Equidistant frequency permutation arrays and related constant composition codes

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Abstract

This paper introduces and studies the notion of an equidistant frequency permutation array (EFPA). An EFPA of length $n = m\lambda$, distance d and size v is defined to be a $v \times n$ array T such that 1) each row is a multipermutation on a multiset of m symbols, each repeated with frequency λ, and 2) the Hamming distance between any two distinct rows of T is precisely d. Such an array generalizes the well-studied equidistant permutation array: it corresponds to a special kind of constant composition code (CCC). Bounds and constructions are obtained for EFPAs and related CCCs (many optimal), and links with other combinatorial structures are explored.

Keywords: permutation arrays, constant composition codes, combinatorial designs

MSC codes: 05B15, 94B25

1 Introduction

Permutation arrays (PAs), and their frequency generalization, are of great interest in combinatorics and coding theory (see for example [3]). One kind of permutation array of special interest is the equidistant permutation array; much work has been

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done to obtain bounds and constructions for such objects (see [6] and [14]). Frequency permutation arrays, generalizations of PAs in which each symbol occurs the same number $k \geq 1$ of times in each row (introduced in [10]) may be considered as a special kind of constant composition code, and here also the notion of an equidistant code is important. For example, it is known that any constant composition code which attains the non-recursive Johnson bound (see [12]) must be an equidistant code. For this reason, a study of equidistant frequency permutation arrays and related codes seems appropriate.

2 Equidistant frequency permutation arrays

Take an alphabet S of size $m \in \mathbb{N}$: in general we shall use $S = \{0, 1, \ldots, m-1\}$ but in certain situations $S = \{1, 2, \ldots, m\}$ may be used instead. Let $\lambda \in \mathbb{N}$ and set $n = m\lambda$. We define a *λ-permutation* to be a word $w_1 w_2 \ldots w_n$ of length n, where the symbols w_i are from the alphabet S and w has precisely λ occurrences of each symbol of S .

When $\lambda = 1$, the set of all λ-permutations is the symmetric group S_n of permutations on n symbols.

Definition 2.1. *Two distinct λ-permutations $\sigma = s_1 \ldots s_n$, $\tau = t_1 \ldots t_n$ have* distance *$d(\sigma, \tau) = d$ if they disagree in d entries, i.e. if $|\{i : s_i \neq t_i\}| = d$.*

This is the Hamming distance familiar from coding theory. In the case when $\lambda = 1$, two permutations $\sigma, \tau \in S_n$ have distance d if $\sigma\tau^{-1}$ has exactly $n - d$ fixed points.

The following definition is well-known.

Definition 2.2. *Let S be a non-empty set of cardinality n. A* permutation array *of length n, minimum distance d and size v, defined on the elements of S, is a $v \times n$ array satisfying the following properties:*

(1) each row is a permutation of the symbols of S;

(2) any two rows agree in at most $n - d$ columns (i.e. the Hamming distance between any two rows is at least d).

Such a PA will be denoted $PA(n, d)$.

If property (2) is replaced by property (2):*

(2) any two rows agree in precisely $n - d$ columns (i.e. the Hamming distance between any two rows is d)*

then the permutation array is said to be an equidistant *permutation array (which will be denoted $EPA(n, d)$).*

In [10], the following definition was introduced.

Definition 2.3. *Let $n = m\lambda$ and let S be a non-empty set of cardinality m. A* frequency permutation array *of length n, frequency λ, minimum distance d and size v, defined on the elements of S, is a $v \times n$ array satisfying the following properties:*

(1) each row is a λ-permutation of the symbols of S;

(2) any two rows agree in at most $n - d$ columns (i.e. the Hamming distance between any two rows is at least d).

Such an object will be denoted $FPA_\lambda(n, d)$.

We will define an equidistant FPA in the natural way, by replacing property (2) of Definition 2.3 by property (2*) from Definition 2.2.

Definition 2.4. *Let $n = m\lambda$ and let S be a non-empty set of cardinality m. An* equidistant frequency permutation array *of length n, frequency λ, distance d and size v, defined on the elements of S, is a $v \times n$ array satisfying the following properties:*

(1) each row is a λ*-permutation of the symbols of* S*;*

(2) any two rows agree in precisely* $n-d$ *columns (i.e. the Hamming distance between any two rows is precisely* d*).*

Such an object will be denoted $EFPA_\lambda(n,d)$*.*

Thus an $EFPA_1(n,d)$ is simply an $EPA(n,d)$. We let $B_\lambda(n,d)$ denote the maximum possible number of rows that can exist in any $EFPA_\lambda(n,d)$.

Example 2.5. *An* $EFPA_3(9,4)$ *of size* 6 *is given by*

$$\begin{array}{ccccccccc}
0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\
0 & 0 & 0 & 1 & 2 & 2 & 1 & 1 & 2 \\
0 & 0 & 2 & 1 & 1 & 2 & 1 & 2 & 0 \\
0 & 0 & 2 & 1 & 2 & 1 & 2 & 1 & 0 \\
0 & 2 & 0 & 1 & 1 & 2 & 2 & 1 & 0 \\
0 & 2 & 0 & 1 & 2 & 1 & 1 & 2 & 0
\end{array}$$

We begin by establishing properties of equidistant frequency permutation arrays for special values of d and m.

Proposition 2.6. *Let* $n = m\lambda$*. Then*

(i) $B_\lambda(n,d) = 1$ *if* $m = 2$ *and* d *is odd;*

(ii) $B_\lambda(n,n) = m$*;*

(iii) $B_\lambda(n,2) = \lambda + 1$.

Proof. In this proof, all EFPAs will be taken on alphabets $\{0,1,\ldots,m-1\}$.

(i) Let $m = 2$ (i.e. n even and $\lambda = \frac{n}{2}$) and let ρ be a binary word of length n with $\frac{n}{2}$ 0's and $\frac{n}{2}$ 1's. Any other binary word ρ_1 of length n with $\frac{n}{2}$ 0's and $\frac{n}{2}$ 1's must be obtainable from ρ by converting k 0's to 1's and k 1's to 0's, for some integer $1 \le k \le \frac{n}{2}$. Hence the distance d between ρ and ρ_1 must be $2k$ for some k.

(ii) Since there are at most m choices for the symbol in the first position of a λ-permutation in an $EFPA_\lambda(n,n)$, we have $B_\lambda(n,n) \leq m$. Take m blocks comprising λ copies of each symbol: $\{0\ldots0\}$, $\{1,\ldots,1\}$, $\ldots$, $\{m-1,\ldots,m-1\}$; applying an m-cycle to these blocks yields m λ-permutations, all of pairwise distance n.

