a 3/4 square yielding the relation $(hg^{-1}) \cdot g = h$. Clearly, as g and h range through G, we obtain the Cayley table of G (considered as a presentation), accomplishing (Rel2). The up-down squares for σ_5 have the form $(2,1;j,l)$, where $Z_j = (1,1,g)$ and $Z_l = (1,1,h)$, and yield the trivial relation $1 = 1$. Likewise, σ_6 yields no further relation. Since S has no further idempotents, we have accomplished (Rel3). This completes the proof of Theorems 3 and 4.

5 Concluding remarks

During the work on this project we have implemented in GAP [3] the Reidemeister–Schreier type rewriting process mentioned in Sections 1 and 2. This has enabled us to gather considerable 'experimental data' and test several early conjectures. The output from any Reidemeister–Schreier type rewriting has a large number of generators and defining relations. Thus, the Tietze Transformations programme, which is a part of the standard GAP distribution, and which is in the GAP manual credited back to the work of Havas, Robertson et al. [7, 8, 17], proved an invaluable tool.

It is well known that there exists a finitely presented group with an unsolvable word problem. Combining such a group with Theorem 2 yields:

Corollary. *There exists a free idempotent generated semigroup F arising from a finite semigroup such that the word problem for F is unsolvable.*

Such a free idempotent generated semigroup F would be non residually finite and non-automatic as well.

The main open question remaining, as mentioned in the Introduction, is:

Problem 1. Is it true that every finitely presented group is a maximal subgroup of some free regular idempotent generated semigroup arising from a finite semigroup?

All our examples were obtained by first fixing a rectangular band R, and then constructing our semigroup S as an ideal extension of R. We remark that for any fixed finite R we can obtain only finitely many maximal subgroups in free idempotent semigroups arising from ideal extensions from R. Indeed, R determines the generators of the maximal subgroups, and the relators arise from singular squares, and there are only finitely many of them. Thus we are lead to ask:

Problem 2. Given a finite 0-simple semigroup R and and idempotent $e \in R$, describe all the groups which arise as maximal subgroups containing e of the free idempotent semigroups $\mathsf{IG}(S)$ where S is a finite ideal extension of R.

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