COMPUTING AUTOMORPHISMS OF SEMIGROUPS

J. ARAÚJO, P. V. BÜNAU, AND J. D. MITCHELL

ABSTRACT. In this paper an algorithm is presented that can be used to calculate the automorphism group of a finite transformation semigroup. The general algorithm employs a special method to compute the automorphism group of a finite simple semigroup. As an application of the algorithm, all the automorphism groups of semigroup of order at most 7 and of the multiplicative semigroups of some group rings are found. We also consider the question of which groups occur as automorphism groups of semigroups of several distinguished types.

1. INTRODUCTION

There is a tremendous amount of literature relating to automorphism groups of mathematical structures of every hue. An algorithm for computing the automorphism group of a finite group was first given in the 1960s and development of procedures with the same purpose continues to the present day; see [8], [9], and [11]. There are numerous papers concerning the automorphism groups of particular classes of semigroups, for example, Schreier [47] and Mal'cev [35] described all automorphisms of the semigroup of all mappings from a set to itself. Similar results have been obtained for various other structures such as orders, equivalence relations, graphs, and hypergraphs; see the survey papers [41] and [42]. More examples are provided, among others, by Gluskĭn [18], Araújo and Konieczny [2], [4], and [5], Fitzpatrick and Symons [13], Levi [29] and [30], Liber [32], Magill [33], Schein [45], Sullivan [49], and Šutov [50]. However, there appears to have been no previous attempt to give an algorithm for computing the automorphisms of an arbitrary finite semigroup. The purpose of this paper is to give such an algorithm. As part of the computation it is necessary to calculate the automorphisms of certain finite groups, partially ordered sets, and graphs associated with the semigroup. The efficiency of the well-developed algorithms used to perform these calculations is thus incorporated in the presented algorithm. The routines presented here have been implemented as part of the MONOID package [40] in the computational algebra system GAP [16].

The most naïve approach to computing the automorphisms of a semigroup S would be to verify, one by one, whether each bijection ϕ from S to S satisfies $(x)\phi(y)\phi = (xy)\phi$ for all $x, y \in S$. To perform this calculation, except for extremely small examples, exceeds human patience. As the examples grow in size, it soon becomes impractical for computers to do the work for us. Although the naïve method is completely impractical, in our algorithm the general strategy is

1991 *Mathematics Subject Classification.* 20M20.

Key words and phrases. Transformation semigroups, automorphism group.

the same. That is, a search is conducted through a relatively small set of bijections and they are tested to see if they are homomorphisms using the relations of a presentation that defines S.

The main algorithm for computing the automorphism group of a semigroup is given in Section 4. This algorithm relies on another procedure for calculating the automorphisms of a special type of semigroup: Rees matrix semigroups. This procedure can be found in Section 2. In Section 3 we give an algorithm to compute the inner automorphisms of a transformation semigroup S. In Section 5 we apply the main algorithm to compute the automorphism groups of the semigroups of order at most 7. In Section 6 we compute the automorphism groups of the multiplicative semigroups of some rings. Finally, in Section 7 we consider which groups can occur as the automorphism groups of semigroups belonging to various standard classes.

Throughout mappings are written on the right with composition from left to right, and all sets, groups, and semigroups are assumed to be finite. The identity mapping on a set X will be denoted by 1_X.

2. Automorphisms of Rees matrix semigroups

In this section we describe how to compute the automorphism group of a finite Rees matrix semigroup.

Let T be a semigroup, I and J be index sets and $P = (p_{ji})_{j\in J, i\in I}$ be a $|J| \times |I|$ matrix with entries in $T \cup \{0\}$. The *Rees matrix semigroup* $\mathcal{M}^0[T; I, J; P]$ is the set $(I \times T \times J) \cup \{0\}$ with multiplication $(i, g, j)(k, h, l) = (i, gp_{jk}h, l)$ and $0(i, g, j) = (i, g, j)0 = 0^2 = 0$. The Rees-Suschkewitz Theorem [27, Theorem 3.2.3] states that a finite semigroup S containing a zero element has exactly one nonzero $\mathcal{D}$-class if and only if it is isomorphic to a Rees matrix semigroup $\mathcal{M}^0[G; I, J; P]$ where G is a group and I, J are finite. Moreover, the proof of the Rees-Suschkewitz Theorem is constructive, and a direct implementation (such as that made in [40]) of that proof can be used to find an isomorphism from a finite simple semigroup with zero to a Rees matrix semigroup $\mathcal{M}^0[G; I, J; P]$.

A characterisation of all homomorphisms between two Rees matrix semigroups is given in [43]; see also [25] and [27].

Theorem 2.1. *Let $M_1 = \mathcal{M}^0[G_1; I_1, J_1; P_1]$ and $M_2 = \mathcal{M}^0[G_2; I_2, J_2; P_2]$ be Rees matrix semigroups, let $\lambda_I : I_1 \to I_2$, $\lambda_J : J_1 \to J_2$, $\gamma : G_1 \to G_2$ be a homomorphism, and $f : I_1 \cup J_1 \to G_2$. Then the mapping $(i, g, j) \mapsto (i\lambda_I, (if)(g\gamma)(jf)^{-1}, j\lambda_J)$ is a homomorphism if and only if*

(i) $p^{(1)}_{ji} = 0$ *if and only if* $p^{(2)}_{j\lambda_J i\lambda_I} = 0$*;*
(ii) $p^{(1)}_{ji}\gamma = (jf)^{-1}(p^{(2)}_{j\lambda_J i\lambda_I})(if)$*, whenever* $p^{(1)}_{ji} \neq 0$.

Furthermore, every homomorphism from M_1 to M_2 can be described in this way.

We require a reformulation of Theorem 2.1 in order to be able to compute the automorphisms of a Rees matrix semigroup. Let $M = \mathcal{M}^0[G; I, J; P]$ be a Rees matrix semigroup over the group G, index sets I and J, and sandwich matrix $P = (p_{ji})_{j\in J, i\in I}$. The automorphism group of M is denoted $\operatorname{Aut} M$.

Let $\Gamma(M)$ be the bipartite graph with vertices $I \cup J$ and edge (i, j) whenever $p_{ji} \neq 0$ and let $\operatorname{Aut}\Gamma(M)$ denote the set of all I and J preserving automorphisms of $\Gamma(M)$. (That is, those bijections α from $\Gamma(M)$ to $\Gamma(M)$ such that $(i\alpha, j\alpha)$ is an edge whenever (i, j) is an edge, $i\alpha \in I$ if $i \in I$, and $j\alpha \in J$ if $j \in J$.) Then it is obvious that any pair of mappings λ_I and λ_J satisfying Theorem 2.1(i) can be replaced by an element of $\operatorname{Aut}\Gamma(M)$. So, the problem of finding mappings λ_I and λ_J satisfying Theorem 2.1(i) is exchanged for the problem of computing $\operatorname{Aut}\Gamma(M)$. The latter problem has been well studied; the implementation in GAP [40] of the algorithm in this section uses the GRAPE package [48] for GAP to compute $\operatorname{Aut}\Gamma(M)$.

Now, every automorphism of M can be represented as a triple of mappings $\lambda \in \operatorname{Aut}\Gamma(M)$, $\gamma \in \operatorname{Aut} G$, and $f : I \cup J \to G$; a more precise formulation of this is given in the next theorem. Let $G^{I\cup J}$ denote the set of all functions from $I \cup J$ to G and M^M denote the monoid of all mappings from M to M under composition.

Theorem 2.2. *Let $\alpha \in M^M$ and let $\Psi : \operatorname{Aut}\Gamma(M) \times \operatorname{Aut} G \times G^{I\cup J} \to M^M$ be defined by*

$$(i, g, j)([\lambda, \gamma, f]\Psi) = (i\lambda, (if)(g\gamma)(jf)^{-1}, j\lambda).$$

Then $\alpha \in \operatorname{Aut} M$ if and only if $\alpha = [\lambda, \gamma, f]\Psi$ for some $[\lambda, \gamma, f] \in \operatorname{Aut}\Gamma(M) \times \operatorname{Aut} G \times G^{I\cup J}$ satisfying

$$p_{ji}\gamma = (jf)^{-1} \cdot (p_{j\lambda i\lambda}) \cdot (if) \tag{1}$$

for all $p_{ji} \neq 0$.

Proof. The proof is an immediate consequence of Theorem 2.1. □

It is straightforward to verify that $\operatorname{Aut}\Gamma(M)\times\operatorname{Aut} G\times G^{I\cup J}$ with multiplication $\diamond$ defined by

$$[\lambda_1, \gamma_1, f_1] \diamond [\lambda_2, \gamma_2, f_2] = [\lambda_1\lambda_2, \gamma_1\gamma_2, \lambda_1 f_2 \star f_1\gamma_2],$$

is a group, where $f \star g : x \mapsto (x)f(x)g$; the identity is $[1_{\operatorname{Aut}\Gamma(M)}, 1_{\operatorname{Aut} G}, x \mapsto 1_G]$ and $[\lambda^{-1}, \gamma^{-1}, x \mapsto (x\lambda^{-1}f\gamma^{-1})^{-1}]$ is the inverse of $[\lambda, \gamma, f]$.

