The primitive permutation groups of degree less than 2500

Colva M. Roney-Dougal*

8th November, 2004

School of Computer Science, University of St Andrews,
North Haugh, St Andrews, Fife, KY16 9SS.

Abstract

In this paper we use the O'Nan–Scott Theorem and Aschbacher's theorem to classify the primitive permutation groups of degree less than 2500.
MSC: 20B15, 20B10

1 Historical Background

The classification of the primitive permutation groups of low degree is one of the oldest problems in group theory. The earliest significant progress was made by Jordan, who in 1871 counted the primitive permutation groups of degree d for $d \leq 17$ [20], and stated that a transitive group of degree 19 is A_{19}, S_{19}, or a group of affine type.

There were various minor omissions in Jordan's work: degree 9 was corrected by Cole [6, 7], and the remaining degrees up to 17 were corrected by Miller in a long series of papers at the end of the 19th century [35, 36, 37, 38, 39, 40, 41]. In these papers Miller also correctly tabulated the number of soluble primitive groups of degree less than 24. By 1912, the classification up to degree 20 had been completed by [34, 3] for degrees 18 and 20, respectively.

After this there was a hiatus: the lists of groups were becoming so long as to be unwieldy for hand calculations, and the chance of an error arising in such an extended calculation was too great. The birth of practical symbolic computation in the 1960s renewed interest in this old problem, and by 1970 Sims had redetermined the list of primitive groups of degree up to 20 [47]. Sims also classified the primitive groups of degree up to 50: this was never published, but the list was widely circulated in manuscript form, and the resulting groups formed one of the earliest databases in computational group theory, becoming part of first CAYLEY [5], and later GAP [13] and MAGMA [2].

Various people worked on this problem in the 1970s and early 1980s, but the next dramatic leap forward came as a result of the announcement of the Classification of Finite Simple Groups (CFSG), after which Dixon and Mortimer used the O'Nan–Scott Theorem to classify the primitive groups with insoluble socles of degree less than 1000 [10].

*The author was partially supported by EPSRC grant number GR/S30580 and partially funded by the Australian Research Council. Part of this work was carried out at the University of Sydney.

The primitive groups with soluble socles are the groups of affine type. There is a natural isomorphism between the point stabilisers of such groups and the irreducible subgroups of the general linear group, so the classification of primitive groups of affine type is equivalent to the classification of irreducible subgroups of $\mathrm{GL}_n(p)$ for prime p.

This latter question has generated considerable interest in its own right. Some of the earliest work was by Harada and Yamaki, who classified the irreducible subgroups of $\mathrm{GL}(n, 2)$ for $n \leq 6$ [14]. Kondratiev classified the insoluble subgroups of $\mathrm{GL}(n, 2)$ for $n \leq 9$ [24, 25, 26, 27], but these calculations were performed by hand and representations were not given. The soluble groups of affine type were classified (with only a couple of omissions) up to degree 255 by Short in 1991 [46]. This list, along with Dixon and Mortimer's list (which also omitted a few groups) were made into a GAP database by Theißen [48]: Short's list was also turned into a Magma database.

Two papers have appeared recently which further extend these classifications. Eick and Höfling classified the primitive soluble groups of degree less than 6561 [12], and the author and Unger classified all primitive groups of degree less than 1000 [45], as well as rechecking much of the work of Dixon and Mortimer. This latter classification is available in Magma, and will be available in GAP shortly.

The purpose of this paper is to classify the primitive groups of degree less than 2500. Our main theoretical tools are CFSG, the O'Nan–Scott Theorem, and Aschbacher's theorem. Our approach is heavily computational, and as far as possible our calculations have been performed with minimal direct intervention, to reduce the risk of error. The list of primitive groups of affine type of degree less than 1000 has been taken as given, but all other groups have been re-calculated. In particular this has found a cohort of primitive groups, of degree 574, that were missing from all previous classifications.

In Section 2 we present the O'Nan–Scott Theorem, and describe Aschbacher's theorem. In Section 3 we use Aschbacher's theorem to classify the primitive groups with soluble socles of degree less than 2500. In Section 4 we classify the almost simple primitive groups of degree less than 2500, then in Section 5 we deal with the groups with nonabelian composite socles. In Section 6 we describe the checks that have been applied to our work, and list the group-theoretic results which we have assumed to be true. Finally in Section 7 we tabulate our results.

One of the main values of this classification is that we have built a database of the resulting groups: this will be released in the near future in both GAP and Magma format. In the meantime, the groups are available from `http://magma.maths.usyd.edu.au/users/colva`.

2 Preliminary Results

In this section we give some key definitions and preliminary lemmas, before stating the O'Nan–Scott Theorem, which describes the structure of primitive groups. We then give a brief description of Aschbacher's theorem.

Definition 2.1 *Let $G \leq \mathrm{S}_n$. Then G is* transitive *if for all $\alpha, \beta \in \{1, \ldots, n\}$ there exists $g \in G$ such that $\alpha^g = \beta$. If G is transitive and there exists a proper nontrivial equivalence relation $\sim$ on $\{1, \ldots, n\}$ such that $\alpha \sim \beta$ if and only if $\alpha^g \sim \beta^g$, for all $g \in G$ and all $\alpha, \beta \in \{1, \ldots, n\}$, then we say that G is* imprimitive. *Otherwise, a transitive group G is* primitive.

We wish to classify the primitive subgroups of S_d for $d \leq 2500$, up to conjugacy in S_d. It is well-known that conjugacy in the symmetric group is equivalent to *permutation isomorphism*. Two permutation groups $G \leq \mathrm{Sym}(\Omega)$ and $H \leq \mathrm{Sym}(\Omega')$ are permutation isomorphic if there exists a group isomorphism ϕ and a bijection $\delta : \Omega \to \Omega'$ such that for all $\alpha \in \Omega$ and $g \in G$ we have $(\alpha^g)\delta = (\alpha\delta)^{(g\phi)}$.

If H is a maximal subgroup of G, then we write $H \leq_{\max} G$. We denote the stabiliser in G of α by G_α The following lemma is classical.

Lemma 2.2 *Let $G \leq S_n$ be transitive. Then $G_\alpha \leq_{\max} G$ if and only if G is primitive.* □

The study of finite primitive permutation groups is therefore equivalent to the study of the maximal subgroups of finite groups. Let G be an almost simple group with socle T, then the maximal subgroups of G are divided into three types. A group $H \leq_{\max} G$ is a *triviality* if $T \trianglelefteq H$. The subgroup H is a *novelty* maximal if $(H \cap T) \not\leq_{\max} T$. Otherwise H is an *ordinary* maximal subgroup of G.

In MagmaV2.11, the maximal subgroups of a group G may be calculated if each non-abelian simple composition factor L of G satisfies at least one of the following:

1. $|L| < 16,000,000$.
2. $L \cong A_n$ for $n < 1000$.
3. $L \cong L_n(q)$ for q a prime power and $n \leq 4$.
4. $L \cong L_5(p)$.
5. $L \cong L_n(2)$ for $n \leq 12$.
6. $L \cong U_3(p)$ for p prime.
7. $L \cong S_4(p)$ for p prime.
8. $L \cong M_{24}, P\Omega_7(3), P\Omega_8^{\pm}(2), HS, J_3, McL, Sz(32)$

If G is in this list, and G is not $L_n(2)$ for $n \geq 10$, then we say that G is *amenable*. The maximal subgroups of the alternating and symmetric groups may be classified up to the same degree as the primitive almost simple groups, using the O'Nan–Scott theorem and work of Liebeck, Praeger and Saxl [31]. We describe the techniques used to classify the maximal subgroups of groups in classes 2, 3, 5 and 6 in [4]. The groups in classes 1 and 7 are well-understood. In [45] we classified the maximal subgroups of $L_8(2)$ and $L_9(2)$; the maximals of $GL_n(2)$ for $n = 10, 11$ will be classified in the course of this article.

A second reason for the study of primitive groups is that they are the basic building blocks of all finite permutation groups. Any intransitive group is a subdirect product of its transitive constituents, and any imprimitive permutation group embeds into the (imprimitive) wreath product of certain primitive subgroups and quotient groups, so any permutation group embeds into a suitable product of primitive groups.

The O'Nan–Scott Theorem describes the possible structures of the finite primitive groups. There are many extant variants of this theorem; the version given below is from [11]. Before stating it we require a few more definitions. The *socle* of a group G is the subgroup generated by the set of minimal normal subgroups of G; we denote it by $\mathrm{Soc}(G)$. A permutation group G is *regular* if G is transitive and the stabiliser of any point is the identity.

Theorem 2.3 (The O'Nan–Scott Theorem) *Let $G \leq \mathrm{S}_d$ be primitive, and let $H := \mathrm{Soc}(G)$. Then one of the following holds.*

1. *$|H| = p^n$ for some prime p, H is regular and elementary abelian. Then $d = p^n$ and G is isomorphic to a subgroup of the affine general linear group $\mathrm{AGL}_n(p)$;*

2. *H is isomorphic to a direct power T^n of a nonabelian simple group T and one of the following holds:*

   (a) $n = 1$ and G is isomorphic to a subgroup of $\mathrm{Aut}(T)$;

   (b) $n \geq 2$ and G is a group of "diagonal type" with $d = |T|^{n-1}$.

   (c) $n \geq 2$. For some proper divisor m of n and some primitive group U with socle isomorphic to T^m, G is isomorphic to a subgroup of the product action wreath product $U \wr \mathrm{S}_{(n/m)}$. The group U is of type 2(a) or 2(b), and $d = l^{n/m}$, where l is the degree of U.

   (d) $n \geq 6$, H is regular, and G is a group of "twisted wreath product" type, with $d = |T|^n$.

We say that groups in class 1 are of *affine type* and groups in class 2(a) are *almost simple*. Note that class 2(a) is different from all of the others, in that no action is specified. More information about diagonal and product action groups is given in Section 5.

A second key theorem which we will make extensive use of in this article is Aschbacher's theorem [1], which may be briefly summarized as follows:

Theorem 2.4 (Aschbacher's Theorem) *Let G be a subgroup of $\mathrm{GL}(n, q)$ and let $Z := Z(\mathrm{GL}(n, q))$. Then one of the following holds:*

$\mathcal{C}_1$ *G is reducible.*

$\mathcal{C}_2$ *G is imprimitive.*

$\mathcal{C}_3$ *G is semilinear.*

$\mathcal{C}_4$ *G is a subfield group.*

$\mathcal{C}_5$ *G is a tensor product group.*

$\mathcal{C}_6$ *G is a subgroup of the normaliser of an extraspecial r-group for some prime r such that $n = r^s$, or the normaliser of a 2-group of symplectic type.*

$\mathcal{C}_7$ *G is a tensor induced group.*

$\mathcal{C}_8$ *G lies between a quasisimple classical group and its normaliser.*

$\mathcal{C}_9$ *$G/(G \cap Z)$ is almost simple, absolutely irreducible, and written over a minimal field.*

In its full generality, this theorem describes subgroups of any classical group except for orthogonal groups of plus type in dimension 8; and symplectic groups of dimension 4 and characteristic 2 that contain a graph automorphism. In particular the 9 classes given in the theorem describe all possible types of maximal subgroups of any classical group (with the stated two exceptions). We say that a subgroup of a classical group is *AS-maximal* if it is a maximal member of its Aschbacher class. See [22] for a considerably expanded statement and explanation of the theorem.