(iii) Let A be an $EFPA_\lambda(n,2)$ of largest possible size (> 2). We may assume that the first row ρ_1 of A is the "standard" λ-permutation

$$0\ldots0\,1\ldots1\ldots(m-1)\ldots(m-1)$$

(otherwise columns may be permuted to achieve this without altering distance) and we shall refer to positions whose entries agree with those of ρ_1 as "fixed positions". Let ρ_2 be the second row of A; two entries of ρ_1 (distinct symbols) must have been swapped to obtain ρ_2. Any other row, ρ, must also be derived from ρ_1 by swapping two distinct symbols. Since ρ has distance 2 from ρ_2, then ρ must share one non-fixed position with ρ_2 and their other non-fixed positions must differ. Moreover, ρ_2 and ρ must contain the same symbol in their shared non-fixed position, which means that the other non-fixed position of ρ can occur only amongst the $\lambda - 1$ remaining positions which contain this symbol in ρ_1. Hence $B_\lambda(n,2) \leq \lambda + 1$. Since this bound can be attained by taking ρ_1 plus the λ multipermutations derived from ρ_1 by swapping (say) a fixed occurence of 0 with each occurence of 1 in turn, we have $B_\lambda(n,2) = \lambda + 1$. □

Theorem 2.7. *For $m > 2$, $B_\lambda(n,3) = \max\{3, m-1\}$.*

Proof. With $m \geq 3$, let A be an $EFPA_\lambda(n,3)$ of largest possible size on alphabet $\{0,1,\ldots,m-1\}$. Let ρ_1 be the "standard" λ-permutation written (with additional subscripts) as $0_1\ldots0_\lambda 1_0\ldots1_\lambda\ldots(m-1)_1\ldots(m-1)_\lambda$ where we refer to each $i_1\ldots i_\lambda$ as a "block" in ρ_1. Let ρ_2 be the second row of A; then it must be derived from ρ_1 by permuting three symbols from different blocks in a 3-cycle. It is immediately clear that taking all three rotations of this 3-cycle yields an $EFPA_\lambda(n,3)$ of size 3 (and that this is the largest possible array such that two non-identity rows have the same set of non-fixed positions). Now let ρ be any other row which does not have the same non-fixed positions as ρ_2. Since ρ_2 and ρ have distance 3, they must share two of their non-fixed positions. Of these, they must agree on one position and disagree on the other, forcing the third position of each to correspond to a different block in ρ_1 (different also from those corresponding to the two shared entries). Hence, given a non-identity row ρ_2, there are at most $m-3$ choices for the block corresponding to the third non-fixed position of any other non-identity row, giving an upper bound of $m-1$ for the size of the EFPA (position within a block is not relevant here). Since an EFPA of size $m-1$ may be constructed by taking ρ_1 plus (say) the $m-2$ multipermutations derived from ρ_1 by applying the 3-cycle $(0_1 1_1 a_1)$, $(a = 2, \ldots, m)$, an array of size $m-1$ can always be obtained. □

Theorem 2.8. *For $n \geq 4$, $B_\lambda(n,4) \geq \lfloor \frac{n}{2} \rfloor$.*

Proof. An $EFPA_\lambda(n,4)$ of size $\lfloor \frac{n}{2} \rfloor$ can be constructed on the alphabet $\{0,1,\ldots,m-1\}$ as follows. Let ρ_1 be the "standard" λ-permutation, written (with additional subscripts) as $0_1 \ldots 0_\lambda 1_1 \ldots 1_\lambda \ldots (m-1)_1 \ldots (m-1)_\lambda$. Let A be the set of all "pairs of transpositions" of the form

$$\{(0_1 1_1)(a_i b_j)\}$$

where the set $\{(a_i, b_j)\}$ comprises $\lfloor \frac{n-2}{2} \rfloor$ pairs from the set of positions

$$\{0_2, \ldots, 0_\lambda, 1_2, \ldots, 1_\lambda, 2_1, \ldots, 2_\lambda, \ldots, (m-1)_1, \ldots, (m-1)_\lambda\}$$

such that all pairs are disjoint and in each pair, a and b are distinct symbols from the alphabet. (By "pairs of transpositions" we mean that λ-permutation $(0_1 1_1)(a_i b_j)$ (written with extra subscripts) is derived from ρ_1 by swapping 0_1 with 1_1 and a_i with b_j in ρ_1.) Then $\rho_1 \cup A$ is the desired EFPA. We show that it is always possible to obtain $\lfloor \frac{n-2}{2} \rfloor$ such pairs.

- For m even, form $(\lambda-1)$ pairs $(0_2, 1_2), \ldots, (0_\lambda, 1_\lambda)$, then (if $m > 2$) form $\frac{(m-2)\lambda}{2}$ pairs $(2_1, 3_1), \ldots, ((m-2)_\lambda, (m-1)_\lambda)$.
- For m odd and λ even, form
  - $\frac{\lambda}{2}$ pairs $(0_i, 2_j)$ ($2 \le i \le \frac{\lambda}{2}+1$ and $1 \le j \le \frac{\lambda}{2}$),
  - $\frac{\lambda}{2}$ pairs $(1_i, 2_j)$ ($2 \le i \le \frac{\lambda}{2}+1$ and $\frac{\lambda}{2}+1 \le j \le \frac{\lambda}{2}$),
  - $\frac{\lambda}{2}-1$ pairs $(0_j, 1_j)$ ($\frac{\lambda}{2}+2 \le j \le \frac{\lambda}{2}$)
  - If $m > 3$, form $\frac{(m-3)\lambda}{2}$ pairs $(2_1, 3_1), \ldots, ((m-2)_\lambda, (m-1)_\lambda)$.
- For m odd and λ odd, the construction is analogous to the previous construction except that there are $\lceil \frac{\lambda}{2} \rceil$ pairs of the form $(0_i, 2_j)$, $\lfloor \frac{\lambda}{2} \rfloor$ pairs of the form $(1_i, 2_j)$, and $\lfloor \frac{\lambda}{2} \rfloor - 1$ pairs of the form $(0_i, 1_j)$.

□

For small values of n, it is possible to construct $EFPA_\lambda(n, 4)$s of size larger than $\lfloor \frac{n}{2} \rfloor$; for example $B_\lambda(n, 4) \ge 7$ for $\lambda \ge 4$ and $m \ge 2$, since an $EFPA_4(8, 4)$ of size 7 may be built from a Hadamard matrix. However, we make the following conjecture:

Conjecture 2.9. *For $n \ge 14$, $B_\lambda(n, 4) = \lfloor \frac{n}{2} \rfloor$.*

Observe that the preceding results generalize those known for EPAs. Let $R(r, \lambda)$ denote the maximum size of an EPA on r symbols whose rows have pairwise agreement in λ positions. It is known (see [5]) that $R(r, r-2) = 2$; for $r > 3$, $R(r, r-3) = r-1$ and for $r > 9$, $R(r, r-4) = \lfloor \frac{r}{2} \rfloor$.

Example 2.10. *An $EFPA_3(12, 8)$ of size 6 arising from the above construction is*

ρ_0	0	0	0	1	1	1	2	2	2	3	3	3
ρ_1	1	1	0	0	0	1	2	2	2	3	3	3
ρ_2	1	0	1	0	1	0	2	2	2	3	3	3
ρ_3	1	0	0	0	1	1	3	2	2	2	3	3
ρ_4	1	0	0	0	1	1	2	3	2	3	2	3
ρ_5	1	0	0	0	1	1	2	2	3	3	3	2

3 Relations between EFPAs and other constant composition codes

It is clear that a permutation array, or frequency permutation array, may be considered as a type of constant composition code, with the rows of the array corresponding to the codewords. An $EFPA_\lambda(n, d)$ may be considered as a special case of an equidistant constant composition code.

Definition 3.1. *Let C be a k-ary code of length n and minimum distance d on the alphabet $\{0, 1, \ldots, k-1\}$. The code C has constant weight composition $[n_0, n_1, \ldots, n_k]$ if every codeword has n_i occurrences of symbol i for $i = 0, 1, \ldots, k-1$. We refer to C as a constant composition code, written $CCC(n, d, [n_0, n_1, \ldots, n_{k-1}])$ (here $n = \sum n_i$). Such a code is said to be equidistant if the distance between any two codewords is precisely d.*

Hence an $EFPA_\lambda(n, d)$ (with $n = m\lambda$) corresponds to an equidistant m-ary $CCC(n, d, [\lambda, \lambda, \ldots, \lambda])$. For more details about constant composition codes in general, see for example [4], [12] and [7].