Lemma 2.3. *The mapping $\Psi : \operatorname{Aut}\Gamma(M) \times \operatorname{Aut} G \times G^{I\cup J} \to M^M$ defined in Theorem 2.2 is a homomorphism.*

Proof. From the definition of Ψ,

$$[\lambda_1, \gamma_1, f_1][\lambda_2, \gamma_2, f_2]\Psi = [\lambda_1\lambda_2, \gamma_1\gamma_2, \lambda_1 f_2 \star f_1\gamma_2]\Psi$$

is the mapping in M^M given by

$$(i, g, j) \mapsto (i\lambda_1\lambda_2, i\lambda_1 f_2 \cdot if_1\gamma_2 \cdot g\gamma_1\gamma_2 \cdot (jf_1\gamma_2)^{-1} \cdot (j\lambda_1 f_2)^{-1}, j\lambda_1\lambda_2).$$

On the other hand, if $\alpha = [\lambda_1, \gamma_1, f_1]\Psi$ and $\beta = [\lambda_2, \gamma_2, f_2]\Psi$, then

$$\begin{aligned}(i, g, j)\alpha\beta &= (i\lambda_1, if_1 \cdot g\gamma_1 \cdot (jf_1)^{-1}, j\lambda_1)\beta \\ &= (i\lambda_1\lambda_2, i\lambda_1 f_2 \cdot [if_1 \cdot g\gamma_1 \cdot (jf_1)^{-1}]\gamma_2 \cdot (j\lambda_1 f_2)^{-1}, j\lambda_1\lambda_2) \\ &= (i\lambda_1\lambda_2, i\lambda_1 f_2 \cdot if_1\gamma_2 \cdot g\gamma_1\gamma_2 \cdot (jf_1\gamma_2)^{-1} \cdot (j\lambda_1 f_2)^{-1}, j\lambda_1\lambda_2),\end{aligned}$$

as required. □

Lemma 2.4. *Let $[\lambda, \gamma, f] \in \operatorname{Aut}\Gamma(M) \times \operatorname{Aut} G \times G^{I\cup J}$. Then $[\lambda, \gamma, f] \in \ker(\Psi)$ if and only if $\lambda = 1_{\Gamma(M)}$, $\gamma : g \mapsto hgh^{-1}$ for some $h \in G$, and $f : x \mapsto h^{-1}$.*

Proof. (⇒) Since $[\lambda, \gamma, f]\Psi = 1_M$, we have that $(i, g, j) = (i\lambda, if \cdot g\gamma \cdot (jf)^{-1}, j\lambda)$ for all $(i, g, j) \in M$. It follows that $\lambda = 1_{\Gamma(M)}$ and $if \cdot g\gamma \cdot (jf)^{-1} = g$ for all $g \in G$. In particular, if $g = 1_G$, then we deduce that $if = jf$ for all $i \in I$ and $j \in J$. Thus f is constant with value h^{-1}, for some $h \in G$. Finally, rearrange $if \cdot g\gamma \cdot (jf)^{-1} = g$ to obtain $g\gamma = (if)^{-1} \cdot g \cdot jf = hgh^{-1}$.

($\Leftarrow$) Let $(i, g, j) \in M$ be arbitrary. Then

$$(i, g, j)([\lambda, \gamma, f]\Psi) = (i\lambda, if \cdot g\gamma \cdot (jf)^{-1}, j\lambda) = (i, h^{-1} \cdot hgh^{-1} \cdot h, j) = (i, g, j),$$

and so $[\lambda, \gamma, f]\Psi \in \ker(\Psi)$. □

It follows from the previous two lemmas that $\operatorname{Aut} M$ is isomorphic to the quotient of elements in $\operatorname{Aut}\Gamma(M) \times \operatorname{Aut} G \times G^{I\cup J}$ satisfying (1) by the normal subgroup of $\operatorname{Aut}\Gamma(M) \times \operatorname{Aut} G \times G^{I\cup J}$ consisting of elements of the form

$$[1_{\Gamma(M)}, g \mapsto hgh^{-1}, x \mapsto h^{-1}],$$

for some $h \in G$.

Roughly speaking, an algorithm to compute $\operatorname{Aut} M$ is now clear; search through a transversal of cosets of $\ker(\Psi)$ in $\operatorname{Aut}\Gamma(M) \times \operatorname{Aut} G \times G^{I\cup J}$ and test if every element satisfies (1). The size of the search space in this case is

$$|\operatorname{Aut}\Gamma(M)|.|\operatorname{Aut} G|.|G|^{|I|+|J|}.$$

With a little more thought we can reduce the size of the search space considerably.

We will start by considering how to find triples in $\operatorname{Aut}\Gamma(M) \times \operatorname{Aut} G \times G^{I\cup J}$ that satisfy (1). If $\lambda \in \operatorname{Aut}\Gamma(M)$ and $\gamma \in \operatorname{Aut} G$ are fixed, then we give a method to construct all the functions $f \in G^{I\cup J}$ such that $[\lambda, \gamma, f]$ satisfies (1).

Let $K_1, K_2, \dots, K_t$ be the connected components of $\Gamma(M)$, for every i let T_i be a fixed spanning tree for K_i and let r_i be a fixed vertex in K_i.

If $\lambda \in \operatorname{Aut}\Gamma(M)$, $\gamma \in \operatorname{Aut} G$, and $g_i \in G$ are arbitrary, then we will define a binary relation $\rho_i = \rho_{K_i}(\lambda, \gamma, g_i) \subseteq K_i \times G$ using a function $\rho' : K_i \to G$ in the following three steps.

Step 1: the definition of ρ' is initiated by letting $r_i\rho' = g_i$.

Step 2: if (x, y) is an edge in T_i with $y\rho' \neq \emptyset$ but $x\rho' = \emptyset$, then assign

$$x\rho' = \begin{cases} p_{y\lambda x\lambda}^{-1} \cdot y\rho' \cdot p_{yx}\gamma & \text{if } x \in I \\ p_{x\lambda y\lambda} \cdot y\rho' \cdot (p_{xy}\gamma)^{-1} & \text{if } x \in J. \end{cases} \tag{2}$$

Repeat Step 2 until $x\rho'$ is defined for all $x \in K_i$. Since T_i is a tree, ρ' is a function.

Step 3: if $x \in I$, then define $x\rho_i$ to be the union of $\{x\rho'\}$ and

$$\{\, p_{y\lambda x\lambda}^{-1} \cdot y\rho' \cdot p_{yx}\gamma \;:\; (x, y) \in K_i \setminus T_i \,\}.$$

Otherwise, $x \in J$ and $x\rho_i$ is defined to be the union of $\{x\rho'\}$ and

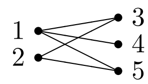
$$\{\, p_{x\lambda y\lambda} \cdot y\rho' \cdot (p_{xy}\gamma)^{-1} \;:\; (x, y) \in K_i \setminus T_i \,\}.$$

The following example demonstrates how the above procedure works in one particular case.

Example 2.5. Let M denote the Rees matrix semigroup $\mathcal{M}^0[C_3; \{1, 2\}, \{3, 4, 5\}; P]$ where $C_3 = \{1, x, x^2\}$ is the cyclic group of order 3 and

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & x \end{pmatrix}.$$

The graph $\Gamma(M)$ is



and so $\operatorname{Aut}\Gamma(M) = \langle (3\,5) \rangle$. The automorphism group of C_3 is $\operatorname{Aut} C_3 = \langle x \mapsto x^2 \rangle$. Let $r = 1$ be the fixed vertex in the unique connected component $K = \Gamma(M)$ and let T be the spanning tree for $\Gamma(M)$ with edges $(1, 3), (1, 4), (1, 5), (3, 2)$.

Now, let $\lambda = (3\,5)$, $\gamma : x \mapsto x$ and $g = 1 \in C_3$. Then from Steps 1 & 2 we obtain

$$\rho' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & x^2 & 1 & 1 & 1 \end{pmatrix}.$$

From Step 3, $1\rho = 1\rho' = 1 = 3\rho = 4\rho$,

$$2\rho = \{2\rho', p^{-1}_{5\lambda 2\lambda} \cdot 5\rho' \cdot p_{52}\gamma\} = \{x, x^2\}$$

and

$$5\rho = \{5\rho', p_{5\lambda 2\lambda} \cdot 2\rho' \cdot (p_{52}\gamma)^{-1}\} = \{1, x\},$$

and the example is complete.

Throughout the remainder of the paper we will denote the relation

$$\bigcup_{i=1}^{t} \rho_{K_i}(\lambda, \gamma, g_i) \subseteq (I \cup J) \times G$$

by $\rho(\lambda, \gamma, \vec{g})$ where $\vec{g} = (g_1, g_2, \ldots, g_t) \in G^t$ (the direct product of t copies of G).

Lemma 2.6. *Let $\lambda \in \operatorname{Aut}\Gamma(M)$, let $\gamma \in \operatorname{Aut} G$, and let $f \in G^{I \cup J}$. Then $[\lambda, \gamma, f]\Psi \in \operatorname{Aut} M$ if and only if the relation $\rho(\lambda, \gamma, \vec{g})$ where $\vec{g} = (r_1 f, r_2 f, \ldots, r_t f) \in G^t$ equals f.*

Proof. Throughout the proof we will denote $\rho(\lambda, \gamma, \vec{g})$ by ρ.

($\Rightarrow$) We start by proving that $f = \rho'$ by a finite recursion on $d(x)$, the least length of a path from any $x \in T_i$ to the fixed vertex $r_i \in K_i$. Starting the recursion with $x \in I \cup J$ where $d(x) = 0$, we get $x = r_i$ and so $x\rho' = r_i\rho' = r_i f = xf$.