We finish this section with some notation. Generally, we follow $\mathbb{ATLAS}$ [8] notation for groups, thus D_n is the dihedral group of order n. The letter p will always denote a prime, and the letter $q := p^e$ will be a prime power. We recall the exceptional isomorphisms $\mathrm{A}_5 \cong \mathrm{L}_2(4) \cong \mathrm{L}_2(5)$, $\mathrm{L}_2(7) \cong \mathrm{L}_3(2)$, $\mathrm{A}_6 \cong \mathrm{L}_2(9)$, $\mathrm{A}_8 \cong \mathrm{L}_4(2)$ and $\mathrm{U}_4(2) \cong \mathrm{S}_4(3)$. We will consider each of these groups to be the first one listed here. The only point where we divert from $\mathbb{ATLAS}$ notation is in our naming of the orthogonal groups, where $\mathrm{P}\Omega^{\epsilon}(n,q)$, $\epsilon \in \{+,-,\circ\}$, denotes a simple orthogonal group.

3 Groups of Affine Type

In this section we classify the primitive groups of affine type of degree d, for $1000 \le d < 2500$.

Let $V := \mathbb{F}_p^{(n)}$. The group $\mathrm{AGL}_n(p)$ can be written as a split extension of a regular elementary abelian p-group of order p^n, which we identify with V, and a point stabiliser, which we identify with $\mathrm{GL}_n(p)$. If $G \le \mathrm{AGL}_n(p)$ satisfies $V \trianglelefteq G$, then G is a group of affine type. Under the map from the point stabiliser of $\mathrm{AGL}_n(p)$ to $\mathrm{GL}_n(p)$, the stabilisers of primitive groups of affine type map to the irreducible subgroups of $\mathrm{GL}_n(p)$.

It is well–known that two irreducible subgroups of $\mathrm{GL}_n(p)$ are conjugate if and only if the corresponding groups of affine type are conjugate in S_{p^n}. Therefore the classification of the primitive groups of affine type of degree less than 2500 is equivalent to the classification of the irreducible subgroups of $\mathrm{GL}_n(p)$ for p prime and $p^n < 2500$.

The algorithm `IrreducibleSubgroups` given in [45] returns the set of irreducible subgroups of $\mathrm{GL}_n(p)$, up to conjugacy in $\mathrm{GL}_n(p)$. Determining conjugacy in the general linear group is the main computational hurdle: see [44] for details.

As input to `IrreducibleSubgroups` we require a list $\mathcal{M}$ of subgroups of $\mathrm{GL}_n(p)$. We use Aschbacher's theorem and other information to construct an initial list $\mathcal{M}$ that contains $\mathrm{GL}_n(p)$ and all irreducible AS-maximal subgroups of $\mathrm{GL}_n(p)$ that might be maximal. It does not matter if some are in fact not maximal. We then go through $\mathcal{M}$ and, for each group G that is not amenable, we find the irreducible maximal subgroups of G, and add to $\mathcal{M}$ those that are *not* known to be properly contained in an amenable subgroup of $\mathcal{M}$. We iterate this procedure until all subgroups in $\mathcal{M}$ have been considered. We call a list $\mathcal{M}$ of subgroups of $\mathrm{GL}_n(p)$ constructed via the above procedure a *complete input list.*

The prime powers p^n in the range $1000 \le p^n < 2500$ are as follows: p^2 for $37 \le p \le 47$; p^3 for $11 \le p \le 13$, 7^4, 3^7, 2^{10}, 2^{11}. It follows from the list of amenable groups in § 2 that we only need to use the algorithms of [45] for the final three cases. We consider each of them separately.

Proposition 3.1 *Let*

$$\mathcal{M} := \quad \{\mathrm{GL}_7(3), \mathrm{SL}_7(3), \Gamma\mathrm{L}_1(3^7), N_{\mathrm{GL}_7(3)}(\mathrm{SO}_7(3)) = 2 \times \mathrm{SO}_7(3)\}.$$

Then $\mathcal{M}$ is a complete input list for $\mathrm{GL}_7(3)$*.*

Proof: We start by letting $\mathcal{M} := \{\mathrm{GL}_7(3), \mathrm{SL}_7(3)\}$. We then use Aschbacher's theorem to find the irreducible AS-maximal subgroups of $\mathrm{GL}_7(3)$. The group $\mathrm{GL}_1(3) \wr \mathrm{S}_7$ preserves a quadratic form, and hence is nonmaximal. We add the unique AS-maximal superfield group to $\mathcal{M}$. There are no $\mathcal{C}_4$ or $\mathcal{C}_7$ groups since 7 is prime, and no subfield groups since 3 is prime. There are no $\mathcal{C}_6$ groups since 7 does not divide $3-1$. The only classical group is $N_{\mathrm{GL}_7(3)}(\mathrm{SO}_7(3))$.

We see in [16] that there are no $\mathcal{C}_9$ groups other than possibly groups of Lie type defined over characteristic 3: consulting [32] we see that this latter possibility does not occur.

The only non-amenable group on the list is $\mathrm{SL}_7(3)$. However, the maximal subgroups of $\mathrm{SL}_7(3)$ are all contained in amenable maximal subgroups of $\mathrm{GL}_7(3)$, so we do not need to add any groups to $\mathcal{M}$. □

Lemma 3.2 *Let G be a maximal irreducible subgroup of $\mathrm{GL}_{10}(2)$. Then G is one of the following: $\mathrm{GL}_5(2) \wr \mathrm{S}_2$, $\Gamma\mathrm{L}_5(2^2)$, $\Gamma\mathrm{L}_2(2^5)$, $\mathrm{Sp}_{10}(2)$, $\mathrm{Aut}(M_{22})$ (2 copies), $\mathrm{PGL}_5(2)$.*

Proof: We start by applying Aschbacher's theorem. There are three AS-maximal imprimitive groups, but it is shown in [22] that $\mathrm{GL}(k,2) \wr \mathrm{S}_t$ is nonmaximal for $k \leq 2$. We list both AS-maximal superfield groups. It is shown in [22] that $\mathrm{GL}_2(2) \circ \mathrm{GL}_k(2)$ is nonmaximal for all k. There are no subfield groups, since 2 is prime, and no $\mathcal{C}_6$ groups, since 10 is not a prime power. There are no tensor induced groups, since 10 is not a proper power. Since 2 is not a square, and is even, the only maximal classical subgroup is $\mathrm{Sp}_{10}(2)$.

We now consider $\mathcal{C}_9$ groups. We consult [16] for a description of the almost simple groups that are not groups of Lie type in characteristic 2, and find that the only group which does not preserve a quadratic form is $M_{22}.2$, of which there are two copies, conjugate under the inverse-transpose automorphism of $\mathrm{GL}_{10}(2)$. We see in [32] to see that the only other $\mathcal{C}_9$ group is $\mathrm{PGL}_5(2)$. □

Lemma 3.3 *Let G be a maximal subgroup of $\Gamma\mathrm{L}_5(2^2)$. Then G is $\Sigma\mathrm{L}_5(2^2)$, $\mathrm{GL}_5(2^2)$ or G is the normaliser in $\Gamma\mathrm{L}_5(2^2)$ of one of the following groups: $\Gamma\mathrm{L}_1(2^{10})$, $\mathrm{GL}_5(2)$, $\mathrm{U}_5(2)$, $\mathrm{L}_2(11)$.*

Proof: If G is a triviality then G is $\mathrm{GL}_5(2^2)$ or $\Sigma\mathrm{L}_5(2^2)$.

The only imprimitive AS-maximal is $\mathrm{GL}_1(2^2) \wr \mathrm{S}_5$, which is nonmaximal. The only superfield AS-maximal is given. Since 5 is prime there are no $\mathcal{C}_4$ or $\mathcal{C}_7$ groups. The only subfield AS-maximal is listed. Since 5 does not divide $4-1$ there are no $\mathcal{C}_6$ groups. Since 5 is odd and 4 is even, the only irreducible classical AS-maximal is $\mathrm{U}_5(2)$.

Next we consider $\mathcal{C}_9$ groups. Consulting [32] we see that there are no groups of Lie type in characteristic 2 that do not preserve classical forms. It then follows from [16] that the only $\mathcal{C}_9$ group is $\mathrm{L}_2(11)$, which has a representation over $\mathbb{F}_4$. □

The normalisers of each of these groups in $\Gamma\mathrm{L}_5(2^2)$ may be readily computed.

Lemma 3.4 *Let G be an irreducible maximal subgroup of $\mathrm{Sp}_{10}(2)$. Then G is one of the following: $\Sigma\mathrm{Sp}_2(2^5)$, $\mathrm{SO}^+_{10}(2)$, $\mathrm{SO}^-_{10}(2)$.*

Proof: It follows from Aschbacher's theorem, and the fact that $\mathrm{Sp}_2(2) \wr \mathrm{S}_5$ preserves a quadratic form, that if G is a maximal irreducible geometric subgroup of $\mathrm{Sp}_{10}(2)$ then G is $\mathrm{Sp}_2(2^5).5$ or $\mathrm{SO}^{\pm}_{10}(2)$. There are no representations of almost simple groups as subgroups of $\mathrm{Sp}_{10}(2)$ that do not preserve quadratic forms. □

We let

$$\begin{aligned}\mathcal{M} := \ & \{\mathrm{GL}_{10}(2), \mathrm{GL}_5(2) \wr \mathrm{S}_2, \Gamma\mathrm{L}_5(2^2), \Gamma\mathrm{L}_2(2^5), \mathrm{Sp}_{10}(2), M_{22} : 2 \text{ (2 copies)}, \\ & \mathrm{PGL}_5(2), \Sigma\mathrm{L}_5(2^2), \mathrm{GL}_5(2^2), \mathrm{SL}_5(2^2), N_{\Gamma\mathrm{L}_5(2^2)}(\mathrm{GL}_5(2)), N_{\Gamma\mathrm{L}_5(2^2)}(\mathrm{U}_5(2)), \\ & N_{\Gamma\mathrm{L}_5(2^2)}(\mathrm{L}_2(11)), \mathrm{SO}^+_{10}(2), \mathrm{SO}^-_{10}(2), \Omega^+_{10}(2), \mathrm{L}_5(2) : 2, \Omega^-_{10}(2), \\ & \mathrm{S}_{12}, M_{12} : 2\}.\end{aligned}$$

Corollary 3.5 *Let $\mathcal{M}$ be as above. Then $\mathcal{M}$ is a complete input list for* $\mathrm{GL}_{10}(2)$.