A useful upper bound for the maximum size of general constant composition codes (CCCs) first appeared in [12]. It is called the non-recursive Johnson bound. Observe that, for a CCC in which all symbols of a codeword occur with equal frequency λ (corresponding to an FPA), this upper bound reduces to the Plotkin bound. An important observation here is that any constant composition code which satisfies the non-recursive Johnson bound must in fact be equidistant.

Proposition 3.2.

- *Denote by $A_k(n, d, [n_0, n_1, \ldots, n_{k-1}])$ the maximum possible size of a k-ary $CCC(n, d, [n_0, n_1, \ldots, n_{k-1}])$. Then (if the denominator is positive) we have the following bound:*

$$A_k(n, d, [n_0, n_1, \ldots, n_{k-1}]) \leq \frac{nd}{nd - n^2 + (n_0^2 + n_1^2 + \cdots + n_{k-1}^2)}$$

- *For $d > n - \lambda$,*

$$M_\lambda(n,d) \leq \frac{d}{d-n+\lambda}.$$

The following upper bound for the distance of any equidistant code is known (see [5] for details):

Proposition 3.3. *Let C be an equidistant k-ary code of length n, size M and (Hamming) distance d. Its distance satisfies*

$$d \leq \frac{nM(k-1)}{(M-1)k}.$$

A constant composition code may be viewed as a kind of constant weight code with additional restrictions (a constant weight code on alphabet $\{0, 1, \ldots, k\}$ is one in which every codeword has the same (fixed) number of non-zero symbols; when the alphabet size is 2 then constant composition is the same as constant weight). Upper bounds on the possible sizes of equidistant CWCs were produced in [15].

Proposition 3.4. *Let $E_k(n,d,w)$ denote the the maximum possible number of codewords in a k-ary equidistant code with constant weight w. Then*

- *$E_k(n,d,w) \leq kn$ if $k > 2$;*
- *$E_k(n,d,w) \leq n$ if $k = 2$.*

We end this section with a result which applies to some basic methods of deriving EFPAs from other CCCs.

Proposition 3.5. *(i) $B_\lambda(n,d) \geq B_\lambda(n-\lambda,d)$;*

(ii) If $n_1 = m\lambda_1$ and $n_2 = m\lambda_2$, then

$$B_{\lambda_1+\lambda_2}(n_1+n_2, d_1+d_2) \geq \min\{B_{\lambda_1}(n_1,d_1), B_{\lambda_2}(n_2,d_2)\}.$$

(iii) If there exists a equidistant k-ary $CCC(n, d, [n_0, n_1, \ldots, n_{k-1}])$ *with*

$$n_0 = n_1 = \ldots = n_{k-r}, n_{k-r+1} = n_0 - s_{r-1}, \ldots, n_{k-1} = n_0 - s_1$$

for some positive integers $s_1, \ldots, s_{r-1}$*, then there exists an* $EFPA_{(n+t)/k}(n + t, d)$ *of the same size, where* $t = \sum_{i=1}^{r-1} s_i$*.*

(iv) Let M be the maximum size of an equidistant binary code with length n and distance d. Then $B_n(2n, 2d) \geq M$*.*

Proof. (i) Adding λ copies of some new symbol to each row of an $EFPA_\lambda(n - \lambda, d)$ yields an $EFPA_\lambda(n, d)$.

(ii) Juxtaposing an $EFPA_{\lambda_1}(n_1, d_1)$ and an $EFPA_{\lambda_2}(n_2, d_2)$ yields an $EFPA_\lambda(n, d)$ with $\lambda = \lambda_1 + \lambda_2$, $n = n_1 + n_2$ and $d = d_1 + d_2$.

(iii) Writing the codewords of the CCC as an $n \times M$ array, then adjoining s_i columns comprising symbol $k - i$ for each $i = 1, \ldots, r - 1$, yields the desired EFPA.

(iv) Let C be an equidistant binary code of size M on alphabet $\{0, 1\}$ of length n and distance d. Then an $EFPA_n(2n, 2d)$ (also of size M) can be constructed by changing 0's to 1's and 1's to 0's in C, then juxtaposing the resulting array with the original array. □

4 An alternative combinatorial viewpoint

The following result is well-known in the context of EPAs (more details may be found in [5]).

Definition 4.1. *Let X be a set of cardinality v. A generalized Room square (GRS) of side r and index λ defined on X is an r×r array F having the following properties:*

(1) every cell of F contains a subset (possibly empty) of X;

(2) each symbol of X occurs once in each row and column of F;

(3) any two distinct symbols occur together in exactly λ cells of F.

Such a GRS is denoted by $S(r, \lambda; v)$.

Theorem 4.2. *An equidistant permutation array of length r, distance $r - \lambda$ and size v exists if and only if an $S(r, \lambda; v)$ exists.*

We need to introduce the following generalization of the definition.

Definition 4.3. *Let X be a set of cardinality v. A* generalized Room rectangle *(GRR) of size $k \times n$, frequency μ (where $n = k\mu$) and index λ defined on X is an $k \times n$ array F having the following properties:*

(1) every cell of F contains a subset (possibly empty) of X;

(2) each symbol of X occurs once in each column of F;

(3) each symbol of X occurs μ times in each row of F;

(4) any two distinct symbols occur together in exactly λ cells of F.

Such an object will be denoted by $GRR(n, \mu, \lambda; v)$.

Clearly, the special case when $\mu = 1$ yields a generalized room square. Analogously to the EPA case, we have an equivalence between EFPAs and these combinatorial objects. The proof below explicitly indicates how to obtain one from the other.

Theorem 4.4. *An equidistant frequency permutation array $EFPA_\mu(n, n - \lambda)$ of size v exists if and only if an $GRR(n, \mu, \lambda; v)$ exists.*

Proof. Let $n = k\mu$, and let C be an $EFPA_\mu(n, n-\lambda)$ of size v on alphabet $\{1, \ldots, k\}$. List its codewords as $c_1, \ldots, c_v$. Form a $GRR(n, \mu, \lambda; v)$ from C as follows: label the k rows of a $k \times n$ array by $\{1, \ldots, k\}$ and the n columns by $\{1, \ldots, n\}$. In the

(i, j)th cell, place symbol r if codeword c_r has symbol i in position j. Clearly, each column will have one occurrence each of $\{1, \ldots, v\}$ (since each codeword has a single symbol in each position), and each row will have μ occurrences each of $\{1, \ldots, v\}$ (since symbol i occurs μ times per codeword). Entries a and b occur together in cell (i, j) of this GRR if and only if codewords c_a and c_b of C both have symbol i in position j, and so any two distinct elements of $\{1, \ldots, v\}$ occur together in exactly $n - (n - \lambda) = \lambda$ cells of the GRR.