Assume that $y\rho' = yf$ for all $y \in I \cup J$ such that $d(y) \leqslant m - 1$. Then let $x \in J$ where $d(x) = m$. It follows, from the construction of ρ', that $x\rho' = p_{x\lambda y\lambda} \cdot y\rho' \cdot (p_{xy}\gamma)^{-1}$ for some $y \in I$ with $d(y) = m - 1$. Thus $x\rho' = p_{x\lambda y\lambda} \cdot yf \cdot (p_{xy}\gamma)^{-1} = xf$ since f satisfies (1). The proof in the case that $x \in I$ follows analogously.

Now, if $x \in I$, then

$$x\rho = \{x\rho'\} \cup \{\, p^{-1}_{y\lambda x\lambda} \cdot y\rho' \cdot p_{yx}\gamma \;:\; (x, y) \in K_i \setminus T_i \,\}$$

But $p^{-1}_{y\lambda x\lambda} \cdot y\rho' \cdot p_{yx}\gamma = p^{-1}_{y\lambda x\lambda} \cdot yf \cdot p_{yx}\gamma = xf$, by (1), and $x\rho' = xf$. Thus $x\rho = xf$, as required. The proof in the case that $x \in J$ follows analogously.

($\Leftarrow$) By the construction of ρ and the fact that ρ is a function, we have that $i\rho = p^{-1}_{j\lambda i\lambda} \cdot j\rho \cdot p_{ji}\gamma$ for all $i \in I$ and $j \in J$ with $p_{ji} \neq 0$. Hence ρ satisfies (1) and so, by Theorem 2.2,

$$[\lambda, \gamma, \rho]\Psi = [\lambda, \gamma, f]\Psi \in \operatorname{Aut} M,$$

as required. □

So, to find the functions in $G^{I \cup J}$ that satisfy (1) it suffices, by Lemma 2.6, to find which of the relations $\rho(\lambda, \gamma, \vec{g})$ are functions. More precisely, let $G_i = \{\, g \in G \;:\; \rho_{K_i}(\lambda, \gamma, g) \text{ is a function} \,\}$. Then $\rho(\lambda, \gamma, \vec{g})$ is a function for some $\vec{g} \in G^t$ if and only if $\vec{g} \in G_1 \times G_2 \times \cdots \times G_t$. Therefore reducing the size of the search space from $|\mathrm{Aut}\,\Gamma(M)|.|\mathrm{Aut}\,G|.|G|^{|I|+|J|}$ to

$$|\mathrm{Aut}\,\Gamma(M)|.|\mathrm{Aut}\,G|.t|G|, \tag{3}$$

where t is the number of connected components of the graph $\Gamma(M)$.

Throughout the remainder of the paper, if G is a group and H is a subgroup of G we will denote a fixed transversal of left cosets of H in G by G/H. The *inner automorphisms* of a group G are denoted by $\mathrm{Inn}\,G(=\{\,\gamma : g \mapsto hgh^{-1} \mid h \in G\,\})$ and the *centre* of G is denoted $Z(G)$. The following lemma allows us to reduce the size of the search space given in (3) further still.

Lemma 2.7. *Let $[\lambda, \gamma, f] \in (\mathrm{Aut}\,M)\Psi^{-1}$ be arbitrary. Then there exists $\delta \in \mathrm{Aut}\,G/\mathrm{Inn}\,G$, $\vec{g} = (g_1, g_2, \ldots, g_t) \in (G/Z(G)) \times G^{t-1}$ such that $[\lambda, \gamma, f]\Psi = [\lambda, \delta, \rho(\lambda, \delta, \vec{g})]\Psi$.*

Proof. We start by proving a related claim. Let $\lambda \in \mathrm{Aut}\,\Gamma$, $\gamma \in \mathrm{Aut}\,G$, $\vec{g} = (g_1, g_2, \ldots, g_t) \in G^t$, and $k \in G$ such that $[\lambda, \gamma, \rho(\lambda, \gamma, \vec{g}k)]\Psi \in \mathrm{Aut}\,M$. Then we will prove that

$$[\lambda, \gamma, \rho(\lambda, \gamma, \vec{g}k)]\Psi = [\lambda, \gamma\phi_k, \rho(\lambda, \gamma\phi_k, \vec{g})]\Psi, \tag{4}$$

where $\phi_k : g \mapsto kgk^{-1} \in \mathrm{Inn}\,G$.

Let $\rho = \rho(\lambda, \gamma, \vec{g}k)$. Then it suffices to prove that $[\lambda, \gamma\phi_k, \rho(\lambda, \gamma\phi_k, \vec{g})]$ and $[\lambda, \gamma, \rho]$ are in the same coset of $\ker(\Psi)$. Consider the product

$$[\lambda, \gamma, \rho] \diamond [1_{\Gamma(M)}, \phi_k, c_{k^{-1}} : x \mapsto k^{-1}] = [\lambda, \gamma\phi_k, \lambda c_{k^{-1}} \star \rho\phi_k].$$

Recall that, by Lemma 2.4, $[1_{\Gamma(M)}, \phi_k, c_{k^{-1}} : x \mapsto k^{-1}] \in \ker(\Psi)$. If $y = \lambda c_{k^{-1}} \star \rho\phi_k : I \cup J \to G$, then $(x)y = x\rho.k^{-1}$. In particular, if $x = r_i$ for some $1 \leqslant i \leqslant t$, then $(x)y = x\rho.k^{-1} = r_i\rho.k^{-1} = g_i$. Thus $[\lambda, \gamma, \rho]\Psi = [\lambda, \gamma\phi_k, y]\Psi \in \mathrm{Aut}\,M$. Hence, by Lemma 2.6, $y = \rho(\lambda, \gamma\phi_k, \vec{g})$, as required.

We may now use (4) to prove the lemma. Let $[\lambda, \gamma, f] \in (\mathrm{Aut}\,M)\Psi^{-1}$ be arbitrary. Then, by Lemma 2.6, $f = \rho(\lambda, \gamma, \vec{h})$ for some $\vec{h} \in G^t$. Now, $\gamma = \delta\phi_k$ for some $\delta \in \mathrm{Aut}\,G/\mathrm{Inn}\,G$. Hence, by (4)

$$[\lambda, \gamma, f]\Psi = [\lambda, \delta\phi_k, \rho(\lambda, \delta\phi_k, \vec{h})]\Psi = [\lambda, \delta, \rho(\lambda, \delta, \vec{h}k)]\Psi.$$

If $\vec{h} = (h_1, h_2, \ldots, h_t)$, then $h_1k = g_1z$ for some $g_1 \in G/Z(G)$ and $z \in Z(G)$. Hence $k^{-1}h_1^{-1}g_1 \in Z(G)$ and so $\phi_{k^{-1}h_1^{-1}g_1}$ is the identity of $\mathrm{Inn}\,G$. Thus

$$[\lambda, \delta, \rho(\lambda, \delta, \vec{h}k)]\Psi = [\lambda, \delta\phi_{k^{-1}h_1^{-1}g_1}, \rho(\lambda, \delta\phi_{k^{-1}h_1^{-1}g_1}, \vec{h}k)]\Psi = [\lambda, \delta, \rho(\lambda, \delta, \vec{g})]\Psi,$$

where $\vec{g} = \vec{h}h_1^{-1}g_1 = (g_1, h_2h_1^{-1}g_1, \ldots, h_th_1^{-1}g_1) \in (G/Z(G)) \times G^{t-1}$. □

Applying Lemma 2.7 the size of the search space becomes

$$|\mathrm{Aut}\,\Gamma(M)|.|\mathrm{Aut}\,G/\mathrm{Inn}\,G|.(|G/Z(G)| + (t-1)|G|), \tag{5}$$

where t is the number of connected components of $\Gamma(M)$ and the number of automorphism is at most

$$|\mathrm{Aut}\,\Gamma(M)|.|\mathrm{Aut}\,G/\mathrm{Inn}\,G|.|G/Z(G)|.|G|^{t-1}.$$

Note that there are some small values where the size of the search space given by (3) is smaller than that given by (5), as can be seen in Example 2.9.

The algorithm used to compute the automorphisms of an arbitrary finite Rees matrix semigroup is given in Algorithm 1.

Algorithm 1 - The automorphism group of a Rees matrix semigroup.

$A \leftarrow \{1_M\}$
for λ in $\operatorname{Aut}\Gamma(M)$ **do**
    for γ in $\operatorname{Aut}G/\operatorname{Inn}G$ **do**
        $B_1, B_2, \ldots, B_t \leftarrow \emptyset$
        for i in $\{1, 2, \ldots, t\}$ **do**
            for $g \in G/Z(G)$ if $i = 1$ or $g \in G$ if $i \neq 1$ **do**
                find $\rho = \rho_{K_i}(\lambda, \gamma, g)$
                if ρ is a function **then**
                    $B_i \leftarrow B_i \cup \{\rho\}$
                end if
            end for
        end for
        $A \leftarrow \langle\, A \cup \{\, [\lambda, \gamma, f] \;:\; f|_{K_i} = \rho \in B_i \;\; \forall i \,\}\, \rangle$
    end for
end for
return A

To conclude the section we give several examples.