PROOF: We construct a complete input list according to the above procedure. First we have $\mathrm{GL}_{10}(2)$ itself. It has no trivialities, so we start with the groups listed in Lemma 3.2.

The first non–amenable group is $\Gamma\mathrm{L}_5(4)$. We append all groups G such that $\mathrm{SL}_5(4) \leq G < \Gamma\mathrm{L}_5(4)$ to $\mathcal{M}$, then adjoin all of the other groups listed in Lemma 3.3, except for $\Gamma\mathrm{L}_1(2^{10}) \leq \Gamma\mathrm{L}_2(2^5)$.

The next non-amenable group is $\mathrm{Sp}_{10}(2)$. It has no trivialities, and we adjoin to $\mathcal{M}$ all of its non-semilinear irreducible maximal subgroups, from Lemma 3.4.

It follows from the proof of Lemma 3.3 that each of the irreducible maximal subgroups of $\Sigma\mathrm{L}_5(4)$, $\mathrm{GL}_5(4)$ and $\mathrm{SL}_5(4)$ is either listed in $\mathcal{M}$ or is contained in an amenable element of $\mathcal{M}$.

The next non-amenable group is $\mathrm{SO}^+_{10}(2)$, whose irreducible maximal subgroups are listed in the $\mathbb{ATLAS}$ (and corrected in [19]). Next we find $\mathrm{SO}^-_{10}(2)$, and append those of its irreducible maximal subgroups that are not obviously contained in an amenable member of $\mathcal{M}$. The group $\Omega^+_{10}(2)$ has no irreducible maximal subgroups, and with the exception of $M_{12} : 2$, the irreducible maximal subgroups of $\Omega^-_{10}(2)$ are contained in S_{12}, $(3 \times \mathrm{U}_5(2)) : 2$ or $\mathrm{GL}_2(5) \wr \mathrm{S}_2$. □

Proposition 3.6 *Let $\mathcal{M} := \{\mathrm{GL}_{11}(2), \Gamma\mathrm{L}_1(2^{11}), M_{24}$ (2 copies)$\}$. Then $\mathcal{M}$ is a complete input list for* $\mathrm{GL}_{11}(2)$.

PROOF: We start by finding the maximal irreducible subgroups of $\mathrm{GL}_{11}(2)$. The imprimitive group $\mathrm{GL}_1(2) \wr \mathrm{S}_{11}$ is reducible, and hence nonmaximal. The unique AS-maximal superfield group is in $\mathcal{M}$. There are no $\mathcal{C}_4$ or $\mathcal{C}_7$ groups since 11 is prime. There are no $\mathcal{C}_5$ groups since 2 is prime. There are no $\mathcal{C}_6$ groups since 11 does not divide $2 - 1$. There are no classical groups since $\mathrm{SO}_{11}(2)$ is reducible.

We consult [16, 32] to find that M_{23} and M_{24} are the only $\mathcal{C}_9$ groups. The subgroup of M_{24} that is isomorphic to M_{23} is acting irreducibly in this representation, we we append only M_{24} to $\mathcal{M}$.

Since all of the irreducible maximal subgroups of $\mathrm{GL}_{11}(2)$ are amenable, we are done. □

Given the complete input lists given in Proposition 3.1, Corollary 3.5 and Proposition 3.6, we can use them as input to the algorithms presented in [45] to compute the irreducible subgroups of $\mathrm{GL}_7(3)$, $\mathrm{GL}_{10}(2)$ and $\mathrm{GL}_{11}(2)$ respectively. Thus the affine primitive groups of degree less than 2500 are known. The results are given in Table 7.

4 Almost Simple Groups

In this section we classify the almost simple primitive groups of degree less than 2500. This is a three stage process. In subsection 4.1 we determine the simple groups that have an action of degree less than 2500. For the alternating groups we determine which groups only have actions on the cosets of intransitive subgroups, and for the classical groups we determine in subsection 4.2 which groups only have actions on the cosets of reducible subgroups. Finally, we discuss on an individual basis the groups which have at least one action of degree less than 2500, that is on the cosets of an irreducible subgroup, provided that these groups are neither described in the $\mathbb{ATLAS}$ nor amenable. Note that the results of this section re-check those of [10].

Let G be an almost simple group. By $P(G)$ we denote the minimal integer d such that G has a faithful primitive permutation action of degree d.

Lemma 4.1 *Let G be an almost simple group with socle S, and suppose that $S < G \leq \mathrm{Aut}(S)$. Then $P(G) \geq P(S)$.*

Proof: Let $H \leq G$ be the point stabiliser in a primitive faithful action for G of degree $P(G)$. Since the action of G is faithful, H must be core-free, and so in particular $S \not\leq H$. The degree of the action is $[G : H] = [S : (S \cap H)]$, so if $H \cap S$ is maximal in S then $P(G) = P(S)$, otherwise $P(G) > P(S)$. □

4.1 The determination of socles

In this subsection we consider each of the families of simple groups in turn, and establish which of these groups could be the socle of a primitive almost simple group of degree less than 2500.

Proposition 4.2 *If A_n or S_n has a faithful primitive action, other than the natural action, of degree less than 2500, then $n \leq 71$. If H, the point stabiliser in this action, is transitive on $\{1, \ldots, n\}$, then $n \leq 14$. If H acts primitively on $\{1, \ldots, n\}$, then $n \leq 12$.*

Proof: Let $X = \mathrm{A}_n$, and assume that $n \geq 7$. Let $X_0 \neq \mathrm{A}_{n-1}$ be a proper subgroup of X, and assume that X_0 is either a point stabiliser in a primitive action of A_n on a set S of size less than 2500, or that $X_0 = Y_0 \cap \mathrm{A}_n$, where Y_0 is a point stabiliser in a primitive action of S_n on S.

We will prove the lemma by examining the action of X_0 on $\{1, \ldots, n\}$. If X_0 has an orbit of length 1 then X_0 is conjugate to a subgroup of A_{n-1}. If the almost simple group is A_n then since $X_0 \leq_{\max} X$ we must have $X_0 \cong \mathrm{A}_{n-1}$, and hence the action is the natural action, a contradiction. If $X_0 = Y_0 \cap \mathrm{A}_n$ then, since $[Y_0 : X_0] = 2$, the group Y_0 must have either 0 or 1 orbits of length 1. If Y_0 has a single orbit of length 1 then, by maximality of Y_0 in S_n, we get $Y_0 \cong \mathrm{S}_{n-1}$, a contradiction. If Y_0 has no orbits of length 1 then $Y_0 \cong \mathrm{S}_2 \times \mathrm{S}_{n-2}$. But then X_0 has no fixed points, a contradiction.

Suppose that X_0 is primitive in its action on $\{1, \ldots, n\}$. By Bochert's Theorem [18, Section 14.2], the index $|\mathrm{S}_n : X_0| > \lfloor (n+1)/2 \rfloor!$ so, since $|\mathrm{A}_n : X_0| < 2500$, we have $|\mathrm{S}_n : X_0| < 4999$, so $n \leq 12$.

Next suppose that X_0 is transitive but imprimitive in its action on $\{1, \ldots, n\}$. If X_0 has a block of size d, and $m := n/d$, then X_0 can be embedded in $(\mathrm{S}_d \wr \mathrm{S}_m) \cap \mathrm{A}_n$. Thus $|X_0| \leq (d!)^m (m!)/2$. Therefore

$$|\mathrm{A}_n : X_0| \geq f(m, d) = (md)!/((d!)^m m!).$$

Table 1: Socles of primitive groups of degree less than 2500

Group	n	q	Not in $\mathbb{ATLAS}$ and not amenable
$\mathrm{L}_n(q)$	$n = 2$	$7 \leq q \leq 2499$	
	$n = 3$	$3 \leq q \leq 47$	
	$n = 4$	$3 \leq q \leq 13$	
	$n = 5$	$2 \leq q \leq 5$	$q = 4$
	$n = 6$	$2 \leq q \leq 4$	$q = 3, 4$
	$n = 7$	$2 \leq q \leq 3$	$q = 3$
$\mathrm{L}_n(2)$	$8 \leq n \leq 11$		$10 \leq n \leq 11$
$\mathrm{U}_n(q)$	$n = 3$	$3 \leq q \leq 13$	
	$n = 4$	$2 \leq q \leq 5$	$q = 4, 5$
	$n = 5$	$2 \leq q \leq 3$	$q = 3$
	$n = 6$	$q = 2$	
$\mathrm{S}_n(q)$	$n = 4$	$4 \leq q \leq 13$	$q = 8, 9$
	$n = 6$	$2 \leq q \leq 4$	$q = 4$
	$8 \leq n \leq 12$	$q = 2$	$n = 10, 12$
$\mathrm{P}\Omega_n(q)$	$n = 7$	$q = 3$	
$\mathrm{P}\Omega_n^+(q)$	$n = 8$	$2 \leq q \leq 3$	
	$10 \leq n \leq 12$	$q = 2$	$n = 12$
$\mathrm{P}\Omega_n^-(q)$	$n = 8$	$2 \leq q \leq 3$	
	$10 \leq n \leq 12$	$q = 2$	$n = 12$

Since $f(m, d)$ increases monotonically in both variables and $f(2, 8) = 6435$, $f(3, 4) = 5775$, $f(4, 3) = 15400$, $f(6, 2) = 10395$, we conclude that $|\mathrm{A}_n : X_0| > 2500$ whenever $n = md > 14$.

Finally, suppose that X_0 is intransitive in its action on $\{1, \ldots, n\}$, but has no orbit of length 1. Let Γ be the smallest orbit of X_0, and suppose that $|\Gamma| = k$. Then $X_0 \leq (\mathrm{S}_k \times \mathrm{S}_{n-k})$, so $|X_0| < k!(n-k)!/2$, and

$$|\mathrm{A}_n : X_0| > n!/(k!(n-k)!) = \binom{n}{k} \geq \binom{n}{2}.$$

Therefore, $n \leq 71$. □

Proposition 4.3 *Let G be an almost simple classical group with a faithful primitive permutation action of degree less than* 2500*. Then the socle S of G appears in Table 4.1.*

Proof: By Lemma 4.1 it suffices to consider the simple classical groups. The result for each of them follows from [22, Table 5.2.A]: we will however discuss each case individually, for the sake of clarity.