Conversely, given a $GRR(n, \mu, \lambda; v)$ on symbols $\{1, 2, \ldots, v\}$, an EFPA with the desired properties can be constructed by reversing the above process. For each symbol $1 \leq r \leq v$, form codeword c_r as follows: position j of c_r contains symbol i precisely if r occurs in cell (i, j) of the GRR. Since each column j contains precisely one copy of each symbol $1 \leq r \leq v$, this is well-defined; since each row contains μ copies of each symbol (no two in the same column), codeword c_r will contain μ occurrences of each symbol i. Appropriate Hamming distance $n - \lambda$ between codewords is guaranteed as each pair of symbols in the GRR occurs together in a cell precisely λ times. □

Example 4.5. *As an example, we exhibit the GRR which corresponds to the EFPA of Example 2.5.*

123456	1234	1256						3456
			123456	135	146	236	245	
	56	34		246	235	145	136	12

The following result shows how infinitely many EFPAs may be immediately constructed from the GRR of one EFPA, and that the resulting EFPAs have optimal size if the original has optimal size.

Proposition 4.6. *Let G be the $GRR(n, \mu, \lambda; v)$ corresponding to an $EFPA_\mu(n, n - \lambda)$ of size v and let $k \in \mathbb{N}$.*

(i) The array obtained by juxtaposing k copies of G side-by-side is a $GRR(kn, k\mu, k\lambda; v)$ corresponding to an $EFPA_{k\mu}(kn, k(n-\lambda))$ of size v.

(ii) If the original $EFPA_{\mu}(n, n-\lambda)$ is of optimal size (i.e. satisfies the upper bound of Proposition 3.2) then the $EFPA_{k\mu}(kn, k(n-\lambda))$ from (i) is also of optimal size.

Proof. The first part is immediate: the new array has $N = kn$ columns, with each symbol occurring once per column and $k\mu$ times per row, while each pair of symbols occurs together in a cell $k\lambda$ times. This clearly corresponds to an $EFPA_L(N, D)$ with $L = k\lambda$ and $D = kn - k\lambda = kd$. For the second part, notice that for the new array, the upper bound of Proposition 3.2 becomes $\frac{D}{D-N+L} = \frac{d}{d-n+\lambda} = v$. □

We note in passing that the concept of the GRR could be naturally generalized as follows.

Definition 4.7. *Let X be a set of cardinality v. Define a* generalized Room rectangle *(GRR) of size $k \times n$, frequency composition $\{\mu_1, \dots, \mu_k\}$ (where $n = \sum_{i=1}^{k} \mu_i$) and index λ defined on X to be an $k \times n$ array F having the properties of Definition 4.3 with (3) replaced by (3^*), where*

(3^*) each symbol of X occurs μ_i times in the ith row of F.

Denote such an object by $GRR(n, \{\mu_1, \dots, \mu_k\}, \lambda; v)$.

The following relationship is then immediately clear (its proof is precisely analogous to that of Theorem 4.4):

Theorem 4.8. *An equidistant k-ary $CCC(n, n-\lambda, [\mu_1, \dots, \mu_k])$ of size v exists if and only if a $GRR(n, \{\mu_1, \dots, \mu_k\}, \lambda; v)$ exists.*

In [7], a similar notion is explored for the case of arbitrary CCCs: *generalized doubly resolvable packings with type* $\{\mu_1, \ldots, \mu_k\}$ are introduced (in which any pair occurs together at most λ times), and their equivalence to general $CCC[n, d, [\mu_1, \ldots, \mu_k]]$s established.

5 EFPAs from k-set partitions of n-sets

We next give a family of constructions for EFPAs of optimal size, in which the corresponding GRRs are built directly.

We shall need the following theorem from combinatorial literature (see [2]).

Theorem 5.1 (Baranyai's Theorem)**.** *If $k|n$, then there exists a partition π of the set of k-subsets of $\{1, 2, \ldots, n\}$ into parallel classes, each of which is a partition of $\{1, 2, \ldots, n\}$.*

Theorem 5.2. *For any even $n \in \mathbb{N}$, an $EFPA_{n-1}(\frac{n(n-1)}{2}, \frac{n(n-2)}{2})$ of optimal size n may be constructed using the set of 2-sets of $\{1, \ldots, n\}$.*

Proof. Let $n \in \mathbb{N}$ be even. There are $\binom{n}{2} = \frac{n(n-1)}{2}$ 2-sets of $\{1, \ldots, n\}$; by Baranyai's Theorem, there is a partition of these 2-sets into parallel classes, each of which is a partition of $\{1, \ldots, n\}$. Label these parallel classes as $\pi_1, \ldots, \pi_{n-1}$, where each π_i comprises $\frac{n}{2}$ 2-sets, which we list as $\pi_i(1), \ldots, \pi_i(\frac{n}{2})$ (the order of the 2-sets does not matter here).

Form $\frac{n}{2}$ arrays of cells $A_1, \ldots, A_{\frac{n}{2}}$, as follows: each A_i is a $1 \times (n-1)$ array of cells, where the $(1, j)$th cell of A_i contains the 2-set $\pi_j(i)$ ($1 \leq i \leq \frac{n}{2}$, $1 \leq j \leq n-1$). Now take a latin square of order $\frac{n}{2}$ on the symbols $\{1, \ldots, \frac{n}{2}\}$, and create an $\frac{n}{2} \times \frac{n(n-1)}{2}$ array of cells by arranging the arrays $A_1, \ldots, A_{\frac{n}{2}}$ according to the latin square (each occurrence of symbol i in the latin square is replaced by A_i in the array).

Then every column comprises some parallel class π_i for some $1 \leq i \leq n-1$, and every row comprises all the 2-sets of $\{1, \ldots, n\}$, arranged one per cell. Altogether, the array contains each 2-set of $\{1,\ldots,n\}$ precisely $\frac{n}{2}$ times. So the array is a $GRR(\frac{n(n-1)}{2}, n-1, \frac{n}{2}; n)$, and hence corresponds to an $EFPA_{n-1}(\frac{n(n-1)}{2}, \frac{n(n-2)}{2})$ of size n. Since this meets the Plotkin bound (here $\frac{d}{d-n+\lambda} = \frac{n(n-2)}{2} \cdot \frac{2}{n-2} = n$), the $EFPA$ so constructed is optimal. □

Example 5.3. *We build an $EFPA_5(15,12)$ of optimal size 6. We form the following 5 parallel classes:*

$$\begin{array}{c|l} \pi_1 & \{(1,2),(3,4),(5,6)\} \\ \pi_2 & \{(1,3),(2,6),(4,5)\} \\ \pi_3 & \{(1,4),(2,5),(3,6)\} \\ \pi_4 & \{(1,5),(2,3),(4,6)\} \\ \pi_5 & \{(1,6),(2,4),(3,5)\} \end{array}$$

We then form the following three 1×5 arrays of cells:

A_1	12	13	14	15	16
A_2	34	26	25	23	24
A_3	56	45	36	46	35

(Note that, in any example, we can arrange for A_1 to comprise all pairs of the form $(1,a)$ for $2 \le a \le n$.)