Example 2.8. Let M denote the Rees matrix semigroup given in Example 2.5, where $G = \{1, x, x^2\}$, $\operatorname{Aut}\Gamma(M) = \langle\, (3\,5)\,\rangle$, $\operatorname{Aut}G = \langle\, x \mapsto x^2\,\rangle$, $G/Z(G) = \{1\}$, and the number of connected components in $\Gamma(M)$ is 1. The number of elements in the search space is

$$|\operatorname{Aut}\Gamma(M)|.|\operatorname{Aut}G/\operatorname{Inn}G|.(|G/Z(G)| + 0.|G|) = 2.2.(1+0) = 4.$$

The identity is represented by the triple $[1_{\Gamma(M)}, 1_{\operatorname{Aut}G}, (1)]$. As we saw in Example 2.5 the triple $[(3\,5), 1_{\operatorname{Aut}G}, (1)]$ does not represent an automorphism of M.

If $\lambda = 1_{\Gamma(M)}$, $\gamma = x \mapsto x^2$, then assigning values to ρ' using the edges in the spanning tree T for $\Gamma(M)$ gives

$$\rho' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Next, to find $\rho = \rho(1_{\Gamma(M)}, x \mapsto x^2, (1))$ we perform Step 3 from above. Considering the single edge $(2,5)$ in $K \setminus T$ yields $2\rho = \{2\rho', p^{-1}_{5\lambda 2\lambda} \cdot 5\rho' \cdot p_{52}\gamma\} = \{1, x\}$. Hence, by Lemma 2.6, $[1_{\Gamma(M)}, x \mapsto x^2, (1)]$ does not represent an automorphism of M.

Finally, if $\lambda = (3\,5)$, $\gamma = x \mapsto x^2$, then assign values to ρ' using the edges in the spanning tree T for $\Gamma(M)$ gives

$$\rho' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & x^2 & 1 & 1 & 1 \end{pmatrix}.$$

Now, again performing Step 3, we obtain

$$2\rho = \{2\rho', p^{-1}_{5\lambda 2\lambda} \cdot 5\rho' \cdot p_{52}\gamma\} = \{x^2\}$$

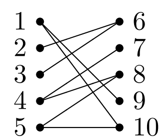
$$5\rho = \{5\rho', p_{5\lambda 2\lambda} \cdot 2\rho' \cdot (p_{52}\gamma)^{-1}\} = \{1\}.$$

Hence $[(35), x \mapsto x^2, (1)]$ does represent an automorphism of M and we conclude that $\operatorname{Aut}M \cong \mathbb{Z}_2$.

Example 2.9. Let M denote the Rees matrix semigroup $\mathcal{M}^0[C_6; 5, 5; P]$ where $C_6 = \{1, x, \ldots, x^5\}$ is the cyclic group of order 6 and

$$P = \begin{pmatrix} 0 & x^4 & x & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & x^4 & x^5 \\ x^4 & 0 & 0 & 0 & 0 \\ x^4 & 0 & 0 & 0 & x \end{pmatrix}.$$

The graph $\Gamma(M)$ is



and so $\operatorname{Aut}\Gamma(M) = \langle\, (1\,4)(7\,9)(8\,10), (2\,3)\,\rangle$. The automorphism group of C_6 is $\operatorname{Aut} C_6 = \langle\, x \mapsto x^5\,\rangle$ and $\operatorname{Inn} C_6$ is trivial. Since C_6 is abelian, $Z(C_6) = C_6$. Thus there are

$$|\operatorname{Aut}\Gamma(M)|.|\operatorname{Aut} G/\operatorname{Inn} G|.(|G/Z(G)| + |G|) = 4.2.(1 + 6) = 56$$

elements in the search space leading to at most

$$|\operatorname{Aut}\Gamma(M)|.|\operatorname{Aut} G/\operatorname{Inn} G|.|G/Z(G)|.|G| = 4.2.6 = 48$$

automorphisms of M. The graph $\Gamma(M)$ is a tree and so all the candidates, in fact, represent automorphisms. It can be shown that $\operatorname{Aut} M \cong C_2 \times C_2 \times C_2 \times \mathcal{S}_3$.

The next example shows that in some sense Algorithm 1 is the best possible, since it can happen that every element in the search space represents an automorphism of M.

Example 2.10. Let M denote the Rees matrix semigroup $\mathcal{M}^0[\mathcal{S}_7; 2, 1; P]$ where P is the 1×2 matrix with all entries equal to the identity of the symmetric group $\mathcal{S}_7$ of degree 7. Then the size of the search space in Algorithm 1 is

$$|\operatorname{Aut}\Gamma(M)|.|\operatorname{Aut} G/\operatorname{Inn} G|.(|G/Z(G)| + 0.|G|) = 2.1.1.7! = 10080.$$

Verifying that every element in this space is in fact an automorphism of M can be done using the MONOID package [40] for GAP.

Although in the previous example every element in the search space is an automorphism there is still considerable room for improvement as the next example shows.

Example 2.11. Let M denote the Rees matrix semigroup $\mathcal{M}^0[G; m, n; P]$ where P is any $n \times m$ matrix with no zero entries. Then the graph $\Gamma(M)$ is the complete bipartite graph with vertex sets of size m and n. Thus $\operatorname{Aut}\Gamma(M) \cong \mathcal{S}_m \times \mathcal{S}_n$ and even for small values of m and n the size of the search space in Algorithm 1 is impractically large.

3. Inner automorphisms

Let S be a semigroup of transformations of the n-element set $\{1, 2, \ldots, n\}$ and let g be an element of $\mathcal{S}_n$, the symmetric group on $\{1, 2, \ldots, n\}$. If the mapping $s \mapsto gsg^{-1}$ is an automorphism of S, then it is called an *inner automorphism*. Note that the notion of an inner automorphism of a semigroup differs from the notion of the same name for groups. The group of all inner automorphisms of S is denoted by $\operatorname{Inn} S$. The purpose of this section is to give an algorithm to compute the inner automorphisms of S.

In what follows $\operatorname{Ims}(S)$ denotes the set of images that elements of S admit. Likewise, $\operatorname{Kers}(S)$ denotes the set of kernels that elements of S admit. Both $\operatorname{Ims}(S)$ and $\operatorname{Kers}(S)$ can be found by using a simple orbit algorithm without computing the elements of S. As usual, if G is a subgroup of $\mathcal{S}_n$ and N is a subset of $\{1, 2, \ldots, n\}$, set of subsets of $\{1, 2, \ldots, n\}$, or subset of S, then $G_{\{N\}}$ denotes the setwise stabilizer of N in G. As in Section 2, we will use G/H to mean a transversal of the left cosets of a subgroup H in a group G.

If $g \in \mathcal{S}_n$ such that $\phi_g : s \mapsto gsg^{-1}$ is an inner automorphism of S, then it is obvious that $\operatorname{Ims}(S)\phi_g = \operatorname{Ims}(S)$ and $\operatorname{Kers}(S)\phi_g = \operatorname{Kers}(S)$. In other words, if $I = \mathcal{S}_n$, then $g \in I_{\{\operatorname{Ims}(S)\}} \cap I_{\{\operatorname{Kers}(S)\}}$. On the other hand, let $g \in I_{\{\operatorname{Ims}(S)\}} \cap I_{\{\operatorname{Kers}(S)\}}$ and let X be any generating set for S. Then $\phi_g \in \operatorname{Inn} S$ if and only if $x \in \langle X\phi_g \rangle$ for all $x \in X$. Algorithm 2 uses these simple facts to compute the inner automorphism group of S.

Algorithm 2 - Inner automorphisms of a transformation semigroup $S = \langle X \rangle$.

$A \leftarrow \{1_S\}$ & $I \leftarrow \mathcal{S}_n$
$I \leftarrow I_{\{\operatorname{Ims}(S)\}}$
if I is not trivial **then**
    $I \leftarrow I_{\{\operatorname{Kers}(S)\}}$
    if I is not trivial **then**
        $A \leftarrow$ automorphisms induced by $I_{\{X\}}$ acting on X
        for a in $I/I_{\{X\}}$(transversal of cosets of $I_{\{X\}}$ in I) **do**
            if not a in A **then**
                if $x \in \langle Xa \rangle$ for all $x \in X$ **then**
                    $a \leftarrow$ the automorphism induced by a
                    $A \leftarrow \langle A, a \rangle$
                end if
            end if
        end for
    end if
end if
return A

Example 3.1. Let R denote the group ring of the cyclic group $\mathbb{Z}_4$ of order 4 over the field with 2 elements. Then using the semigroup theoretic analogue of Cayley's theorem we can find a transformation semigroup S with generating set

$$X = \left\{ x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 3 & 7 & 9 & 1 & 15 & 5 & 11 & 13 & 11 & 13 & 3 & 5 & 15 & 7 & 9 \end{pmatrix}, \right.$$

$$y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 16 & 9 & 6 & 5 & 8 & 13 & 12 & 15 & 2 & 3 & 4 & 7 & 10 & 11 & 14 \end{pmatrix},$$

$$z = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 8 & 11 & 2 & 5 & 16 & 13 & 14 & 3 & 6 & 15 & 10 & 7 & 4 & 9 & 12 \end{pmatrix} \Bigg\}$$

that is isomorphic to the multiplicative semigroup of R.

The setwise stabilizer J of $\text{Ims}(S)$ in $\mathcal{S}_{16}$ has 1935360 elements, and the setwise stabilizer I of $\text{Kers}(S)$ in J has 2048 elements. There are 2 elements in the stabilizer $I_{\{X\}}$ of the generators X in I and a further 128 elements in $I/I_{\{X\}}$.

As it turns out, $\text{Inn}\, S = \text{Aut}\, S \cong C_2 \times D_8$; see Section 6 for more details.