In the linear case, for $(n, q) \notin \{(2, 5), (2, 7), (2, 9), (2, 11), (4, 2)\}$ we have $P(\mathrm{L}_n(q)) = (q^n - 1)/(q - 1)$. The exceptions all have representations of degree less than 2500.

In the unitary case, for $q \neq 5$ we have $P(\mathrm{U}_3(q)) = q^3 + 1$; whilst $P(\mathrm{U}_3(5)) = 50$. We have $P(\mathrm{U}_4(q)) = (q + 1)(q^3 + 1)$. For $n \geq 5$

$$P(\mathrm{U}_n(q)) = (q^n - (-1)^n)(q^{n-1} - (-1)^{n-1})/(q^2 - 1),$$

unless $q = 2$ and n is divisible by 6, in which case $P(\mathrm{U}_n(2)) = 2^{n-1}(2^n - 1)/3$.

In the symplectic case, for $q \neq 2$ and $n \geq 4$, $P(\mathrm{S}_n(q)) = (q^n - 1)/(q - 1)$. For $q = 2$ we have $P(\mathrm{S}_n(2)) = 2^{n/2-1}(2^{n/2} - 1)$. Recall that $\mathrm{S}_2(q) \cong \mathrm{L}_2(q)$.

Finally we consider the orthogonal groups, namely $\mathrm{P}\Omega^{\epsilon}_{2m}(q)$ for $\epsilon \in \{+, -\}$ and $m \geq 4$, and $\mathrm{P}\Omega_{2m+1}(q)$ for $m \geq 3$. Recall that $\mathrm{P}\Omega_{2m+1}(2^i) \cong \mathrm{S}_{2m}(2^i)$ for $i > 1$, so in odd dimensions we may assume that q is even. In odd dimensions, for $q \geq 5$, $P(\mathrm{P}\Omega_{2m+1}(q)) = (q^{2m} - 1)/(q - 1)$. We have $P(\mathrm{P}\Omega_{2m+1}(3)) = 3^m(3^m - 1)/2$. In even dimensions, if $q \geq 3$ then

$$P(\mathrm{P}\Omega^{+}_{2m}(q)) = (q^m - 1)(q^{m-1} + 1)/(q - 1),$$

whilst $P(\mathrm{P}\Omega^{+}_{2m}(2)) = 2^{m-1}(2^m - 1)$. For groups of minus type we have $P(\mathrm{P}\Omega^{-}_{2m}(q)) = (q^m + 1)(q^{m-1} + 1)/(q - 1)$, for all q. □

Proposition 4.4 *Let G be an almost simple exceptional group with a faithful primitive permutation action of degree less than* 2500*. Then $\mathrm{Soc}(G)$ is one of $G_2(3), G_2(4), \mathrm{Sz}(8), \mathrm{Sz}(32)$, ${}^3D_4(2)$ or ${}^2F_4(2)'$: all of these groups are described in the* $\mathbb{ATLAS}$.

Proof: By Lemma 4.1, it suffices to consider the simple exceptional groups. We deal first with the untwisted groups: $E_6(q), E_7(q), E_8(q), F_4(q), G_2(q)$. We then examine the twisted groups ${}^2B_2(2^{2m+1}) = \mathrm{Sz}(2^{2m+1})$, ${}^3D_4(q)$, ${}^2E_6(q)$, ${}^2F_4(q)$ and ${}^2G_2(3^{2m+1})$.

If a group has a primitive permutation representation of degree d then its representation as permutation matrices over a field of characteristic coprime to the group order has a constituent of degree $d - 1$.

The minimal degree of a projective irreducible representation of $E_6(q)$ over a field of characteristic other than p is $q^9(q^2 - 1)$ [28], so $P(E_6(q)) > q^9(q^2 - 1)$. In particular, if $q \geq 3$ then $P(E_6(q)) > 2500$. None of the maximal subgroups of $E_6(2)$ have index less than 2500 [23].

The corresponding bounds for $E_7(q)$ and $E_8(q)$ are $q^{15}(q^2 - 1)$ and $q^{27}(q^2 - 1)$ [28]. Thus, for all q, $P(E_7(q)) > 2500$ and $P(E_8(q)) > 2500$.

The minimal degree projective representation of $F_4(q)$ is at least $q^9/2$ for $q > 2$ [28], so $P(F_4(q)) > 2500$ for all $q > 2$. The $\mathbb{ATLAS}$ shows that $P(F_4(2)) = 69615$.

For $q > 4$ the largest maximal subgroup of $G_2(q)$ has order less than $q^6(q^2 - 1)(q - 1)$ [29, Thm 5.2i]. This implies that $P(G_2(q)) > 2500$ for all $q \geq 5$. The group $G_2(2)$ is not simple. The remaining groups, $G_2(3)$ and $G_2(4)$, appear in the $\mathbb{ATLAS}$, and are shown to have maximal subgroups of index less than 2500.

Next we consider the twisted groups. The minimal degree for a projective representation of ${}^2B_2(2^{2m+1}) = \mathrm{Sz}(2^{2m+1})$ in coprime characteristic is $2^m(2^m - 1)$ [28]. Thus if $\mathrm{Sz}(2^{2m+1})$ has a primitive permutation representation of degree less than 2500 then $m \leq 3$. In [33] the work of Suzuki is used to give a precise description of the maximal subgroups of $\mathrm{Sz}(q)$, which eliminates $\mathrm{Sz}(128)$. The remaining groups, $\mathrm{Sz}(8)$ and $\mathrm{Sz}(32)$, are described in the $\mathbb{ATLAS}$: both have maximal subgroups of index less than 2500.

For all q, the order of a maximal subgroup of ${}^3D_4(q)$ is bounded above by $q^{12}(q^6-1)(q-1)$ [29]. Therefore, for $q \geq 3$ we have $P({}^3D_4(q)) > 2500$. The group ${}^3D_4(2)$ is described in the $\mathbb{ATLAS}$; it has maximal subgroups of index less than 2500.

The minimal degree of a projective representation in comprime characteristic of ${}^2E_6(q)$ is q^{14} [28]. This is greater than 2500 for all q.

For all $q > 2$, the order of a maximal subgroup of ${}^2F_4(q)$ is bounded above by $q^{12}(q^2 + 1)(q - 1)^2$ [29]. This implies that for all $q > 3$ we have $P({}^2F_4(q)) > 2500$. The group ${}^2F_4(2)$

is not simple. Its socle, ${}^2F_4(2)'$, is described in the $\mathbb{ATLAS}$: it has maximal subgroups of index less than 2500.

Finally, we have that the minimal degree of a projective representation of ${}^2G_2(3^{1+2m}) = R(3^{1+2m})$ is $3^{1+2m}(3^{1+2m}-1)$ [28]. We require $m > 0$ as ${}^2G_2(3) = L_2(8) : 3$. This leaves only ${}^2G_2(3^3)$, which is described in the $\mathbb{ATLAS}$: it has no subgroups of index less than 2500. □

The maximal subgroups of all sporadic groups S for which $P(S) < 2500$ are described in the $\mathbb{ATLAS}$.

4.2 Reduction of actions

Proposition 4.5 *Let G be an almost simple group of Lie type and suppose that (a) G has a primitive permutation action of degree less than 2500, and (b) G is not described in the $\mathbb{ATLAS}$. Then the point stabiliser of G is reducible unless the socle of G is one of:* $\mathrm{L}_2(p)$ *for* $37 \leq p \leq 71$, $\mathrm{L}_2(p^2)$ *for* $7 \leq p \leq 17$, $\mathrm{L}_2(p^e)$ *for* $e \geq 3$ *and* $p^e \in \{32, 64, 81\}$, $\mathrm{L}_4(q)$ *for* $4 \leq q \leq 5$, $\mathrm{U}_4(4)$, $\mathrm{U}_6(2)$, $\mathrm{S}_4(q)$ *for* $5 \leq q \leq 8$, $\mathrm{S}_6(4)$, $\mathrm{S}_{10}(2)$ *or* $\mathrm{S}_{12}(2)$.

Proof: We discuss only the nonamenable groups.

Suppose that the socle of G is $\mathrm{L}_5(q)$, for some q with $3 \leq q \leq 5$. If q is odd then the largest irreducible maximal subgroup of G is almost simple with socle $\Omega_5(q)$ [30, Thm 5.1]. We have $N_{\mathrm{L}_5(q)}(\Omega_5(q)) = \mathrm{SO}_5(q)$, and for $q > 3$ we have $[\mathrm{L}_5(q) : \mathrm{SO}_5(q)] > 2500$. It follows from [30, Thm 5.1] that the largest irreducible subgroup of $\mathrm{L}_5(4)$ is $\mathrm{U}_5(2)$, which has index greater than 2500.

Suppose that the socle of G is $\mathrm{L}_d(q)$ with $d > 5$ even and $q \leq 4$. The largest irreducible maximal subgroup of G is $N_G(\mathrm{S}_d(q)) \leq \mathrm{PGSp}_d(q)$ [30, Thm 5.1]. In each case this has index greater than 2500.

Now suppose that the socle of G is $\mathrm{L}_n(2)$ with $n \in \{7, 9, 11\}$. If $n = 9, 11$ then the maximal irreducible subgroups of G have index greater than $2^{(n-2)(n-3)/2} > 2500$ [21, Thm 1]. If $n = 7$ then the largest irreducible maximal subgroup of G has order less than 2^{18} [30, Thm 5.1], and hence index greater than 2500.

We complete the linear case by supposing that G is $\mathrm{L}_7(3)$. Then [21, Thm 1] states that the largest irreducible maximal subgroup of G has index at least 3^{15}.

We consider the unitary case next. The largest irreducible subgroup of $\mathrm{U}_4(5)$ is $\mathrm{S}_4(5)$ [30, Thm 5.3], which has index 3150. The largest irreducible subgroup of $\mathrm{U}_5(3)$ is $N_{\mathrm{U}_5(3)}(\Omega_5(3)) = \mathrm{SO}_5(3)$ [30, Thm 5.3], which has index greater than 2500.

Theorem 5.2 of [30] states that the largest irreducible subgroup of $\mathrm{S}_4(9)$ is $\mathrm{L}_2(81).2$, which has index greater than 2500.