Next, we take a latin square of order 3:

$$\begin{array}{ccc} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{array}$$

and arrange A_1, A_2, A_3 according to this square:

12	13	14	15	16	56	45	36	46	35	34	26	25	23	24
34	26	25	23	24	12	13	14	15	16	56	45	36	46	35
56	45	36	46	35	34	26	25	23	24	12	13	14	15	16

Labelling the rows of this GRR by symbols $\{0,1,2\}$, and taking the entries i of the GRR to correspond to the entries of the codeword C_i, the corresponding $EFPA_5(15,12)$

can then be read off as:

$$\begin{array}{c|l} C_1 & 000001111122222 \\ C_2 & 011111222220000 \\ C_3 & 102122102002101 \\ C_4 & 120212010201210 \\ C_5 & 221020021011021 \\ C_6 & 212200200110112 \end{array}$$

In fact, this construction can be generalized for any $k|n$:

Theorem 5.4. *For any $n \in \mathbb{N}$ with $k|n$, an $EFPA_{\binom{n-1}{k-1}}(\binom{n}{k}, \frac{n-k}{n-1}\binom{n}{k})$ of optimal size n may be constructed using the set of k-sets of $\{1, \ldots, n\}$.*

Proof. The construction is exactly analogous to the proof of Theorem 5.2. Let $n \in \mathbb{N}$ be divisible by k. There are $\binom{n}{k}$ k-sets of $\{1, \ldots, n\}$; by Baranyai's Theorem, there is a partition of these k-sets into parallel classes, each of which is a partition of $\{1, \ldots, n\}$. Label these parallel classes as $\pi_1, \ldots, \pi_{\binom{n-1}{k-1}}$, where each π_i comprises $\frac{n}{k}$ k-sets, which we list as $\pi_i(1), \ldots, \pi_i(\frac{n}{k})$ (the order of the k-sets does not matter here). Form $\frac{n}{k}$ arrays of cells $A_1, \ldots, A_{\frac{n}{k}}$, as follows: each A_i is a $1 \times \binom{n-1}{k-1}$ array of cells, where the $(1, j)$th cell of A_i contains the k-set $\pi_j(i)$ $(1 \leq i \leq \frac{n}{k}, 1 \leq j \leq \binom{n-1}{k-1})$. We take a latin square of order $\frac{n}{k}$ on the symbols $\{1, \ldots, \frac{n}{k}\}$, and create an $\frac{n}{k} \times \binom{n}{k}$ array of cells by arranging the arrays $A_1, \ldots, A_{\frac{n}{k}}$ according to the latin square.

We can check that every column comprises some parallel class π_i for some $1 \leq i \leq \binom{n-1}{k-1}$, and every row comprises all the k-sets of $\{1, \ldots, n\}$, arranged one per cell. Each symbol occurs $\binom{n-1}{k-1}$ times per row, and each pair occurs together $\frac{n}{k}\binom{n-2}{k-2}$ times in the array. So the array is a $GRR(\binom{n}{k}, \binom{n-1}{k-1}, \frac{n}{k}\binom{n-2}{k-2}; n)$, and hence corresponds to an $EFPA_{\binom{n-1}{k-1}}(\binom{n}{k}, \frac{n-k}{n-1}\binom{n}{k})$ of size n. Since this meets the Plotkin bound, the $EFPA$ so constructed is optimal. □

For certain values of n and k, the k-sets of $\{1, \ldots, n\}$ may be arranged in other ways to form GRRs and hence EFPAs.

Example 5.5. *The set of* $\binom{6}{3}$ *3-sets of* $\{1,\ldots,6\}$ *may be arranged into the following* $GRR(10,5,4;6)$*, which corresponds to an* $EFPA_5(10,6)$ *of optimal size* 6.

123	124	135	146	156	236	245	256	345	346
456	356	246	235	234	145	136	134	126	125

6 EFPAs from resolvable BIBDs

In the previous section, we built optimal EFPAs from the full set of k-sets of an n-set. We now show that a similar construction can be used to construct an optimal EFPA from any balanced incomplete block design (BIBD) which possesses the property of being resolvable. Properties and construction methods of resolvable BIBDs have been extensively studied; we refer the reader to [5] for information and further references.

Definition 6.1.

- *A balanced incomplete block design (BIBD) is a pair* (V, B) *where* V *is a* v*-set and* B *is a collection of* b k*-subsets (blocks) of* V*, such that each element of* V *is contained in exactly* r *blocks and any* 2*-subset of* V *is contained in exactly* λ *blocks. Here* $r = \frac{\lambda(v-1)}{k-1}$ *and* $b = \frac{vr}{k}$*.*
- *A parallel class is a set of blocks of a BIBD that partition the point set, and a resolvable BIBD is one whose blocks can be partitioned into parallel classes. Necessary conditions for the existence of a* (v, k, λ)*-RBIBD are* $(k-1)|\lambda(v-1)$ *and* $k|v$*.*

Theorem 6.2. *If there exists a resolvable* (v, k, λ)*BIBD, then there exists an optimal* $EFPA_r(b, \frac{(r-\lambda)v}{k})$ *of size* v*.*

Proof. The proof is analogous to that of Theorem 5.2. Denote the design by D. Then D has r parallel classes, each containing $\frac{v}{k}$ blocks. Label these parallel classes as $\pi_1, \ldots, \pi_r$, where each π_i comprises $\frac{v}{k}$ k-sets, which we list as $\pi_i(1), \ldots, \pi_i(\frac{v}{k})$ (the order of the k-sets does not matter here).

Form $\frac{v}{k}$ arrays of cells $A_1, \ldots, A_{\frac{v}{k}}$, as follows: each A_i is a $1 \times r$ array of cells, where the $(1, j)$th cell of A_i contains the k-set $\pi_j(i)$ $(1 \leq i \leq \frac{v}{k}, 1 \leq r)$. Now take a latin square of order $\frac{v}{k}$ on the symbols $\{1, \ldots, \frac{v}{k}\}$, and create an $\frac{v}{k} \times b$ array of cells by arranging the arrays $A_1, \ldots, A_{\frac{v}{k}}$ according to the latin square.

Then every column comprises some parallel class π_i for some $1 \leq i \leq r$, and every row comprises all the blocks of the design, arranged one per cell. Altogether, the array contains each block of the design precisely $\frac{v}{k}$ times. So the array is a $GRR(b, r, \frac{\lambda v}{k}; v)$, and hence corresponds to an $EFPA_r(b, \frac{(r-\lambda)v}{k})$ of size v. Since this meets the Plotkin bound (here the expression for the maximum possible size simplifies to $\frac{\lambda(v-k)}{k-1} \cdot \frac{v(k-1)}{\lambda(v-k)} = v$), the $EFPA$ so constructed is optimal. □

Example 6.3. *A $GRR(12, 4, 3; 9)$ corresponding to an $EFPA_4(12, 9)$ of optimal size 9, constructed from the method of Theorem 6.2 from the $(9, 3, 1)$ RBIBD given in [5].*

123	147	159	168	456	258	267	249	789	369	348	357
789	369	348	357	123	147	159	168	456	258	267	249
456	258	267	249	789	369	348	357	123	147	159	168

We note that in [8], a construction method is given for a family of optimal constant composition codes which are examples of EFPAs, using RBIBDs in combination with difference matrices.

7 EFPAs from Skolem sequences

In this section, we give construction methods for CCCs and EFPAs using cyclic shifts of Skolem sequences.