The overall aim is to compute $\text{Aut}\, S$ for an arbitrary S. In conjunction with Algorithm 2, the following theorem gives us a method to do this in one special case.

Theorem 3.2. *Let S be a subsemigroup of $\mathcal{T}_n$ such that for all $s, t \in S$ there exists a constant mapping $k \in S$ such that $ks \neq kt$. Then* $\text{Aut}\, S = \text{Inn}\, S$.

Proof. For a proof see [49, Theorem 1] . □

Corollary 3.3. *If S contains all the constant mappings, then* $\text{Aut}\, S = \text{Inn}\, S$.

The converse of Theorem 3.2 is not true. For example, if S is the semigroup from Example 3.1, then the mapping with constant value 1 is the only constant in S. However, the generators do not satisfy the condition of Theorem 3.2 and $\text{Aut}\, S = \text{Inn}\, S$.

4. THE MAIN ALGORITHM

In this section we give the main algorithm for computing the automorphism group $\text{Aut}\, S$ of a finite transformation semigroup S. Throughout the remainder of this section we assume that S is a finite transformation semigroup. Of course, since every finite semigroup is isomorphic to a finite transformation semigroup the algorithm described in this section can be used to compute the automorphism group of an arbitrary finite semigroup.

As mentioned in Section 1 the algorithm consists of searching through a space of candidates and testing if the elements are automorphisms. Our principal focus in this section is to reduce the size of the search space by considering certain structural aspects of S that are preserved by automorphisms. The main aspect we consider is Green's $\mathcal{D}$-relation. Let S^1 denote the semigroup S with a new identity adjoined, that is, an element 1 that acts as an identity on the elements of S. Then *Green's $\mathcal{L}$-relation* is the set of pairs $(x, y) \in S \times S$ such that $S^1x = \{\ sx\ :\ s \in S^1\ \} = S^1y$; denoted by $x\mathcal{L}y$. *Green's $\mathcal{R}$-relation* is defined analogously and denoted by $x\mathcal{R}y$. Although both Green's $\mathcal{L}$- and $\mathcal{R}$-relations are preserved by automorphisms of S (see Lemma 4.1(i)), we are interested in their composition $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$. Like $\mathcal{L}$ and $\mathcal{R}$, *Green's $\mathcal{D}$-relation* is an equivalence relation and as such partitions the set of elements of S into $\mathcal{D}$-classes.

Using the fact that S is finite, it can be shown (see [27, Proposition 2.1.4]) that $x\mathcal{D}y$ if and only if $S^1xS^1 = S^1yS^1$. This alternative formulation leads to a natural partial order on the $\mathcal{D}$-classes of S: $D_1 \leqslant D_2$ if $S^1xS^1 \subseteq S^1yS^1$ for some $x \in D_1$ and $y \in D_2$.

So far, S has been partitioned into $\mathcal{D}$-classes and arranged in a partial order $\leqslant$. Let us now inspect the individual $\mathcal{D}$-classes more closely. Let D be a $\mathcal{D}$-class of S.

Then define D^* such that $D^* = D$ if $st \in D$ for all $s, t \in D$ and $D^* = D \cup \{0\}$ otherwise and define multiplication on D^* by

$$s * t = \begin{cases} st & \text{if } s, t, \text{and } st \in D \\ 0 & \text{if } s, t, \text{or } st \notin D. \end{cases}$$

Then D^* is a semigroup, called the *principal factor* of D. What is more, D^* is either a zero semigroup or a completely simple semigroup with or without a zero [27, Theorem 3.1.6]. It follows by the Rees-Suschkewitz Theorem [27, Theorem 3.2.3] that a $\mathcal{D}$-class of S can be thought of as a Rees matrix semigroup as described in Section 2. If $st \in D$ for all $s, t \in D$, then the construction in the proof of the Rees-Suschkewitz Theorem yields a isomorphism from D to $\mathcal{M}^0[G; I, J; P] \setminus \{0\}$. However, in this case $\operatorname{Aut} \mathcal{M}^0[G; I, J; P] \cong \operatorname{Aut} \mathcal{M}^0[G; I, J; P] \setminus \{0\}$ and so without loss of generality we can ignore the distinction.

Let D_1 and D_2 be $\mathcal{D}$-classes of S. Then $\phi : D_1 \to D_2$ is an *isomorphism* if it is the restriction to D_1 of an isomorphism between D_1^* and D_2^*; we will denote this by $D_1 \cong D_2$.

The following simple lemma is our main tool for reducing the size of the search space in Algorithm 3.

Lemma 4.1. *Let S be a semigroup, let D_1 and D_2 be $\mathcal{D}$-classes of S, and $\phi \in \operatorname{Aut} S$. Then the following hold*

(i) *ϕ preserves Green's $\mathcal{D}$-relation ($x\phi \mathcal{D} y\phi$ if and only if $x \mathcal{D} y$);*
(ii) *ϕ preserves the partial order $\leqslant$ of $\mathcal{D}$-classes ($D_1\phi \leqslant D_2\phi$ if and only if $D_1 \leqslant D_2$);*
(iii) *if $D_1\phi = D_2$, then D_1 and D_2 are isomorphic.*

Let S be an arbitrary finite semigroup generated by a set X with $\mathcal{D}$-classes $D_1, D_2, \ldots, D_t$. Using Lemma 4.1, we can now define the group inside which $\operatorname{Aut} S$ lives. Let $\operatorname{Aut} \mathfrak{P}$ denote the automorphism group of the partial order $\mathfrak{P}$ of $\mathcal{D}$-classes of S such that $D\psi \cong D$ for all $\psi \in \operatorname{Aut} \mathfrak{P}$ and all $\mathcal{D}$-classes D, and let $\phi_{i,j} : D_i \to D_j$ be a fixed isomorphism for every pair of isomorphic $\mathcal{D}$-classes D_i and D_j such that $\phi_{i,j} \circ \phi_{j,k} = \phi_{i,k}$ for all i, j, k. Let $\Psi : \operatorname{Aut} \mathfrak{P} \to \operatorname{Aut} (\operatorname{Aut} D_1 \times \cdots \times \operatorname{Aut} D_t)$ be defined by

$$(\psi)\Psi : (\delta_1, \ldots, \delta_t) \mapsto (\phi_{1,1\psi^{-1}} \delta_{1\psi^{-1}} \phi^{-1}_{1,1\psi^{-1}}, \ldots, \phi_{t,t\psi^{-1}} \delta_{t\psi^{-1}} \phi^{-1}_{t,t\psi^{-1}}),$$

where we follow the convention that $D_i\psi = D_{i\psi}$. Then form the semidirect product of $\operatorname{Aut} D_1 \times \cdots \times \operatorname{Aut} D_t$ by $\operatorname{Aut} \mathfrak{P}$ via Ψ; denoted $(\operatorname{Aut} D_1 \times \cdots \times \operatorname{Aut} D_t) \rtimes \operatorname{Aut} \mathfrak{P}$. An element $f = (\delta_1, \delta_2, \ldots, \delta_t, \psi)$ of $(\operatorname{Aut} D_1 \times \cdots \times \operatorname{Aut} D_t) \rtimes \operatorname{Aut} \mathfrak{P}$ acts on S as follows

$$sf = s\delta_i \phi_{i,i\psi} \text{ if } s \in D_i.$$

Theorem 4.2. $\operatorname{Aut} S$ *is isomorphic to a subgroup of* $(\operatorname{Aut} D_1 \times \cdots \times \operatorname{Aut} D_t) \rtimes \operatorname{Aut} \mathfrak{P}$.

Proof. This is a straightforward corollary of Lemma 4.1. □

Of course, in order to compute $\operatorname{Aut} S$ we only have to consider the images of the generators of S under elements of $(\operatorname{Aut} D_1 \times \cdots \times \operatorname{Aut} D_t) \rtimes \operatorname{Aut} \mathfrak{P}$. Moreover, there may be elements of $\operatorname{Aut} D_i$, beside the identity, that fix the generators $X \cap D_i$ in D_i pointwise. Let $G_{(N)}$ denote the pointwise stabilizer of the set N with respect to the group G. So, we will search through the elements of the set

$$[\operatorname{Aut} D_1 / (\operatorname{Aut} D_1)_{(X \cap D_1)}] \times \cdots \times [\operatorname{Aut} D_r / (\operatorname{Aut} D_r)_{(X \cap D_r)}] \times \operatorname{Aut} \mathfrak{P},$$

where $D_1, D_2, \ldots, D_r$ are the $\mathcal{D}$-classes contained in the orbits in $\operatorname{Aut}\mathfrak{P}$ of $\mathcal{D}$-classes containing a generator. The elements in the search space will be tested to see if they induce automorphisms. This is done by verifying that the generators of S are mapped to a new generating set and by finding a presentation that defines S and testing if the new generators satisfy the relations. The Froidure-Pin Algorithm [15] conveniently allows the $\mathcal{D}$-classes of S, the partial order of $\mathcal{D}$-classes of S, and a presentation that defines S to be calculated more or less simultaneously. Thus nothing is lost by requiring that we know a presentation for S.

The automorphism group of the partial order of $\mathcal{D}$-classes can be computed using the method given in [38] implemented in nauty [39] and available through the GAP package GRAPE [48]. Finally, since S is a transformation semigroup it is possible to verify if it is completely simple using [21, Proposition 2.3]. Algorithm 3 describes how to compute the automorphism group of S.