Finally, we consider the orthogonal groups. The largest irreducible maximal subgroup of a group with socle $\mathrm{P}\Omega^{\epsilon}_{12}(2)$ is $N_G(\mathrm{GU}_6(2).2)$ in type $+$ and A_{13} in type $-$ [30, Thm 5.4 + discussion] . Each of these has index greater than 2500. □

Proposition 4.6 *Let G be a primitive almost simple classical group of degree less than 2500. If G is not described in the $\mathbb{ATLAS}$, is not amenable, and all faithful primitive permutation actions of G of degree less than 2500 are on the cosets of reducible subgroups, then the primitive permutation representations of G are as described in § 7.*

Proof: Let G be a group satisfying the hypotheses of the proposition. Then the socle of G is one of

$$\mathrm{L}_5(4), \mathrm{L}_6(3), \mathrm{L}_6(4), \mathrm{L}_7(3), \mathrm{L}_9(2), \mathrm{L}_{10}(2), \mathrm{L}_{11}(2), \mathrm{U}_4(5), \mathrm{S}_4(9), \mathrm{P}\Omega^{\epsilon}_{12}(2).$$

For each of the above groups $\mathrm{L}_n(q)$ we use the algorithms of [17] to construct the k-space stabiliser for $1 \leq k \leq d/2$ and, if the corresponding permutation representation has degree less than 2500, we insert the corresponding permutation representation, with normaliser $\mathrm{P\Gamma L}_n(q)$. We use [17] to construct the novelty maximals of $\mathrm{L}2_n(q)$, which extend to $\mathrm{P\Gamma L}\,2_n(q)$, and include these as appropriate.

In the unitary case, we use [17] to construct the stabiliser of an isotropic point, a non-isotropic point, and an isotropic 2-space. Only the action on the cosets of the first of these has degree less than 2500: this action extends to $\mathrm{P\Gamma U}_4(5)$.

In a symplectic geometry all points are isotropic, so we use [17] to construct the stabiliser of a point and a totally isotropic 2-space. Both of these have index less than 2500, and the corresponding permutation action extends to the full automorphism group.

Finally, in the orthogonal cases we find that the stabilisers of an isotropic and of a nonisotropic point have indices less than 2500, but that the stabilisers of any larger subspaces have indices greater than 2500. The point stabilisers extend to maximal subgroups of $\mathrm{Aut}(\mathrm{P}\Omega^{\epsilon}_{12}(2)) = \mathrm{PSO}^{\epsilon}_{12}(2)$. □

Corollary 4.7 *Let G be an almost simple primitive group of degree less than 2500. If G is not a classical group acting on the cosets of a reducible subgroup, then either G is described in the* $\mathbb{ATLAS}$*, or G is amenable, or the socle of G is one of:* $\mathrm{U}_4(4)$, $\mathrm{U}_6(2)$, $\mathrm{S}_4(8)$, $\mathrm{S}_6(4)$, $\mathrm{S}_{10}(2)$, $\mathrm{S}_{12}(2)$.

Proposition 4.8 *If G is a group whose socle is one of the exceptions listed in the previous corollary, then the faithful primitive permutation actions of G are as described in § 7.*

Proof: First we examine $\mathrm{U}_4(4)$. The $\mathcal{C}_2$ AS-maximals have index greater than 2500. There are no AS-maximals in $\mathcal{C}_3$, $\mathcal{C}_4$, $\mathcal{C}_6$, $\mathcal{C}_7$ or $\mathcal{C}_8$. The unique AS-maximal subfield group is $N_{\mathrm{U}_4(4)}(\mathrm{S}_4(4)) = \mathrm{S}_4(4)$, of index 1040. Consultation of [16, 32] shows that there are no maximal $\mathcal{C}_9$ groups. Thus the only entries in Table 5 are on cosets of reducible or symplectic subgroups.

Next we discuss $\mathrm{U}_6(2)$. The indices of all geometric irreducible AS-maximals are greater than 2500. Consulting [16] we find the following candidates for maximal $\mathcal{C}_9$ groups: A_7, M_{22} and $\mathrm{U}_4(3)$. The normalisers of the first two of these have index greater than 2500 in $\mathrm{U}_6(2)$, the normaliser of the third has index 1048. Consulting [32] we find $\mathrm{L}_3(2).2$ which preserves an orthogonal form, $\mathrm{L}_3(4).2$ which has index greater than 2500, $\mathrm{L}_4(2)$ which is a subgroup of $\mathrm{L}_6(2)$, $\mathrm{U}_4(2)$ which preserves a quadratic form, $\mathrm{S}_6(2)$ which is a subfield group, and $G_2(2)$ which preserves a symplectic form. Thus the only entries in Table 5 are on cosets of reducible groups, or on the cosets of $N_{\mathrm{U}_6(2)}(\mathrm{U}_4(3))$.

We move on now to the symplectic groups. The first group which we must consider is $\mathrm{S}_4(8)$. The symplectic groups in dimension 4 and characteristic 2 have an exceptional outer automorphism, and hence have a different maximal subgroup structure from the other symplectic groups, see [1] for details. The maximal imprimitive group $\mathrm{Sp}_2(8) \wr \mathrm{S}_2$ has index 2080, and is conjugate in $\mathrm{Aut}(\mathrm{S}_4(8))$ to $\mathrm{PSO}^{+}_4(8)$. The maximal superfield group has index 2016 and is conjugate in $\mathrm{Aut}(\mathrm{S}_4(8))$ to $\mathrm{PSO}^{-}_4(8)$. The maximal subfield group has index greater than 2500. The largest $\mathcal{C}_9$ group is $Sz(8)$, but it has index greater than 2500 in $\mathrm{S}_4(8)$. There are assorted novelty maximal subgroups of $\mathrm{S}_4(8).2$ (graph automorphism) but, since they must have index at least 2 in the maximal subgroups of $\mathrm{S}_4(8)$, we need only consider the novelty reducible. However, this has index 5265.

The next symplectic group on our list is $\mathrm{S}_6(4)$. The $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_5$ AS-maximals all have indices greater than 2500. It is shown in [22] that $\mathcal{C}_4$ groups are nonmaximal in symplectic groups for even q. The classical groups $\mathrm{PSO}_6^+(4) \cong \mathrm{L}_4(4):2$ and $\mathrm{PSO}_6^-(4) \cong \mathrm{U}_4(4):2$ have indices 2080 and 2016 respectively, and hence are shown in §7. From [16] we see that $\mathrm{U}_3(3)$, J_2 and $\mathrm{L}_2(13)$ might be maximal $\mathcal{C}_9$ subgroups of $\mathrm{S}_6(4)$, but all of these have index greater than 2500. From [32] we see that $\mathrm{L}_3(4)$, $\mathrm{U}_3(4)$ and $G_2(4)$ are potentially maximal. Of these, $\mathrm{L}_3(4)$ has no symplectic representations, $G_2(4)$ has index 16320 in $\mathrm{S}_6(4)$ and $|\mathrm{S}_6(4)|/|\,\mathrm{P\Gamma U}_3(4)| > 2500$.

Now we turn to $\mathrm{Sp}_{10}(2)$. It follows from the main theorem of [30] that the irreducible maximal subgroups of $\mathrm{Sp}_{10}(2)$ are geometric, or have socle A_n for $n = 11, 12$. The AS-maximal imprimitive subgroups of $\mathrm{Sp}_{10}(2)$ have index greater than 2500, as does the AS-maximal superfield subgroup. The classical groups $\mathrm{PSO}_{10}^+(2)$ and $\mathrm{PSO}_{10}^-(2)$ have indices 528 and 426 respectively. The remaining geometric Aschbacher classes are empty. Since $|\,\mathrm{Sp}_{10}(2)|/12! > 2500$ the results in Table 6 follow.

Finally we turn to $\mathrm{Sp}_{12}(2)$. Similarly to the previous case, the imprimitive and semilinear AS-maximals have index greater than 2500, and there are no groups in class $\mathcal{C}_i$ for $4 \le i \le 7$. In $\mathcal{C}_8$ we find $\mathrm{PSO}_{12}^+(2)$ and $\mathrm{PSO}_{12}^-(2)$, of indices 2080 and 2016 respectively. It follows from [30] that there are no $\mathcal{C}_9$ groups of index less than $|\,\mathrm{Sp}_{12}(2)|/14! > 2500$. □

Thus the almost simple primitive groups of degree less than 2500 are known.

5 Nonabelian composite socles

The final possibility for the socle of a finite primitive permutation group is that it is nonabelian and not simple. There are three types of groups that fall into this category: twisted wreath product groups, product action groups and diagonal action groups. In this section we classify the primitive groups with nonabelian composite socles of degree less than 2500.

For the twisted wreath product groups, it follows immediately from the statement of the O'Nan–Scott Theorem that the degree of any group with twisted wreath product action is at least 60^6.

If G is a group of product action type then we may denote the socle of G by K^m, where K is the socle of a primitive permutation group U of degree n of almost simple or diagonal type. The degree of G is n^m and it follows from [10, Lemma 5] that $K^m \le G \le N_{\mathrm{S}_n}(U) \wr \mathrm{S}_m$. Since we are assuming that $n^m < 2500$, only restricted values of n and m may occur. In particular, since $60^2 > 2500$, U must be almost simple, and so K is a nonabelian simple group. The degree n of U is therefore at least 5, so $m \le 4$. If $m = 2$ then $n \le 49$; if $m = 3$ then $n \le 13$; and if $m = 4$ then $n \le 7$.

Our strategy to construct the groups is computational. We make extensive use of the fact that the primitive groups of degree less than 50 are well-known.

For $2 \le m \le 4$, and $5 \le n \le \sqrt[m]{2500}$ do

1. For each primitive almost simple group U of degree n that is maximal in its cohort do:

   (a) Let $M := U \wr \mathrm{S}_m$, with the product action.

(b) Let $Q := M/\mathrm{Soc}(M)$.

(c) For each subgroup S of Q (up to Q-conjugacy) do

i. If the preimage S' of S in M is primitive, add S' to the list of primitive product action groups of degree n^m.

It is clear that if two groups are conjugate in Q then their preimages are conjugate in M. We discuss our techniques for checking that all groups in the tables are pairwise inconjugate in the symmetric group in §6. The results are given in Tables 10 and 11.

Finally, we construct the diagonal action groups. Since $60^2 > 2500$, we need only consider groups with two factors in the socle. The simple groups of order less than 2500 are

$$\mathrm{A}_5, \mathrm{L}_3(2), \mathrm{A}_6, \mathrm{L}_2(8), \mathrm{L}_2(11), \mathrm{L}_2(13), \text{ and } \mathrm{L}_2(17).$$

For each of these groups, we proceed as follows:

1. Let $M := \mathrm{Aut}(T) \wr \mathrm{S}_2$.
2. For each $G \leq M$ with $|M : N| = |\mathrm{Out}(T)|$ do:
   (a) If G has a maximal subgroup K of index $|T|$, construct the primitive permutation action of G on K.
   (b) Return the list of primitive groups H such that $\mathrm{Soc}(G) = T^2 \leq H \leq G$.