We will require the following definition:

Definition 7.1. *A Skolem sequence of order n is a sequence $S = (s_1, s_2, \ldots, s_{2n})$ of $2n$ integers satisfying:*

- *for every $k \in \{1, 2, \ldots, n\}$ there exists exactly two elements $s_i, s_j \in S$ such that $s_i = s_j = k$;*
- *if $s_i = s_j = k$ with $i < j$ then $j - i = k$.*

An extended Skolem sequence of order n is a sequence $ES = (s_1, s_2, \ldots, s_{2n+1})$ of $2n + 1$ integers satisfying the two conditions above plus a third condition:

- *there is exactly one $s_i \in ES$ such that $s_i = 0$.*

A Skolem sequence of order n exists if and only if $n \equiv 0, 1 \bmod 4$; an extended Skolem sequence of order n exists for any n ([1]). This latter fact can easily be seen from the following construction, which provides an extended Skolem sequence of order n for any $n \geq 0$:

$$S = (r, (r-2), \ldots, 3, 1, 1, 3, \ldots, (r-2), r, s, (s-2), \ldots, 2, 0, 2, \ldots, (s-2), s)$$

where $r = n$ and $s = n - 1$ if n is odd, and $r = n - 1$ and $s = n$ if n is even. The first few sequences are given by:

$$(1, 1, 0), (1, 1, 2, 0, 2), (3, 1, 1, 3, 2, 0, 2), (3, 1, 1, 3, 4, 2, 0, 2, 4), \ldots$$

Theorem 7.2. *For any integer n, we can construct:*

(a) an optimal $CCC[2n + 1, 2n, [2, 2, \ldots, 2, 1]]$ of size $2n + 1$;

(b) an $EFPA_2(2n + 2, 2n)$ of size $2n + 1$.

Proof. For part (a), we will prove: given an extended Skolem sequence S of order n (length $2n + 1$), the array formed by taking as rows S and all its cyclic shifts is a code of the required type. For part (b), we simply take the array in (a) and add a column consisting entirely of 0's.

Let $S = (s_1, s_2, \ldots, s_{2n+1})$, and let A be the $(2n+1) \times (2n+1)$ array whose rows are S and all its cyclic shifts. Denote the rows of A by ρ_j $(0 \le j \le 2n)$ where $\rho_0(i) = s_i$ $(1 \le i \le 2n+1)$ and ρ_j is the rightward shift of ρ_0 by j positions; more precisely

$$\rho_k(i) = \begin{cases} s_{i-k}, \text{ if } k < i \le 2n+1 \\ s_{2n+1+i-k}, \text{ if } 1 \le i \le k \end{cases}$$

Let ρ_j and ρ_k be any two rows of A $(j < k)$. There are three cases for a position i, $1 \le i \le 2n+1$:

Case 1: $i > k$, i.e. $j < i \le 2n+1$ and $k < i \le 2n+1$. Then

$$\rho_j(i) = \rho_k(i) \Rightarrow s_{i-j} = s_{i-k} = x \text{ for some symbol } x \ \Rightarrow x = (i-j) - (i-k) = k - j$$

Since $1 \le x \le n$, this case occurs only if $k - j \le n$.

Case 2: $j < i \le k$, i.e. $j < i \le 2n+1$ and $1 \le i \le k$. Then

$$\rho_j(i) = \rho_k(i) \Rightarrow s_{i-j} = s_{2n+1+i-k} = x \text{ for some symbol } x$$

$$\Rightarrow x = (2n+1+i-k) - (i-j) = 2n+1+j-k$$

Since $1 \le x \le n$, this case occurs only if $k - j \ge n+1$.

Case 3: $i \le j$, i.e. $1 \le i \le j$ and $1 \le i \le k$. Then

$$\rho_j(i) = \rho_k(i) \Rightarrow s_{2n+1+i-j} = s_{2n+1+i-k} = x \text{ for some symbol } x$$

$$\Rightarrow x = (2n+1+i-j) - (2n+1+i-k) = k - j$$

Since $1 \le x \le n$, this case occurs only if $k - j \le n$.

So, if $1 \le k - j \le n$, rows ρ_j and ρ_k can agree only in the symbol $k - j$, while if $n \le k - j \le 2n$, rows ρ_j and ρ_k can agree only in the symbol $2n+1+j-k$. In both cases, any agreement position must contain a symbol uniquely defined by j and k, giving Hamming distance at least $2n-1$. We shall show that in fact ρ_j and ρ_k have precisely one position of agreement (i.e. distance $2n$).

Suppose ρ_j and ρ_k agree in positions $i_1 < i_2$. If $n \leq k-j \leq 2n$ then the symbol is $2n+1+j-k$ and i_1, i_2 are both Case 2 elements. ρ_j has symbol $2n+1+j-k$ in positions i_1 and i_2; these entries are s_{i_1-j} and s_{i_2-j} of S respectively. We should also have that $\rho_j(i_1)$ and $\rho(i_2)$ contain symbol $2n+1+j-k$. Suppose $\rho_k(i_1) = s_{2n+1+i_1-k}$ contains symbol $2n+1+j-k$; we shall show that $\rho_k(i_2)$ cannot. Since a full set of $2n+1$ cycles have not elapsed, the symbol in $\rho_k(i_1)$ must arise from the sequence entry in $\rho_j(i_2)$, namely s_{i_2-j}, i.e. $2n+1+i_1-k = i_2-j$. Now, $\rho(i_2) = s_{2n+1+i_2-k}$. But then $i_1 - j < i_2 - j < 2n+1+i_2-k$, and symbol $2n+1-j+k$ occurs only twice in our Skolem sequence - a contradiction.

Otherwise, $1 \leq k-j \leq n$ then the symbol is $k-j$; either i_1, i_2 are both Case 1 elements, or they are both Case 3 elements, or we have one of each. Arguments analogous to the above establish that this is impossible. For example, when both are Case 1: say ρ_j and ρ_k $(j < k)$ agree in positions $i_1 < i_2$. Now $\rho_j(i_1) = s_{i_1-j}$, $\rho_j(i_2) = s_{i_2-j}$, $\rho_k(i_1) = s_{i_1-k}$ and $\rho_j(i_2) = s_{i_2-k}$, and we must have $i_2 - k = i_1 - j$. But then $i_1 - k < i_2 - k < i_2 - j$, giving three positions in S which must contain symbol $k-j$ - a contradiction. The situation with both elements Case 3 is similar. Finally, when i_1 is Case 1 and i_2 is Case 3, $\rho_j(i_1) = s_{2n+1+i_1-j}$, $\rho_j(i_2) = s_{i_2-j}$, $\rho_k(i_1) = s_{2n+1+i_1-k}$ and $\rho_j(i_2) = s_{i_2-k}$, and we must have $i_2 - j = 2n+1+i_1-k$. But then $i_2-k < i_2-j < 2n+1+i_1-j$ are three positions in S which must contain symbol $k-j$ - a contradiction. □

Corollary 7.3. *An $EFPA_2(2n+2, 2n)$ of size $2n+1$ is given by:*

$$(r, r-2, \ldots, 3, 1, 1, 3, \ldots, r-2, r, s, s-2, \ldots, 2, 0, 2, \ldots, s-2, s)0$$

where $r = n$ and $s = n-1$ if n is odd, and $r = n-1$ and $s = n$ if n is even, and *the elements within the brackets are cyclically shifted.*

Example 7.4. *An $EFPA_2(10, 8)$ of size 9 arising from the extended Skolem sequence $S = [3, 1, 1, 3, 4, 2, 0, 2, 4]$ is given by*

ρ_0	3	1	1	3	4	2	0	2	4	0
ρ_1	4	3	1	1	3	4	2	0	2	0
ρ_2	2	4	3	1	1	3	4	2	0	0
ρ_3	0	2	4	3	1	1	3	4	2	0
ρ_4	2	0	2	4	3	1	1	3	4	0
ρ_5	4	2	0	2	4	3	1	1	3	0
ρ_6	3	4	2	0	2	4	3	1	1	0
ρ_7	1	3	4	2	0	2	4	3	1	0
ρ_8	1	1	3	4	2	0	2	4	3	0

8 EFPAs from Odd Balanced Tournament Designs

In this section, we show how odd balanced tournament designs can be used to obtain CCCs and EFPAs.