Algorithm 3 - The automorphism group of a finite transformation semigroup $S = \langle X \rangle$.

```
if S is completely simple then
    apply Algorithm 1 to S
else
    A ← Inn S from Algorithm 2 (automorphisms)
    if S satisfies Theorem 3.2 then
        return A
    else
        R ← relations of presentation defining S
        compute \mathcal{D}-classes D_1, D_2, ..., D_t & Aut \mathfrak{P}
        find D_1, D_2, ..., D_r - orbits in Aut \mathfrak{P} of \mathcal{D}-classes of generators X
        Ω ← [Aut D_1/(Aut D_1)_{(X∩D_1)}] × ··· × [Aut D_r/(Aut D_r)_{(X∩D_r)}] × Aut \mathfrak{P}
        i ← 0 & B ← {} (non-automorphisms)
        while 2|A| + |B| ⩽ |Ω| & i ⩽ |Ω| do
            i ← i + 1 & Ω_i ← the i^{th} element of Ω
            if not Ω_i in A or B then
                if XΩ_i satisfies R then
                    if x in ⟨ XΩ_i ⟩ for all x ∈ X then
                        A ← ⟨ A, Ω_i ⟩
                    end if
                end if
            end if
            B ← B ∪ Bt
        end while
    end if
end if
return A
```

Examples 4.3, 4.4, 4.5, and 4.6 are examples of the algorithm at work; the unexplained steps can be verified using GAP.

Example 4.3. Let us return to the multiplicative semigroup S of the group ring R defined in Example 3.1. The semigroup S is not completely simple and S does not satisfy the hypothesis of Theorem 3.2. Hence using the Froidure-Pin Algorithm we compute the following presentation that defines S

$$\langle\, x, y, z \quad | \quad yx = xy,\ zx = xz,\ zy = yz,\ z^2 = y^2,\ x^2z = x^2y,\ xyz = x,\ x^3y = x^3,\ x^2y^2 = x^2,\ xy^3 = xz,\ xy^2z = xy,\ x^5 = x^4,\ y^5 = y,\ y^4z = z \,\rangle.$$

The $\mathcal{D}$-classes in S are: D_1 containing the generators y and z, D_2 containing the generator x, D_3 and D_4 respectively containing the mappings

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 7 & 5 & 13 & 1 & 7 & 1 & 13 & 5 & 13 & 5 & 7 & 1 & 7 & 5 & 13 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 5 & 1 & 5 & 1 & 5 & 1 & 5 & 1 & 5 & 1 & 5 & 1 & 5 & 1 & 5 \end{pmatrix}$$

and D_5 containing the constant mapping with value 1.

The partial order P of the $\mathcal{D}$-classes is just a chain with $D_1 \geq_{\mathcal{D}} D_2 \geq_{\mathcal{D}} D_3 \geq_{\mathcal{D}} D_4 \geq_{\mathcal{D}} D_5$. Hence $\operatorname{Aut}\mathfrak{P}$ is trivial and so the orbits of $\mathcal{D}$-classes of the generators of S only contain D_1 and D_2. Now, D_1^* is isomorphic to the group $C_4 \times C_2$ and D_2^* is isomorphic to a zero semigroup with 5 elements. It can then be shown that $\mathrm{Aut}\, D_1$ is isomorphic to the dihedral group with 8 elements and $\mathrm{Aut}\, D_2$ is isomorphic to $\mathcal{S}_4$. The stabilizer of the generators in $S \cap D_1$ with respect to $\mathrm{Aut}\, D_1$ is trivial and the stabilizer of the generator in $S \cap D_2$ contains 6 elements. Thus the search space contains

$$|\mathrm{Aut}\, D_1 \times [\mathrm{Aut}\, D_2/(\mathrm{Aut}\, D_2)_{(X\cap D_2)}] \times \mathrm{Aut}\, \mathfrak{P}| = 8.4.1 = 32$$

elements.

Recall from Example 3.1 that $\mathrm{Inn}\, S \cong C_2 \times D_8$. Therefore $\mathrm{Aut}\, S = \mathrm{Inn}\, S$ if and only if there is a single element in $(\mathrm{Aut}\, D_1 \times [\mathrm{Aut}\, D_2/(\mathrm{Aut}\, D_2)_{(X\cap D_2)}] \times \mathrm{Aut}\, \mathfrak{P}) \setminus \mathrm{Inn}\, S$ that does not induce an automorphism of S.

As it turns out, running through the elements of the search space we find such an element and so $\mathrm{Aut}\, S = \mathrm{Inn}\, S \cong C_2 \times D_8$.

Example 4.4. Let S be the semigroup generated by the following transformations

$$X = \left\{ x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 4 & 8 & 8 & 8 & 8 & 4 & 8 \end{pmatrix}, y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 8 & 2 & 5 & 5 & 8 & 8 \end{pmatrix}, \right.$$

$$\left. z = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 8 & 3 & 7 & 8 & 3 & 7 & 8 \end{pmatrix}, t = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 6 & 6 & 8 & 6 & 8 & 8 & 8 \end{pmatrix} \right\}.$$

Incidently, the semigroup S is Knast's example of a semigroup that lies in the variety **LJ** (locally $\mathcal{J}$-trivial semigroups) but not in the variety B_1 (the variety of semigroups corresponding to the dot-depth one languages) as given in the manual for [44]. The semigroup S has 30 elements.

The set $\mathrm{Ims}(S)$ of images that elements of S admit is

$$\{\{2,5,8\},\{2,8\},\{3,7,8\},\{3,8\},\{4,8\},\{5,8\},\{6,8\},\{7,8\},\{8\}\}$$

and the setwise stabilizer $I_{\{\mathrm{Ims}(S)\}}$ of $\mathrm{Ims}(S)$ in $I = \mathcal{S}_8$ is the permutation group generated by $\{(4\,6),(2\,3)(5\,7),(2\,3\,5\,7)\}$ ($I_{\{\mathrm{Ims}(S)\}} \cong C_2 \times D_8$). The set $\mathrm{Kers}(S)$ of kernels that elements of S admit is

$$\{\{\{1,2,3,4,5,6,7,8\}\},\{\{1,2,3,5,6,8\},\{4,7\}\},\{\{1,2,4,5,7,8\},\{3,6\}\},$$
$$\{\{1,2,5,8\},\{3,6\},\{4,7\}\},\{\{1,2,7\},\{3,4,5,6,8\}\},\{\{1,3,5,6,7,8\},\{2,4\}\},$$
$$\{\{1,3,7,8\},\{2,4\},\{5,6\}\},\{\{1,3,7,8\},\{2,4,5,6\}\},\{\{1,4,6,7,8\},\{2,3,5\}\}\}$$

and the setwise stabilizer of $\mathrm{Kers}(S)$ in $I_{\{\mathrm{Ims}(S)\}}$ is trivial. Thus $\mathrm{Inn}\,(S)$ is trivial.

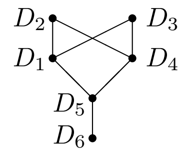
The presentation

$$\begin{aligned}\langle\, x,y,z,t \quad | \quad & xt = x^2, y^2 = y, yz = x^2, tx = x^2, t^2 = x^2, zy = x^2, z^2 = z, x^3 = x^2, \\ & x^2y = x^2, x^2z = x^2, xyx = x, xzx = x, xzt = x^2, yx^2 = x^2, tyx = x^2, \\ & tyt = t, tzx = x^2, tzt = t, zx^2 = x^2 \,\rangle\end{aligned}$$

defines S. The generators x,y,z,t of S lie in the distinct $\mathcal{D}$-classes $D_1,\ldots D_4$, respectively, and there are 2 further $\mathcal{D}$-classes D_5 and D_6 with representatives

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 6 & 8 & 8 & 8 & 8 & 6 & 8 \end{pmatrix}$$

and the constant function with value 8, respectively. The Hassé diagram of the partial order of $\mathcal{D}$-classes is



and so we see that $\operatorname{Aut}\mathfrak{P} = \langle (2\,3), (1\,4)\rangle (\cong C_2 \times C_2)$. Now, $D_1 \cong D_4 \cong \mathcal{M}^0[\langle()\rangle; 3, 3; P]$ where P is the matrix

$$\begin{pmatrix} () & () & 0 \\ () & 0 & () \\ 0 & () & () \end{pmatrix}$$

and both D_2 and D_3 are trivial groups. The automorphism groups of D_1 and D_4 are isomorphic to $\mathcal{S}_6$ and clearly both $\operatorname{Aut} D_2$ and $\operatorname{Aut} D_3$ are trivial. It turns out that the stabilizers of X in D_1 and D_4 under the action of their respective automorphism groups have size 2. Thus the search space has size

$$|[\operatorname{Aut} D_1/(\operatorname{Aut} D_1)_{(X\cap D_1)}]| \cdot |[\operatorname{Aut} D_4/(\operatorname{Aut} D_4)_{(X\cap D_4)}]| \cdot |\operatorname{Aut}\mathfrak{P}| = 3.3.4 = 36.$$

As it turns out none of the non-identity elements in the search space are automorphisms and so $\operatorname{Aut} S$ is trivial.

The next example shows that in some cases it can be better not to compute the inner automorphisms in Algorithm 3.

Example 4.5. Let S be the semigroup generated by the following set of transformations

$$X = \left\{ x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 1 \end{pmatrix}, y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix} \right\}$$

Then S has 40266 elements.