If two groups are conjugate in G, then they are conjugate in the symmetric group. We discuss how we double–checked the converse in the next section. There are only two cohorts of groups of degree greater than 1000: those with socle $\mathrm{L}_2(13) \times \mathrm{L}_2(13)$ and those with socle $\mathrm{L}_2(17) \times \mathrm{L}_2(17)$. They are described in the discussion at the beginning of Section 7.

6 Accuracy

In this section we describe the various checks that were applied to our data, and give an explicit list of which results of other people have been assumed as true without rechecking. There are three main types of errors which we wish to eliminate: mathematical, computational, and what we will term *clerical*, namely those errors that result from processing large amounts of data into complex tables. We will first discuss the most general checks that were applied, before considering each of the three types of error in turn.

The primitive groups of degree less than 1000 were compared with the lists in [10], as corrected in [45]. This showed that we have found one additional cohort of primitive groups, an action of $\mathrm{L}_2(41)$ on the cosets of A_5, as well as rediscovering various previously-known errors, mostly concerning the rank of the normaliser.

The numbers of primitive soluble groups were compared with the lists in [12]. Our numbers agreed in all cases with theirs.

Within each degree, all groups were checked to be pairwise inconjugate. To do this we used the techniques of [45]. First we computed, for each group G, a *signature* consisting of $|G|$, the transitivity k of G, the multiset of orbit lengths of the k-point stabiliser, the multiset of chief factors of G, and the orders of all groups in the derived series of G. The groups were put into equivalence classes by signature, and those in classes of size one were discarded.

We then computed the Sylow 2-subgroup of each remaining group, and partitioned the groups yet further using their *extended signature*: the signature extended by an extra coordinate containing the multisets of isomorphism types of all abelian groups of order not divisible by 512 that occur as quotients in the derived series of G. Again, groups in equivalence classes of size 1 were discarded.

Next, we computed the point stabiliser and the derived subgroup of each remaining group, and partitioned each class yet further by the extended signature of each of these groups group. Very few groups remained after this procedure, and none were in equivalence classes of size greater than 2. We simply checked that each pair of groups were nonisomorphic.

As far as other sources of information are concerned, we assumed all that of the results cited in Propositions 4.3, 4.4 and 4.5 were correct. We have assumed the accuracy of [32] and of the corrected version of [16]. When using the $\mathbb{ATLAS}$, we have checked the current version of Norton's "Improvements to the Atlas", from `http://web.mat.bham.ac.uk/atlas/v2.0/`. We have obviously made extensive use of [1], [19] and [22].

Next we consider the computational accuracy. The most basic test was to re-run all of the code several times, often with minor adjustments to the actual coding, and to check that the results agreed. The algorithms of [17] were used to make many of the geometric maximal subgroups: each time we did this we checked that the groups constructed were equal to their own normalisers. The code for the amenable groups has now been in Magma for at least a year: the fact that it is in regular use by many people, and that no bugs have been found after the initial period, gives us a reasonable degree of confidence that the implementation is reliable. We also performed various of our calculations with almost simple groups in GAP as well.

We finish with a discussion of the clerical checks. Firstly, we have repeatedly examined the tables, comparing descriptions with those in the $\mathbb{ATLAS}$, and ensuring that the content of the tables matches the descriptions in the relevant section of the paper. Since the groups that we have constructed are to go into a database, we have electronic files of all of them, and have checked that the totals from the paper copy agree with the files. For all of the amenable groups, a Magma run was done to print out the degrees of the primitive actions, and the number of groups in each cohort. This was then compared with the typeset version.

7 The Tables

In this section we give the tables describing the primitive groups of degree at least 1000 and less than 2500. See the online version of this article for extended tables which describe all primitive groups of degree less than 2500. For the groups of degree less than 1000, our results agree with the lists given in [45], with a single exception. This is an additional cohort of almost simple groups of degree 574, which consists of a single group that is equal to its own normaliser in S_{574}. This group is $L_2(41)$, acting on the cosets of A_5. The action has rank 16.

Each table of almost simple groups follows a similar format. The first column describes the minimal primitive group G in the cohort, generally in $\mathbb{ATLAS}$ notation. Underneath we state the structure of the outer automorhism group of the socle of G. The column labelled "Conditions", if it exists, gives any parameters for which whole families of groups may occur: we continue with the convention that p is prime and $q = p^e$. The next column is the degree d of all of the groups in the cohort. The next column gives the structure of the point stabiliser in the group G, again in $\mathbb{ATLAS}$ notation. After this we describe the structure of

Table 2: Type A: Alternating groups (excluding the natural action)

Primitive Group G	Conditions	Degree d	Stabiliser in G	$H = \mathrm{Soc}(G)$ $N = N_{\mathrm{S}_d}(H)$	Rank of N	Size of Cohort
A_n	$46 \leq n \leq 71$	$\binom{n}{2}$	S_{n-2}	$H.2$	3	2
Out $= 2$	$20 \leq n \leq 25$	$\binom{n}{3}$	$(\mathrm{A}_{n-3} \times 3):2$	$H.2$	4	2
$n > 6$	$14 \leq n \leq 17$	$\binom{n}{4}$	$(\mathrm{A}_{n-4} \times \mathrm{A}_4):2$	$H.2$	5	2
	$13 \leq n \leq 14$	$\binom{n}{5}$	$(\mathrm{A}_{n-5} \times \mathrm{A}_5):2$	$H.2$	6	2
A_{13}		1716	$(\mathrm{A}_7 \times \mathrm{A}_6):2$	$H.2$	7	2
A_{14}		1716	$(\mathrm{A}_7 \times \mathrm{A}_7):4$	$H.2$	4	2

N, the normaliser in S_d of the socle H of the primitive group G, as an extension of H. The penultimate column gives the rank of N in this degree d action, and the final column gives the number of groups in the cohort.

When describing the product action groups, the structure of the normaliser of G in S_d is clear, and the structure of the point stabiliser can be deduced by examining the tables of almost simple primitive groups. Thus these columns are omitted.

We do not give a table for the diagonal action groups, as there are only two cohorts of degree at least 1000 and less than 2500. There is a cohort with socle $\mathrm{L}_2(13) \times \mathrm{L}_2(13)$, of degree 1092. The rank of the normaliser in the symmetric group is 8, and there are 5 groups in the cohort. There is a cohort with socle $\mathrm{L}_2(17) \times \mathrm{L}_2(17)$, of degree 2448. The rank of the normaliser in the symmetric group is 10, and there are 5 groups in the cohort.

When describing the groups of affine type of degree p^d we simply list the numbers of soluble and insoluble primitive subgroups of $\mathrm{AGL}_d(p)$, for all $p^d < 2500$ with $d > 1$. If $d = 1$ then the number of primitive groups of affine type is equal to the number of divisors of $p-1$, as each group has the form $p : x$ with $x|(p-1)$.

Representations of the groups of degree greater than 1000 will shortly be available as both MAGMA and GAP databases. In the meantime they may be downloaded from `http://magma.maths.usyd.edu.au/users/colva`.

Table 3: Type B: $L_2(p^e)$, p prime, $e \geq 1$

Primitive Group G	Conditions	Degree d	Stabiliser in G	$H = \mathrm{Soc}(G)$ $N = N_{\mathrm{S}_d}(H)$	Rank of N	Size of Cohort
$L_2(p)$	$999 \leq p \leq 2499$	$p+1$	$p : \frac{p-1}{2}$	$H.2$	2	2
Out $= 2$	$45 \leq p \leq 71$	$p(p-1)/2$	D_{p+1}	$H.2$	$\frac{p+1}{2}$	2
	$45 \leq p \leq 67$	$p(p+1)/2$	D_{p-1}	$H.2$	$\frac{p+3}{2}$	2
$L_2(29).2$		1015	S_4	$H.2$	52	1
$L_2(37)$		2109	A_4	$H.2$	100	2
$L_2(41)$		1435	S_4	H	69	1
$L_2(47)$		2162	S_4	H	101	1
$L_2(59)$		1711	A_5	H	38	1
$L_2(61)$		1891	A_5	H	41	1
$L_2(p^2)$	$32 \leq p \leq 49$	p^2+1	$p^2 : \frac{p^2-1}{2}$	$H.2^2$	2	5
Out $= 2^2$						
$L_2(7^2)$		1176	D_{50}	$H.2^2$	16	5
		1225	D_{48}	$H.2^2$	17	5
$L_2(13^2)$		1105	$PGL_2(13)$	$H.2$	8	2
$L_2(17^2)$		2465	$PGL_2(17)$	$H.2$	10	2
$L_2(p^3)$	$11 \leq p \leq 13$	p^3+1	$p^3 : \frac{p^3-1}{2}$	$H.6$	2	4
Out $= 6$						
$L_2(7^4)$		2402	$7^4 : 1200$	$H.(2 \times 4)$	2	8
Out $= 2 \times 4$						
$L_2(2^6)$		2016	D_{130}	$H.6$	8	4
Out $= 6$		2080	D_{126}	$H.6$	9	4
$L_2(3^7)$		2188	$3^7 : 1093$	$H.14$	2	4
Out $= 7$						
$L_2(2^{10})$		1025	$2^{10} : 1023$	$H.10$	2	4
Out $= 10$						
$L_2(2^{11})$		2049	$2^{11} : 2047$	$H.11$	2	2
Out $= 11$						