Definition 8.1. *An odd balanced tournament design, $OBTD(n)$, defined on a $2n+1$-set V, is an arrangement of the $\binom{2n+1}{2}$ distinct unordered pairs of the elements of V into an $n \times (2n + 1)$ array such that*

- *each column of the array contains $2n$ distinct elements of V;*
- *each element of V occurs precisely twice in each row.*

It is known that an $OBTD(n)$ exists for every positive integer n. Moreover, by appropriate choice of construction methods, it can be ensured that no two columns of the array are missing the same element of the $2n + 1$-set V.

Theorem 8.2. *For any $k \in \mathbb{N}$,*

- *an equidistant* $CCC(2k+1, 2k, [2,2,\dots,2,1]; 2k+1)$ *(i.e. optimal size) can be constructed;*

- *an* $EFPA_2(2k+2, 2k)$ *of size* $2k+1$ *can be constructed.*

Proof. Let A be an $OBTD(k)$, i.e. a $k\times(2k+1)$ array with each cell containing a 2-set of elements from $\{1,\dots,2k+1\}$. From the remark preceding the theorem, we may assume that no two columns of the array are missing the same element of $\{1,\dots,2k+1\}$. Add an extra row to A, whose ith cell contains the single symbol not occurring in the ith column of A. The resulting array is a $GRR(2k+1, \{2,\dots,2,1\}, 1; 2k+1)$ equivalent to a $k+1$-ary $CCC(2k+1, 2k, [2,2,\dots,2,1])$ of size $2k+1$. Optimal size follows as it satisfies the bound of Proposition 3.2. Adjoining an additional column whose cells are empty apart from the cell in row $k+1$, which contains a copy of each symbol, yields a $GRR(2k+2, 2, 2; 2k+1)$ equivalent to the desired EFPA. □

Example 8.3. *Here is an example of an* $EFPA_2(8,6)$ *of size* 7*, constructed as above.*

36	47	51	62	73	14	25	
27	31	42	53	64	75	16	
45	56	67	71	12	23	34	
1	2	3	4	5	6	7	1, 2, 3, 4 5, 6, 7

We next present a construction for a specific choice of parameters, which illustrates one way in which additional properties of OBTDs may be used to obtain CCCs and EFPAs. It is hoped that this may be representative of a range of different constructions in special cases. We will need an additional definition and property, introduced in [11].

Definition 8.4. *Let* B *be an* $OBTD(n)$ *defined on* V *(*$|V| = 2n+1$*). Let* $R_1,\dots,R_n$ *be the rows of* B *and let* $C_1,\dots,C_{2n+1}$ *be the columns of* B*. Then* $C = \{C_1,\dots,C_{2n+1}\}$

is a resolution of B. A resolution $D = \{D_1, \ldots, D_{2n+1}\}$ is called an orthogonal resolution to C if

- $|C_i \cap D_j| \leq 1$ *for* $1 \leq i, j \leq 2n+1$, *and*
- $|D_j \cap R_i| = 1$ *for* $j = 1, \ldots, 2n+1$ *and* $i = 1, \ldots, n$.

In fact, we need the following refinement of the definition:

Definition 8.5. *Let $V = \{x_i : 1 \leq i \leq 2n+1\}$. Suppose C_i contains every element of $V \setminus \{x_i\}$ precisely once, and suppose D_i contains every element of $V \setminus \{x_i\}$ precisely once, for $1 \leq i \leq 2n+1$. Let $C'_i = C_i \cup \{x_i\}$ and let $D'_i = D_i \cup \{x_i\}$. Let $C' = \{C'_1, \ldots, C'_{2n+1}\}$ and $D' = \{D'_1, \ldots, D'_{2n+1}\}$. If C and D are a pair of orthogonal resolutions for B and if $|C'_i \cap D'_j| \leq 1$ for $1 \leq i, j \leq 2n+1$, then we call C and D a pair of $*$-orthogonal resolutions for B.*

Example 8.6. *From an $OBTD(3)$ with two $*$-orthogonal resolutions, we can construct an $EFPA_3(15, 12)$ of size 14 (and an optimal $CCC(14, 12, [3, 3, 3, 2, 3])$) of size 14).*

Proof. For this construction, take an $OBTD(3)$, B say, on symbol set $\{0, 1, \ldots, 6\}$ with resolutions C and D (where C corresponds to the columns of B).

Extend B by adding a 4th row to B such that the $(4, i)$th cell of the new array contains that element from the symbol set which does not appear in column i of B. Call this array E. Resolution D of B can be uniquely extended to D' of E by taking each D_i together with the cell in row 4 of E which contains the single symbol unused in D_i. Note that each pair (i, j) with $i \neq j$ in $\{0, 1, \ldots, 6\} \times \{0, 1, \ldots, 6\}$ occurs together in a cell precisely once in this array.

Now enter symbol $7 + i$ into the cells of E which occur in D'_i. This yields an 4×7 array in which the first 3 rows have 3 entries per cell, while the 4th row has 2 entries per cell; each pair (i, j) in $\{0, 1, \ldots, 6\} \times \{7, \ldots, 13\}$ occurs in a cell precisely once in this array.

Form a 5th row by taking, in cell $(5, i)$ of this row $(1 \leq i \leq 7)$, the three symbols from $\{7, 8, \ldots, 13\}$ which have not hitherto been used in column i. Observe that these cells form a set S of 3-sets such that each pair (i, j) with $i \neq j \in \{7, 8, \ldots, 13\}$ occurs together precisely once in a member of S. Call this 5×7 array F. Each pair (i, j) of symbols $(i \neq j)$ occurs together in some cell of F precisely once, and each symbol occurs once in each column of F. Now, form a new array F' by swapping each symbol i with symbol $(7 + i)\, mod\, 14$. Place F and F' side-by-side, and add an 15th column whose cells are empty except in the 4th row, where the cell $(4, 15)$ contains a copy of each symbol $\{0, 1, \ldots, 13\}$. The resulting GRR corresponds to an $EFPA_3(15, 12)$ of size 14. Without the extra 15th column, the array is a GRR corresponding to a $CCC(14, 12, [3, 3, 3, 2, 3]))$ of size 14 (optimal since it meets the bound of Proposition 3.2). The array is shown below. □

2 5 10	3 6 11	4 0 12	5 1 13	6 2 7	0 3 8	1 4 9	9 12 3	10 13 4	11 7 5	12 8 6	13 9 0	7 10 1	8 11 2	
1 6 12	2 0 13	3 1 7	4 2 8	5 3 9	6 4 10	0 5 11	8 13 5	9 7 6	10 8 0	11 9 1	12 10 2	13 11 3	7 12 4	
3 4 13	4 5 7	5 6 8	6 0 9	0 1 10	1 2 11	2 3 12	10 11 6	11 12 0	12 13 1	13 7 2	7 8 3	8 9 4	9 10 5	
0 7	1 8	2 9	3 10	4 11	5 12	6 13	7 0	8 1	9 2	10 3	11 4	12 5	13 6	$0, 1, 2, 3, 4,$ $5, 7, 8, 9,$ $10, 11, 12, 13$
8 9 11	9 10 12	10 11 13	7 11 12	12 13 8	13 7 9	7 8 10	1 2 4	2 3 5	3 4 6	0 4 5	1 5 6	2 6 0	0 1 3	

9 EFPAs from orthogonal arrays

An orthogonal array $OA(k,s)$ is a $k \times s^2$ array with entries from an s-set S such that in any two rows, each (ordered) pair of symbols from S occurs exactly once. It is clear that any orthogonal array $OA(k,s)$ is an example of an $EFPA_s(s^2, s^2-s)$. We can improve upon a basic orthogonal array.