If $I = \mathcal{S}_9$, then $I_{\{\mathrm{Ims}(S)\}}$ has 1296 elements and so does the stabilizer of $\mathrm{Kers}(S)$ in $I_{\{\mathrm{Ims}(S)\}}$. Since this is a relatively large number, let us first consider the size of the search space constructed from the $\mathcal{D}$-classes.

The number of $\mathcal{D}$-classes in S is 11 with the generators x and y is different $\mathcal{D}$-classes D_x and D_y. The orbit of each of these $\mathcal{D}$-classes in $\operatorname{Aut}\mathfrak{P}$ is trivial. Now, $\operatorname{Aut}(D_x) \cong C_6$ and $\operatorname{Aut}(D_y)$ is a group with 93312 elements. The stabilizer of x in $\operatorname{Aut}(D_x)$ is trivial but thankfully the size of the pointwise stabilizer of the generator y in $\operatorname{Aut}(D_y)$ has 5184 elements. Thus

$$|[\operatorname{Aut} D_1/(\operatorname{Aut} D_1)_{(X\cap D_1)}]| \cdot |[\operatorname{Aut} D_2/(\operatorname{Aut} D_2)_{(X\cap D_2)}]| \cdot |\operatorname{Aut}\mathfrak{P}| = 6 \cdot 18 \cdot 1 = 108.$$

As it turns out, exactly half of the elements in this space are automorphisms and $\operatorname{Aut} S \cong (C_9 \rtimes C_3) \rtimes C_2$ (the group with identification number $[54, 6]$ used in the Small Group library [6] available in GAP and MAGMA [34]).

Example 4.6. Let S be the semigroup generated by the following set of transformations

$$X = \left\{ x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 9 & 10 & 11 & 12 & 8 \end{pmatrix}, \right.$$

$$y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 1 & 4 & 5 & 6 & 7 & 3 & 8 & 9 & 10 & 11 & 12 \end{pmatrix},$$

$$\left. z = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 1 & 1 & 4 & 3 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{pmatrix} \right\}.$$

Then S is a Clifford semigroup, that is, a strong semilattice Y of groups G_y, $y \in Y$, with respect to the homomorphisms $\phi_{x,y} : G_x \to G_y$ (multiplication is defined by $st = (s)\phi_{s,st}(t)\phi_{t,st}$). In particular, the semilattice in this case has 2 elements $a > b$, the groups $G_a \cong C_5$ and $G_b \cong C_5 \times \mathcal{S}_5$ correspond to the $\mathcal{D}$-classes D_x of x and $D_{y,z}$ of y and z, respectively, and the homomorphism $\phi_{a,b} : G_a \to G_b$ is defined by $x \mapsto xz^2$.

Clearly the automorphism group of the partial order $\mathfrak{P}$ of $\mathcal{D}$-classes of S is trivial. Now, $\operatorname{Aut} C_5 \cong C_4$ and $\operatorname{Aut}(C_5 \times \mathcal{S}_5) \cong C_4 \times \mathcal{S}_5$, the stabilizer of x in $\operatorname{Aut} D_x$ is trivial and the stabilizer of y and z in $\operatorname{Aut} D_{y,z}$ contains 4 elements. Thus the size of the search space is

$$|[\operatorname{Aut} D_x/(\operatorname{Aut} D_x)_{(X\cap D_x)}]|\cdot|[\operatorname{Aut} D_{y,z}/(\operatorname{Aut} D_{y,z})_{(X\cap D_{y,z})}]|\cdot|\operatorname{Aut}\mathfrak{P}| = 4\cdot 120\cdot 1 = 480.$$

It can be shown using the implementation of Algorithm 3 in GAP that every element in the search space is indeed an automorphism.

5. Small semigroups

In this section we give the automorphism groups of all semigroups up to isomorphism and anti-isomorphism of orders up to 7 and the number of semigroups with a given automorphism group.

Note that the number of semigroups of order 8 is 1843120128 and the number of semigroups of order 9 is unknown, and this is the reason we do not attempt to compute the automorphism groups of semigroups of orders greater than 7. The semigroups were computed using the method described in [10] and are available in the MONOID package [40] for GAP.

n	Automorphism groups	Number of semigroups
2	trivial	3
	C_2	1
3	trivial	12
	C_2	5
	$\mathcal{S}_3$	1
4	trivial	78
	C_2	39
	$C_2 \times C_2$	3
	$\mathcal{S}_3$	5
	$\mathcal{S}_4$	1
5	trivial	746
	C_2	342
	C_3	2
	C_4	1

n	Automorphism groups	Number of semigroups
5	$C_2 \times C_2$	26
	$\mathcal{S}_3$	33
	D_8	1
	D_{12}	4
	$\mathcal{S}_4$	4
	$\mathcal{S}_5$	1
6	trivial	10965
	C_2	4121
	$C_2 \times C_2$	441
	$C_2 \times C_2 \times C_2$	6
	$C_2 \times \mathcal{S}_4$	4
	C_3	26
	C_4	7
	D_{12}	49
	D_8	17
	$\mathcal{S}_3$	300
	$\mathcal{S}_3 \times \mathcal{S}_3$	2
	$\mathcal{S}_4$	30
	$\mathcal{S}_5$	4
	$\mathcal{S}_6$	1
7	trivial	746277
	$(\mathcal{S}_3 \times \mathcal{S}_3) \rtimes C_2$	1
	C_2	76704
	$C_2 \times C_2$	7314
	$C_2 \times C_2 \times C_2$	172
	$C_2 \times C_2 \times \mathcal{S}_3$	14
	$C_2 \times D_8$	10
	$C_2 \times \mathcal{S}_4$	45
	$C_2 \times \mathcal{S}_5$	4
	C_3	412
	C_4	82
	$C_4 \times C_2$	4
	C_5	6
	C_6	37
	D_{10}	2
	D_{12}	790
	D_8	169
	$\mathcal{S}_3$	3638
	$\mathcal{S}_3 \times \mathcal{S}_3$	24
	$\mathcal{S}_3 \times \mathcal{S}_4$	2
	$\mathcal{S}_4$	277
	$\mathcal{S}_4 \times \mathcal{S}_3$	2
	$\mathcal{S}_5$	30
	$\mathcal{S}_6$	4
	$\mathcal{S}_7$	1

6. Group rings

Note that Algorithm 3 can be easily modified to compute the automorphism group of a near-ring, or indeed any algebra with associative binary operations. To illustrate we compute the automorphism groups of the multiplicative semigroup of some group rings.

In the following table, G denotes the group, R the ring and S the multiplicative semigroup of the group ring over G and R. The fourth column in the table contains the group identification number used in the Small Group library [6] available in GAP and MAGMA [34].

G	R	$\operatorname{Aut} S$	Group Id.
C_2	$GF(2)$	trivial	
C_3	$GF(2)$	C_2	
C_4	$GF(2)$	$C_2 \times D_8$	
$C_2 \times C_2$	$GF(2)$	$C_2 \times (((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_3) \rtimes C_2)$	$(192, 1538)$
C_5	$GF(2)$	$C_4 \times C_2$	
C_6	$GF(2)$	$\mathcal{S}_3 \times \mathcal{S}_3$	
$\mathcal{S}_3$	$GF(2)$	$\mathcal{S}_3$	
C_7	$GF(2)$	$C_3 \times ((C_6 \times C_2) \rtimes C_2)$	$(72, 30)$

7. What groups?

In this section we consider the class of groups that occur as automorphism groups of semigroups. It might be imagined that if this class is restricted, then we could use this fact to our advantage in the procedures described above. Alas such speculation is irrelevant as the following well-known theorem shows that the class of automorphism groups of semigroups is not in general restricted. Furthermore, our conjecture remains irrelevant even if we restrict our attention to some of the most important special classes of semigroup. It is worth noting that, in contrast to Theorem 7.1, it is known that certain groups do not occur as the automorphism groups of any groups; for example see [22].

Theorem 7.1. *Every finite group is isomorphic to the automorphism group of a finite semigroup of any of the following types: nilpotent, commutative, Clifford, and simple.*

Proof. We begin by proving that every finite group is isomorphic to the automorphism group of a finite semigroup with no further conditions. Frucht's Theorem [7, Section 14.7] states that every group G is the automorphism group of some simple graph Γ with vertices V. Let b and r be elements that are not in V. Form a semigroup from the set $S = V \cup \{b, r\}$ by defining the product of adjacent elements of V to equal b and all other products to equal r. The mapping $\phi : S \to S$ is an automorphism of S if and only if $\phi|_V$ is an automorphism of Γ, $b\phi = b$, and $r\phi = r$. Thus $\operatorname{Aut} S \cong \operatorname{Aut} \Gamma$. Note that the semigroup S constructed above is nilpotent, and commutative.

In order to prove the theorem for Clifford semigroups it suffices to prove that every group is the automorphism group of a Clifford semigroup. In [20] it was shown that every finite group is isomorphic to the automorphism group of a finite bounded lattice. (Here automorphism means order automorphism.) A lattice can be thought of as a Clifford semigroup over trivial groups. The automorphisms of this semigroup are precisely the order automorphisms of the lattice. It follows that every finite group occurs as the automorphism group of a Clifford semigroup. Clifford semigroups are a special type of inverse semigroups, which in turn are a special type of regular semigroups.

To conclude the proof we consider the case of simple semigroups. Every finite simple semigroup is completely simple and by the Rees Theorem [27, Theorem 3.2.3] every completely simple semigroup is isomorphic to a Rees matrix semigroup. Let $M = \mathcal{M}^0[G; I, J; P]$ be a Rees matrix semigroup over the trivial group G. We will use the same notation used in Section 2.