Table 4: Type C: $\mathrm{L}_n(q)$, $n \neq 2$

Primitive Group G	Degree d	Stabiliser in G	$H = \mathrm{Soc}(G)$ $N = N_{\mathrm{S}_d}(H)$	Rank of N	Size of Cohort
$\mathrm{L}_3(11).2$ Out $= 2$	1596	$11_+^{1+2} : 10^2 : 2$	$H.2$	4	1
$\mathrm{L}_3(37)$ Out $= \mathrm{S}_3$	1407	$37^2 : 12.\mathrm{L}_2(37) : 2$	$H.3$	2	2
$\mathrm{L}_3(41)$ Out $= 2$	1723	$41^2 : \mathrm{GL}_2(41)$	H	2	1
$\mathrm{L}_3(43)$ Out $= \mathrm{S}_3$	1893	$43^2 : 14.\mathrm{L}_2(43) : 2$	$H.3$	2	2
$\mathrm{L}_3(47)$ Out $= 2$	2257	$47^2 : \mathrm{GL}_2(47)$	H	2	1
$\mathrm{L}_3(7^2)$ Out $= D_{12}$	2451	$7^4 : 16.\mathrm{L}_2(49).2$	$H.3.2_2$	2	4
$\mathrm{L}_3(2^5)$ Out $= 10$	1057	$2^{10} : \mathrm{GL}_2(32)$	$H.5$	2	2
$\mathrm{L}_4(3)$ Out $= 2^2$	2106	$(4 \times \mathrm{A}_6) : 2$	$H.2^2$	8	5
$\mathrm{L}_4(3).2$	1080	$\mathrm{L}_3(3).2$	$H.2^2$	6	3
$\mathrm{L}_4(5)$ Out $= D_8$	1550	$\mathrm{S}_4(5)$	$H.2^2$	4	4
$\mathrm{L}_4(11)$ Out $= 2^2$	1464	$11^3 : 5 : \mathrm{L}_3(11)$	$H.2$	2	2
$\mathrm{L}_4(13)$ Out $= D_8$	2380	$13^3 : 3.\mathrm{L}_3(13) : 3$	$H.4$	2	3
$\mathrm{L}_4(2^2)$ Out $= 2^2$	1008	$\mathrm{S}_4(4)$	$H.2^2$	4	5
$\mathrm{L}_4(2^2).2$	1785	$2^{2+8} : (3 \times \mathrm{S}_3 \times \mathrm{A}_5)$	$H.2^2$	5	3
$\mathrm{L}_5(2).2$ Out $= 2$	1085	$2^{4+4} : (\mathrm{S}_3 \times \mathrm{S}_3) : 2$	$H.2$	7	1
$\mathrm{L}_5(3)$ Out $= 2$	1210	$3^6 : 2.(\mathrm{L}_2(3) \times \mathrm{L}_3(3)) : 2$	H	3	1
$\mathrm{L}_6(2)$	1395	$2^9 : (\mathrm{GL}_3(2) \times \mathrm{GL}_3(2))$	$H.2$	4	2
$\mathrm{L}_6(2).2$	1953	$2_+^{1+8} : \mathrm{GL}_4(2) : 2$	$H.2$	5	1
Out $= 2$	2016	$\mathrm{GL}_5(2) : 2$	$H.2$	5	1
$\mathrm{L}_6(4)$ Out $= D_{12}$	1365	$2^{10} : \mathrm{L}_5(4)$	$H.\mathrm{S}_3$	2	4
$\mathrm{L}_7(3)$ Out $= 2$	1093	$3^6 : \mathrm{GL}_6(3)$	H	2	1
$\mathrm{L}_{10}(2)$ Out $= 2$	1023	$2^9 : \mathrm{GL}_9(2)$	H	2	1
$\mathrm{L}_{11}(2)$ Out $= 2$	2047	$2^{10} : \mathrm{GL}_{11}(2)$	H	2	1

Table 5: Type D: $\mathrm{U}_n(q)$, $n > 2$

Primitive Group G	Degree d	Stabiliser in G	$H = \mathrm{Soc}(G)$ $N = N_{\mathrm{S}_d}(H)$	Rank of N	Size of Cohort
$\mathrm{U}_3(4)$	1600	$13 : 3$	$H.4$	15	3
Out $= 4$					
$\mathrm{U}_3(5).3$	1750	$3^2 : 2\mathrm{A}_4$	$H.\mathrm{S}_3$	12	2
Out $= \mathrm{S}_3$	1750	$6^2 : \mathrm{S}_3$	$H.\mathrm{S}_3$	15	2
$\mathrm{U}_3(7)$	2107	$2(\mathrm{L}_2(7) \times 4).2$	$H.2$	8	2
Out $= 2$					
$\mathrm{U}_3(11)$	1332	$11_+^{1+2} : 40$	$H.\mathrm{S}_3$	2	4
Out $= \mathrm{S}_3$					
$\mathrm{U}_3(13)$	2198	$13_+^{1+2} : 168$	$H.2$	2	2
Out $= 2$					
$\mathrm{U}_4(3)$	1296	A_7	$H.2$	6	2
Out $= D_8$					
$\mathrm{U}_4(4)$	1040	$\mathrm{S}_4(4)$	$H.4$	4	3
Out $= 4$	1105	$2^{2+8} : (15 \times \mathrm{A}_5)$	$H.4$	3	3
$\mathrm{U}_5(2)$	1408	$3^4 : \mathrm{S}_5$	$H.2$	6	2
Out $= 2$					
$\mathrm{U}_5(3)$	2440	$3^{1+6} : 8 : \mathrm{U}_3(3)$	$H.2$	3	2
Out $= 2$					
$\mathrm{U}_6(2)$	1408	$\mathrm{U}_4(3).2_2$	$H.2$	3	2
Out $= 2$					

Table 6: Type E: $S_n(q)$, $n > 2$

Primitive Group G	Degree d	Stabiliser in G	$H = \mathrm{Soc}(G)$ $N = N_{S_d}(H)$	Rank of N	Size of Cohort
$S_4(4)$	1360	S_6	$H.4$	7	3
Out = 4					
$S_4(7)$	1176	$L_2(49) : 2$	$H.2$	5	2
Out = 2	1225	$2.(L_2(7) \times L_2(7)) : 2$	$H.2$	5	2
$S_4(8)$	2016	$L_2(64) : 2$	$H.3$	3	2
Out = 6	2080	$(L_2(8) \times L_2(8)) : 2$	$H.3$	3	2
$S_4(11)$	1464	$11^{1+2} : 10.L_2(11)$	$H.2$	3	2
Out = 2	1464	$11^3 : 5 : L_2(11) : 2$	$H.2$	3	2
$S_4(13)$	2380	$13^{1+2} : 12.L_2(13)$	$H.2$	3	2
Out = 2	2380	$13^3 : 6.L_2(13).2$	$H.2$	3	2
$S_6(3)$	1120	$3^6 : L_3(3)$	$H.2$	4	2
Out = 2					
$S_6(4)$	2016	$P\Omega_6^-(4) : 2$	$H.2$	3	2
Out = 2	2080	$P\Omega_6^+(4) : 2$	$H.2$	3	2
$S_8(2)$	2295	$2^{10} : A_8$	H	5	1
Out = 1					
$S_{10}(2)$	1023	$2^{1+8} : S_8(2)$	H	3	1
Out = 1					
$S_{12}(2)$	2016	$P\Omega_{12}^-(2) : 2$	H	2	1
Out = 1	2080	$P\Omega_{12}^+(2) : 2$	H	2	1

Table 7: Type F: $P\Omega_n^\epsilon(q)$, $n > 6$

Primitive Group G	Degree d	Stabiliser in G	$H = \mathrm{Soc}(G)$ $N = N_{S_d}(H)$	Rank of N	Size of Cohort
$P\Omega_7(3)$	1080	$G_2(3)$	H	3	1
Out = 2	1120	$3^{3+3} : L_3(3)$	$H.2$	4	2
$P\Omega_8^+(2)$	1120	$(3 \times U_4(2)) : 2$	$H.2$	5	2
Out = S_3	1575	$2_+^{1+8} : (S_3 \times S_3 \times S_3)$	$H.S_3$	5	4
$P\Omega_8^+(2).2$	2025	$2^{3+6} : (L_3(2) \times 2)$	$H.2$	7	1
$P\Omega_8^-(2)$	1071	$2_+^{1+8} : (S_3 \times A_5)$	$H.2$	5	2
Out = 2	1632	$(3 \times A_8) : 2$	$H.2$	5	2
$P\Omega_8^+(3)$	1080	$P\Omega_7(3)$	$H.2^2$	3	5
Out = S_4	1120	$3^6 : L_4(3)$	$H.D_8$	3	8
$P\Omega_8^-(3)$	1066	$3^6 : 2U_4(3).2$	$H.2^2$	3	5
Out = 2^2	1107	$P\Omega_7(3) : 2$	$H.2$	3	2
$P\Omega_{10}^+(2)$	2295	$2^{10} : L_5(2)$	H	3	1
Out = 2					
$P\Omega_{12}^+(2)$	2016	$S_{10}(2)$	$H.2$	3	2
Out = 2	2079	$2^{10} : P\Omega_{10}^+(2)$	$H.2$	3	2
$P\Omega_{12}^-(2)$	2015	$2^{10} : P\Omega_{10}^-(2)$	$H.2$	3	2
Out = 2	2080	$S_{10}(2)$	$H.2$	3	2

Table 8: Type G: Exceptional Groups of Lie Type

Primitive Group G	Degree d	Stabiliser in G	$H = \mathrm{Soc}(G)$ $N = N_{S_d}(H)$	Rank of N	Size of Cohort
$G_2(3) : 2$	1456	$3^2.(3 \times 3_+^{1+2}) : D_8$	$H.2$	7	1
Out = 2					
$G_2(4)$	1365	$2^{2+8} : (3 \times A_5)$	$H.2$	4	2
Out = 2	1365	$2^{4+6} : (A_5 \times 3)$	$H.2$	4	2
	2016	$U_3(4) : 2$	$H.2$	3	2
	2080	$3^{\cdot} L_3(4) : 2_3$	$H.2$	4	2
${}^2B_2(8)$	1456	$5 : 4$	$H.3$	27	2
Out = 3	2080	D_{14}	$H.3$	59	2
${}^2B_2(32)$	1025	$2^{5+5} : 31$	$H.5$	2	2
Out = 5					
${}^3D_4(2)$	2457	$2^2.[2^9] : (7 \times S_3)$	$H.3$	4	2
Out = 3					
${}^2F_4(2)'$	1600	$L_3(3) : 2$	H	4	1
Out = 2	1755	$2.[2^8] : 5 : 4$	$H.2$	5	2
	2304	$L_2(25)$	$H.2$	6	2

Table 9: Type H: Sporadic Groups

Primitive Group G	Degree d	Stabiliser in G	$H = \mathrm{Soc}(G)$ $N = N_{\mathrm{S}_d}(H)$	Rank of N	Size of Cohort
M_{12}	1320	$\mathrm{A}_4 \times \mathrm{S}_3$	$H.2$	22	2
Out = 2					
$M_{12}:2$	1584	S_5	$H.2$	27	1
J_1	1045	$2^3:7:3$	H	11	1
Out = 1	1463	$2 \times \mathrm{A}_5$	H	22	1
	1540	$19:6$	H	21	1
	1596	$11:10$	H	19	1
J_2	1008	$\mathrm{A}_5 \times D_{10}$	$H.2$	8	2
Out = 2	1800	$\mathrm{L}_3(2):2$	$H.2$	14	2
	2016	$5^2:D_{12}$	$H.2$	12	2
M_{23}	1288	M_{11}	H	4	1
Out = 1	1771	$2^4:(3 \times \mathrm{A}_5):2$	H	8	1
HS	1100	$\mathrm{L}_3(4).2_1$	$H.2$	5	2
Out = 2	1100	S_8	$H.2$	5	2
M_{24}	1288	$M_{12}:2$	H	3	1
Out = 1	1771	$2^6:3^{\cdot}\mathrm{S}_6$	H	4	1
	2024	$\mathrm{L}_3(4):\mathrm{S}_3$	H	5	1
McL	2025	M_{22}	H	4	1
Out = 2					
He	2058	$\mathrm{S}_4(4):2$	$H.2$	4	2
Out = 2					
Suz	1782	$G_2(4)$	$H.2$	3	2
Out = 2					
Co_2	2300	$\mathrm{U}_6(2):2$	H	3	1
Out = 1					