Proposition 9.1. *Given an $OA(k,s)$ with M rows, an $EFPA_s(s^2, s^2-s)$ of size $(s-1)M$ can be constructed. In particular, for $s = q$ (a prime power),*

- *an $EFPA_q(q^2, q^2-q)$ of size q^2-1 can be constructed*
- *an optimal $CCC(q^2-1, q^2-q, [q,\ldots,q,q-1])$ of size q^2-1 can be constructed.*

Proof. Given an $OA(k,s)$ A on symbol set $S = \{1,\ldots,s\}$, we fix one symbol (say 1) and perform $s-1$ substitutions on the others to obtain a new array B, in which each row of A yields $s-1$ rows of B. More precisely, let $\pi \in S_s$ be the permutation $(2\ldots s)$. Then for each row $\rho_i = a_{i,1}a_{i,2}\ldots a_{i,s^2}$ of A $(1 \le i \le M)$, and for each $1 \le j \le s-1$, replace ρ_i by ${\rho_i}^j = (\pi^j(a_{i,1})\pi^j(a_{i,2})\ldots\pi^j(a_{i,s^2}))$ to obtain array B of size $(s-1)M$.

It is clear that each row of B has the appropriate number and type of symbols; we check that the pairwise distance between rows is s^2-s. Let α and β be different rows of B. If α and β arise from the same row of A under different substitutions, i.e. $\alpha = {r_i}^j$ and $\beta = {r_i}^k$ for some $1 \le i \le M$ and $1 \le j \ne k \le s-1$, then they agree on the s copies of symbol 1 and disagree in all other positions. If α and β are two distinct rows of A under the same substitution, i.e. $\alpha = {r_h}^j$ and $\beta = {r_i}^j$ for some $1 \le j \le s-1$ and $1 \le h \ne i \le M$, then all agreements are as in the original rows. Now let α and β be different rows of A under different substitutions, i.e. $\alpha = {r_i}^k$ and $\beta = {r_j}^l$, some $1 \le i \ne j \le M$ and $1 \le k \ne l \le s-1$. Then it is still the case that each ordered pair in $S \times S$ occurs precisely once in these two rows, and hence pairwise distance is still $s^2 - s$.

For the second part, use a set of mutually orthogonal latin squares on symbols $\{1, \ldots, s\}$ to form the orthogonal array (with the squares in standard form so that the first column of the orthogonal array is an all-1 column). Perform substitutions as described above; to obtain the CCC from the resulting array, delete the first (all-1) column. Its optimality follows as it achieves the bound of Proposition 3.2. $\square$

Example 9.2. *An* $EFPA_3(9,6)$ *of size* 8 *arising from a set of MOLS plus substitutions with* $\pi = (23)$ *is given by*

$$\begin{array}{ccccccccc}
1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\
1 & 1 & 1 & 3 & 3 & 3 & 2 & 2 & 2 \\
1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\
1 & 3 & 2 & 1 & 3 & 2 & 1 & 3 & 2 \\
1 & 2 & 3 & 2 & 3 & 1 & 3 & 1 & 2 \\
1 & 3 & 2 & 3 & 2 & 1 & 2 & 1 & 3 \\
1 & 2 & 3 & 3 & 1 & 2 & 2 & 3 & 1 \\
1 & 3 & 2 & 2 & 1 & 3 & 3 & 2 & 1
\end{array}$$

Observe that deleting the first column yields a $CCC(8, 6, [3, 3, 2])$ *of optimal size* 8.

We observe that, while it is not possible in general to convert one EFPA to another with a smaller or larger symbol set by simple maps on the symbols, it can be done in the case where the EFPA possesses an extra property, which is posssessed for example by orthogonal arrays. (The following result appeared in [10] for general FPAs.)

Proposition 9.3. *Let* $n = m\lambda$. *Let* A *be an* $EFPA_\lambda(n, d)$ *such that, between any two rows, each of the* m^2 *pairs* (i, j) *occurs precisely* t *times. Then* A *may be converted, by reduction modulo* r *(where* $r|n$*) to an* $EFPA_{\frac{n}{r}}(n, n - \frac{tm^2}{r})$.

10 Binary equidistant frequency permutation arrays

It has been shown that, for $m = 2$, EFPAs of size greater than 1 can exist only for even distances d. Moreover, the maximum size of an $EFPA_{n/2}(n, d)$ is at most n, by Proposition 3.4. Binary FPAs are binary constant weight codes, and have been much-studied. Below, for reference, we list a few construction methods which are specific to binary EFPAs: since these are easily derived from known results, we omit details of proofs (see references such as [13] for further details).

Definition 10.1. *A Hadamard matrix of order n is an $n \times n$ matrix with entries $+1, -1$ satisfying $HH^t = nI$, i.e. its rows are pairwise orthogonal. For a Hadamard matrix of order n to exist, n must be $1, 2$ or $4k$ for some positive integer k.*

Construction 10.2. *For any n for which there exists a Hadamard matrix of order N, an $EFPA_{\frac{n}{2}}(n, \frac{n}{2})$ can be constructed of size $n - 1$.*

Take a normalized Hadamard matrix $H(n)$ (i.e. all entries in its first row and first column are equal to 1) and remove the first row, then convert each occurrence of -1 to 0.

Definition 10.3. *A Legendre sequence is a binary sequence $v = [v_0, v_1, \ldots, v_{p-1}]$ of length p (prime), where $\frac{p-1}{2}$ is odd, such that $v_i = 0$ if i is a quadratic non-residue modulo p, and $v_i = 1$ if i is a quadratic residue modulo p; the digit v_0 can be either 0 or 1.*

Construction 10.4. *For a prime p with $\frac{p-1}{2}$ odd (i.e. $p \equiv 3 \bmod 4$), an $EFPA_{(p+1)/2}(p+1, (p+1)/2)$ can be constructed of size p.*

Form a $p \times p$ array by taking as rows a Legendre sequence v, and its $(p-1)$ rightward cyclic shifts. It can be shown that the Hamming distance between each row is $\frac{p+1}{2}$.

To make the EFPA, append a column to the array consisting entirely of 0's if $v_0 = 1$, or 1's if $v_0 = 0$.

Example 10.5. *An* $EFPA_4(8,4)$ *of size* 7 *arising from the Legendre sequence* $v = [0,0,0,1,0,1,1]$ *is given by*

$$L_1 = \begin{matrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{matrix}$$

A maximal length feedback shift register sequence (m-sequence for short) is a type of pseudorandom sequence possessing many useful properties (for more details see [9]). There exists a binary m-sequence of length $2^n - 1$ for any integer $n > 1$.

Construction 10.6. *For a Mersenne prime* p *(a prime of the form* $2^n - 1$*), an* $EFPA_{(p+1)/2}(p+1,(p+1)/2)$ *can be constructed of size* p*.*

Let v be a binary m-sequence of length p. Then the array obtained from v together with all its $(p-1)$ cyclic rightward shifts is equidistant as a code. Each row has $\frac{p+1}{2}$ 1's, $\frac{p-1}{2}$ 0's and distance $\frac{p+1}{2}$. To make the EFPA, append a column of 0's to this array.

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