Let $\lambda \in \operatorname{Aut}\Gamma(M)$ and 1_G be the unique automorphism of the trivial group $G = \{e\}$. Then there is only one possible function $c : I \cup J \to G$, the constant mapping with value e. The equality $p_{ji} = (jc)(p_{j\lambda^{-1}i\lambda^{-1}})1_G(ic)^{-1}$ holds for all $p_{ji} \neq 0$, since both sides equal e. Thus if $M = \mathcal{M}^0[G; I, J; P]$ is a Rees matrix semigroup over the trivial group, then $\operatorname{Aut} M \cong \operatorname{Aut}\Gamma(M)$.

Now, any bipartite graph can occur as the graph $\Gamma(M)$ of some Rees matrix semigroup M. Thus the class of automorphism groups of simple semigroups contains the class of automorphism groups of bipartite graphs. In [23] it is shown that every group is the automorphism group of a bipartite graph; also see [24, Section 4.8]. □

Corollary 7.2. *Every finite group is isomorphic to the automorphism group of a finite semigroup of any of the following types: orthodox, regular, completely regular, and inverse.*

Acknowledgements

The authors would like to thank Prof. J. Neubüser and Prof. E. F. Robertson who originally suggested trying to find an algorithm to compute the automorphism group of a semigroup. We would also like to thank Dr M. R. Quick and V. Maltcev for their helpful suggestions.

References

[1] A. Ja. Aĭzenštat, On homomorphisms of semigroups of endomorphisms of ordered sets, (Russian) *Leningrad. Gos. Ped. Inst. Učen. Zap.* **238** 1962 38–48.
[2] J. Araújo and J. Konieczny, Automorphism groups of centralizers of idempotents, *J. Algebra* **269** (2003), 227–239.
[3] J. Araújo and J. Konieczny, Dense relations are determined by their endomorphism monoids, *Semigroup Forum* **70** (2005), 302–306.
[4] J. Araújo, and J. Konieczny, A Method of Finding Automorphism Groups of Endomorphism Monoids of Relational Systems, *Discrete Math.* **307** (2007) 1609–1620.
[5] J. Araújo and J. Konieczny, Automorphisms of Endomorphism Monoids of Relatively Free Bands, *Proc. Edinburgh Math. Soc.* **50** (2007) 1–21
[6] H. U. Besche, B. Eick and E. O'Brien, *The small groups library*, `http://www-public.tu-bs.de:8080/~beick/soft/small/small.html`, 2002.
[7] P. J. Cameron, *Combinatorics: topics, techniques, algorithms*, Cambridge University Press, Cambridge, 1994.
[8] J. J. Cannon and D. Holt, Automorphism group computation and isomorphism testing in finite groups, *J. Symbolic Comput.* **35** (2003), 241–267.
[9] B. Eick, C. R. Leedham-Green, and E. A. O'Brien, Constructing automorphism groups of p-groups, Comm. Algebra **30** (2002), 2271–2295.
[10] A. Distler, T. Kelsey, and J. D. Mitchell, Computing finite semigroups and monoids, in preparation.
[11] V. Felsch and J. Neubüser, Über ein Programm zur Berechnung der Automorphismengruppe einer endlichen Gruppe, *Numer. Math.* **11** (1968) 277–292.

[12] V. H. Fernandes, Presentations for some monoids of partial transformations on a finite chain: a survey, *Semigroups, algorithms, automata and languages (Coimbra, 2001)* 363–378, World Sci. Publishing, River Edge, NJ, 2002.
[13] S. P. Fitzpatrick and J. S. Symons, Automorphisms of transformation semigroups, *Proc. Edinburgh Math. Soc.* **19** (1974/75), 327-329.
[14] E. Formanek, A question of B. Plotkin about the semigroup of endomorphisms of a free group, *Proc. Amer. Math. Soc.* **130** (2002), 935–937.
[15] V. Froidure and J.-E. Pin, Algorithms for computing finite semigroups, *Foundations of computational mathematics (Rio de Janeiro, 1997)*, 112–126, Springer, Berlin, 1997.
[16] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*; 2006, `http://www.gap-system.org`.
[17] L. M. Gluskĭn, Semi-groups of isotone transformations, *Uspehi Mat. Nauk* **16** (1961), 157–162. (Russian)
[18] L. M. Gluskĭn, Semigroups and rings of endomorphisms of linear spaces I, *Amer. Math. Soc. Transl.* **45** (1965), 105–137.
[19] G. M. S. Gomes and J. M. Howie, On the ranks of certain semigroups of order-preserving transformations, *Semigroup Forum* **45** (1992), 272–282.
[20] G. Grätzer and J. Sichler, On the endomorphism semigroup (and category) of bounded lattices, *Pacific J. Math.* **35** 1970, 639–647.
[21] R. Gray and J. D. Mitchell, Largest subsemigroups of the full transformation monoid, to appear in *Discrete Math.*.
[22] Z. Hedrlín and J. Lambek, How comprehensive is the category of semigroups? *J. Algebra* **11** 1969 195–212.
[23] P. Hell and J. Nešetřil, Groups and monoids of regular graphs (and of graphs with bounded degrees), *Canad. J. Math.* **25** (1973) 239-251.
[24] P. Hell and J. Nešetřil, *Graphs and homomorphisms*, Oxford Lecture Series in Mathematics and its Applications, **28**, Oxford University Press, Oxford, 2004.
[25] C. H. Houghton, Completely 0-simple semigroups and their associated graphs and groups, *Semigroup Forum* **14** (1977) 41–67.
[26] J. M. Howie, Products of idempotents in certain semigroups of transformations, *Proc. Edinburgh Math. Soc.* (2) **17** (1970/71) 223–236.
[27] J. M. Howie, *Fundamentals of semigroup theory*, London Mathematical Society Monographs, New Series, 12, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1995.
[28] N. Iwahori and H. Nagao, On the automorphism group of the full transformation semigroups. Proc. Japan Acad. **48** (1972), 639–640.
[29] I. Levi, Automorphisms of normal transformation semigroups, *Proc. Edinburgh Math. Soc. (2)* **28** (1985), 185–205.
[30] I. Levi, Automorphisms of normal partial transformation semigroups, *Glasgow Math. J.* **29** (1987), 149–157.
[31] I. Levi, On the inner automorphisms of finite transformation semigroups, *Proc. Edinburgh Math. Soc. (2)* **39** (1996), 27–30.
[32] A. E. Liber, On symmetric generalized groups, *Mat. Sbornik N. S.* **33** (1953), 531–544. (Russian)
[33] K. D. Magill, Semigroup structures for families of functions, I. Some homomorphism theorems, *J. Austral. Math. Soc.* **7** (1967), 81–94.
[34] W. Bosma, J. Cannon, and C. Playoust, The Magma algebra system. I. The user language, *J. Symbolic Comput.* **24** (1997) 235–265.
[35] A. I. Mal'cev, Symmetric groupoids, *Mat. Sbornik N.S.* **31** (1952), 136–151. (Russian)
[36] G. Mashevitzky, B. Plotkin, and E. Plotkin, Automorphisms of categories of free algebras of varieties, *Electron. Res. Announc. Amer. Math. Soc.*, **8** (2002), 1–10.
[37] G. Mashevitzky and B.M. Schein, Automorphisms of the endomorphism semigroup of a free monoid or a free semigroup, *Proc. Amer. Math. Soc.* **131** (2003) 1655–1660.
[38] B. McKay, Practical graph isomorphism, *Congressus Numerantium* **30** (1981) 45–87.
[39] B. McKay, *nauty*; 2004, `(http://cs.anu.edu.au/~bdm/nauty/)`.
[40] J. D. Mitchell, *The MONOID package v3*, `http://www-groups.mcs.st-andrews.ac.uk/~jamesm/semigroups/Monoid.html`.
[41] V. A. Molchanov, Semigroups of mappings on graphs, *Semigroup Forum* **27** (1983), 155–199.

[42] A. V. Molchanov, On definability of hypergraphs by their semigroups of homomorphisms, *Semigroup Forum* **62** (2001), 53–65.
[43] W. D. Munn, *Semigroups and their algebras*, Dissertation, Cambridge University, 1955.
[44] J.-E. Pin, *Semigroupe : a program for computing finite semigroups*; 2002, `http://www.liafa.jussieu.fr/~jep/Logiciels/Semigroupe/semigroupe.html`.
[45] B. M. Schein, Ordered sets, semilattices, distributive lattices and Boolean algebras with homomorphic endomorphism semigroups, *Fund. Math.* **68** (1970), 31–50.
[46] B. Schein and B. Teclezghi, Endomorphisms of finite full transformation semigroups, *Proc. Amer. Math. Soc.* **126** (1998), 2579–2587.
[47] J. Schreier, Über Abbildungen einer abstrakten Menge Auf ihre Teilmengen, *Fund. Math.* **28** (1936), 261–264.
[48] L. Soicher, *GRAPE: Graph algorithms using permutation groups*; version 4.2, `http://www.maths.qmul.ac.uk/~leonard/grape/`.
[49] R. P. Sullivan, Automorphisms of transformation semigroups, *J. Austral. Math. Soc.* **20** (1975), 77–84.
[50] È. G. Šutov, Homomorphisms of the semigroup of all partial transformations, *Izv. Vysš. Učebn. Zaved. Matematika* **3** (1961), 177–184. (Russian)