Table 10: Product action groups with socle factors alternating

Primitive Group G	Degree	Conditions	Rank of normaliser	Size of Cohort
$A_n \times A_n$	n^2	$32 \leq n \leq 49$	3	4
	$\binom{n}{2}^2$	$9 \leq n \leq 10$	6	4
$(A_6 \times A_6).2^2$	1296		10	20
	2025		15	20
$A_7 \times A_7$	1225		10	4
$A_8 \times A_8$	1225		6	4
$A_n \times A_n \times A_n$	n^3	$10 \leq n \leq 13$	4	10
$A_5 \times A_5 \times A_5$	1000		10	10
$A_6 \times A_6 \times A_6$	1000		4	85
$A_n \times A_n \times A_n \times A_n$	n^4	$6 \leq n \leq 7$	5	45
$A_5 \times A_5 \times A_5 \times A_5$	1296		5	45

Table 11: Other product action groups

Primitive Group G	Degree	Conditions	Rank of normaliser	Size of Cohort
$L_2(p) \times L_2(p)$	$(p+1)^2$	p prime, $31 \leq p \leq 47$	3	4
$L_2(2^3) \times L_2(2^3)$	1296		6	4
$L_2(2^5) \times L_2(2^5)$	1089		3	4
$L_2(11) \times L_2(11) \times L_2(11)$	1331		4	2
	1728		4	10
$L_2(7) \times L_2(7) \times L_2(7) \times L_2(7)$	2401		5	5
$L_4(3) \times L_4(3)$	1600		3	4
$L_3(3) \times L_3(3) \times L_3(3)$	2197		4	2
$U_3(3) \times U_3(3)$	1296		6	4
$U_4(2) \times U_4(2)$	1296		6	4
	1600		6	4
	1600		6	4
	2025		6	4
$S_6(2) \times S_6(2)$	1296		3	1
$M_{11} \times M_{11} \times M_{11}$	1331		4	2
	1728		4	2
$M_{12} \times M_{12} \times M_{12}$	1728		4	2

Table 12: Groups of affine type, split into soluble (S) and insoluble (I)

n		$p = 2$	3	5	7	11	13	17	19	23	29	31	37	41	43	47
2	S	2	7	19	29	42	62	75	77	54	100	114	127	174	118	66
	I	0	0	3	4	6	6	5	9	4	10	12	9	14	8	4
3	S	2	9	22	62	54	136									
	I	1	2	11	14	22	22									
4	S	10	108	509												
	I	10	37	138												
5	S	2	16													
	I	1	18													
6	S	40	324													
	I	24	147													
7	S	2	18													
	I	1	53													
8	S	129														
	I	109														
9	S	21														
	I	15														
10	S	50														
	I	55														
11	S	6														
	I	6														

References

[1] M. Aschbacher, On the maximal subgroups of the finite classical groups, Invent. Math., 76, (1984), 469–514.

[2] W. Bosma, J. Cannon, C. Playoust, The Magma algebra system I: The user language, J. Symbolic Comput., 24, (3), (1997), 235–265.

[3] E.R. Bennett, Primitive groups with a determination of the primitive groups of degree 20, Amer. J. Math., 34, (1912), 1–20.

[4] J.N. Bray, C.M. Roney-Dougal, The maximal subgroups of the low rank classical groups, In preparation.

[5] J.J. Cannon, An introduction to the group theory language, Cayley, in: Computational Group Theory, ed. Michael D. Atkinson, Academic Press, London, (1984), 145–183.

[6] F.N. Cole, The transitive substitution-groups of nine letters, Bull. New York Math. Soc., 2, (1893), 250–258.

[7] F.N. Cole, List of the substitution groups of nine letters, Quart. J. Pure Al. Math., 26, (1893), 372–388.

[8] J.C. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, Atlas of Finite Groups, Clarendon Press, Oxford, 1985.

[9] L.E. Dickson, Linear Groups: with an exposition of the Galois field theory, 1900, Reprinted Dover, New York, 1958.

[10] J.D. Dixon, B. Mortimer, The primitive permutation groups of degree less than 1000, Math. Proc. Cambridge Philos. Soc., 103, (1988), 213–238.

[11] J.D. Dixon, B. Mortimer, Permutation Groups, Springer, New York, 1996.

[12] B. Eick, B. Höfling, The solvable primitive permutation groups of degree at most 6560, LMS J. Comput. Math., 6, (2003), 29–39.

[13] The GAP Group, GAP – Groups, Algorithms and Programming, Version 4.4, 2004. (`http://www.gap-system.org`).

[14] K. Harada, H. Yamaki, The irreducible subgroups of $\mathrm{GL}_n(2)$ with $n \leq 6$, C. R. Math. Rep. Acad. Sci. Canada, 1, (1978/79), 75–78.

[15] R.W. Hartley, Determination of the ternary linear collineation groups whose coefficients lie in $\mathbb{F}_{2^n}$, Ann. of Math., 27, (1926), 140–158.

[16] G. Hiss, G. Malle, Low-dimensional representations of quasi-simple groups, LMS J. Comput. Math., 4, (2001), 22–63. Erratum: LMS J. Comput. Math., 5, (2002), 95–126.

[17] D.F. Holt, C.M. Roney-Dougal, Constructing maximal subgroups of classical groups, LMS J. Comput. Math., to appear.

[18] B. Huppert, Endliche Gruppen I, Springer, Berlin, 1967.

[19] C. Jansen, K. Lux, R. Parker, R. Wilson, An Atlas of Brauer Characters, Oxford University Press, New York, 1995.

[20] C. Jordan, Traité des Substitutions et des Equations Algébriques, Gauthier-Villers, Paris, 1871.

[21] W.M. Kantor, Permutation representations of the finite classical groups of small degree or rank, J. Algebra, 60, (1979), 158–168.

[22] P.B. Kleidman, M. Liebeck, The subgroup structure of the finite classical groups, Cambridge University Press, Cambridge, 1990.

[23] P.B. Kleidman, R.A. Wilson, The maximal subgroups of $E_6(2)$ and $\mathrm{Aut}(E_6(2))$, Proc. London Math. Soc., 60, (1990), 266–294.

[24] A.S. Kondrat'ev, Irreducible Subgroups of $\mathrm{GL}(7, 2)$, Mathematical notes of the academy of sciences of the USSR, 37, (1985), 317–321.

[25] A.S. Kondrat'ev, Linear Groups of Small Degree over a field of order 2 (Russian), Algebra i Logika, 25, (1986), 544–565.

[26] A.S. Kondrat'ev, The irreducible subgroups of the group $\mathrm{GL}(8, 2)$, Communications in Algebra, 15, (5), (1987), 1039–1093.

[27] A.S. Kondrat'ev, Irreducible subgroups of $\mathrm{GL}(9, 2)$, Mathematical notes of the academy of sciences of the USSR, 39, (1986), 173–178.

[28] V. Landazuri, G.M. Seitz, On the minimal degrees of projective representatations of the finite Chevalley groups, J. Algebra, 32, (1974), 418–443.

[29] M.W. Liebeck, J. Saxl, Primitive permutation groups containing an element of large prime order, J. London Math. Soc., 31, (1985), 237–249.

[30] M.W. Liebeck, On the orders of maximal subgroups of the finite classical groups, Proc. London Math. Soc., (3), 50, (1985), 426–446.

[31] M.W. Liebeck, C.E. Praeger, J. Saxl, A classification of the maximal subgroups of the finite alternating and symmetric groups, J. Algebra, 111, (1987), 365-383.

[32] F. Lübeck, Small degree representations of finite Chevalley groups in defining characteristic, LMS J. Comput. Math., 4, (2001), 135–169.

[33] H. Lüneberg, Translation Planes, New York, Springer-Verlag, 1980.

[34] E.N. Martin, On the imprimitive substitution groups of degree fifteen and the primitive substitution groups of degree eighteen, Amer. J. Math., 23, (1901), 259–286.

[35] G.A. Miller, Note on the transitive substitution groups of degree 12, Bull. Amer. Math. Soc., 1, (2), (1895), 255–258.

[36] G.A. Miller, List of transitive substitution groups of degree 12, Quart. J. Pure Appl. Math., 28, (1896), 193–231. Erratum: Quart. J. Pure Appl. Math., 29, (1898), p.249.

[37] G.A. Miller, On the primitive substitution groups of degree fifteen, Proc. London Math. Soc., 28, (1), (1897), 533–544.

[38] G.A. Miller, Sur l'énumeration des groupes primitifs dont le degré est inférieur à 17, C. R. Acad. Sci., 124, (1897), 1505–1508.

[39] G.A. Miller, On the transitive substitution groups of degrees 13 and 14, Quart. J. Pure Appl. Math., 29, (1898), 224–249.

[40] G.A. Miller, On the primitive substitution groups of degree 16, Amer. J. Math., 20, (1898), 229–241.

[41] G.A. Miller, On the transitive substitution groups of degree seventeen, Quart. J. Pure Appl. Math., 31, (1900), 49–57.

[42] H.H. Mitchell, Determination of the ordinary and modular ternary linear groups, Trans. Amer. Math. Soc., 12, (1911), 207–242.

[43] H.H. Mitchell, The subgroups of the quaternary abelian linear groups, Trans. Amer. Math. Soc., 15, (1914), 379–396.

[44] C.M. Roney-Dougal, Conjugacy of subgroups of the general linear group, Exp. Math., 13, (2), (2004), 151–164.

[45] C.M. Roney-Dougal, W.R. Unger, The primitive affine groups of degree less than 1000, J. Symbolic Comput., 35, (2003), 421–439.

[46] M.W. Short, The Primitive Soluble Permutation Groups of Degree less than 256, Springer–Verlag, Berlin, 1991.

[47] C.C. Sims, Computational methods for permutation groups, in: Computational Problems in Abstract Algebra, ed. J. Leech, Pergamon, (1970), 169–183.

[48] H. Theißen, Eine Methode zur Normalisatorberechnung in Permutationsgruppen mit Anwendungen in der Konstruktion primitiver Gruppen, Dissertation, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, 1997.