On the Fibonacci length of powers of dihedral groups

C. M. Campbell*, P. P. Campbell*, H. Doostie†
and E. F. Robertson*
**Mathematical Institute, University of St Andrews,*
North Haugh, St Andrews, KY16 9SS, Scotland.
† *Mathematics Department, University for Teacher Education,*
49 Mofateh Avenue, Tehran, 15614, Iran.
E-mail: cmc@st-andrews.ac.uk, peterc@mcs.st-andrews.ac.uk,
doostih@saba.tmu.ac.ir, edmund@mcs.st-andrews.ac.uk

Abstract. For a finitely generated group $G = \langle A \rangle$, where $A = \{a_1, a_2, \ldots, a_n\}$, the sequence $x_i = a_i$, $1 \leq i \leq n$, $x_{i+n} = \prod_{j=1}^{n} x_{i+j-1}$, $i \geq 1$, is called the Fibonacci orbit of G with respect to the generating set A, denoted $F_A(G)$. If $F_A(G)$ is periodic we call the length of the period of the sequence the Fibonacci length of G with respect to A, written $LEN_A(G)$. In this paper we examine the Fibonacci lengths of D_{2m}^i, $i > 1$ where D_{2m} is the dihedral group of order $2m$.

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1 Introduction

The idea of forming a sequence of group elements based on a Fibonacci-like recurrence relation was initially introduced by Wall in [14] and later developed by other authors, see [1], [7], [16]. This idea was then refined in [3] and [5] to that of a Fibonacci orbit and Fibonacci length of a given group $G = \langle A \rangle$ where $A = \{a_1, a_2, \ldots, a_n\}$ as follows:

Definition 1.1 For the finitely generated group $G = \langle A \rangle$, where $A = \{a_1, \ldots, a_n\}$, the *Fibonacci orbit* of G with respect to the generating set A, written $F_A(G)$, is the sequence $x_1 = a_1, \ldots, x_n = a_n, x_{i+n} = \prod_{j=1}^{n} x_{i+j-1}$, $i \geq 1$. If $F_A(G)$ is periodic then the length of the period of the sequence is called the *Fibonacci length* of G with respect to the generating set A, written $LEN_A(G)$. When it is clear which generating set is being investigated we will write $LEN(G)$ for $LEN_A(G)$. It is clear that $F_A(G)$ is periodic if G is an epimorphic image of a Fibonacci group; see [13] for more information on Fibonacci groups.

As can be seen from the above definitions the Fibonacci length of a group depends in general on the chosen generating set. It was noted in [3] that, for any generating pair $\{a,b\} \in D_{2m}$, $LEN_{\{a,b\}}(D_{2m}) = 6$, where D_{2m} is the dihedral group of order $2m$. In this paper we intend to investigate this phenomenon further by calculating the Fibonacci length of D_{2m}^i, for any integer i, with respect to the natural generating set. We use the natural generating set for D_{2m}, as in [4], defined as satisfying $\langle\ a_1,\ b_1 \mid a_1^2 = 1,\ b_1^m = 1,\ (a_1b_1)^2 = 1\ \rangle$. This is extended to the direct product by using the following well known method of construction: if $G = \langle\ X \mid R\ \rangle$ and $N = \langle\ Y \mid S\ \rangle$ then $G \times N = \langle\ X, Y \mid R, S, [X,Y]\ \rangle$ where $[X,Y] = \{[x,y] : x \in X, y \in Y\}$, see [9]. In the above we have used relations and relators as appropriate and will continue to do so in this paper. When rewriting words we use the convention found in [11], namely the use of underscores to highlight the subwords which are replaced in passing from one word to the next.

Over the years there has been interest in particular types of presentations for direct powers of groups, with one strand of this interest motivated by questions of Wiegold in [15]; see for example [2] and [4]. Other strands are discussed in [10] and [12]. In this paper we find the structure of the Fibonacci orbit of D_∞^i and use this to find the Fibonacci length of D_{2m}^i. We show that for $i > 1$ the Fibonacci length of D_{2m}^i on the natural generating set no longer remains constant but varies with i and m. We also examine the Fibonacci length of a restricted set of D_{2m}^i, for odd m, on a different generating set.

The results in this paper were suggested by data from computer programs written in the computational algebra system GAP [8].

2 The Fibonacci Orbit of D_∞^i, $i \geq 2$

D_∞^i has a presentation with $2i$ generators and $2i^2$ relations:

$$\begin{aligned} D_\infty^i = \langle\ a_1,\ b_1,\ a_2,\ b_2,\ \ldots,\ a_i,\ b_i \ \mid\ & a_l^2 = (a_lb_l)^2 = 1, \\ & [a_j,a_k] = [a_j,b_k] = [b_j,b_k] = 1, \\ & 1 \leq l \leq i,\ 1 \leq j < k \leq i\ \rangle. \end{aligned}$$

The elements of the group defined by the above presentation can be written in the normal form $a_i^{q_i}a_{i-1}^{q_{i-1}}\ldots a_1^{q_1}b_i^{r_i}b_{i-1}^{r_{i-1}}\ldots b_1^{r_1}$ where the q_l are either 0 or 1. Throughout this paper we will always reduce elements of the Fibonacci orbit to this normal form.

In order to elucidate the main proof of this section we start by considering the Fibonacci type sequence of elements of D_∞^2 as

$$x_1 = a_1, x_2 = b_1, x_3 = a_2, x_4 = b_2, x_n = x_{n-4}x_{n-3}x_{n-2}x_{n-1}, (n \geq 5).$$

We have:

Lemma 2.1 *Every element of the Fibonacci orbit* $\{x_j\}$ *of* $D^2_\infty = \langle a_1, b_1, a_2, b_2\rangle$ *may be represented by:*

$$x_j = \begin{cases} a_1, & j \equiv 1, -4 \bmod 10 \\ b_1^{\pm 1}, & j \equiv 2, -3 \bmod 10 \\ a_2 b_1^{\pm 2(j-3)/5}, & j \equiv 3, -2 \bmod 10 \\ b_2^{\pm 1} b_1^{\pm 2(j-4)(j+1)/5^2}, & j \equiv 4, -1 \bmod 10 \\ a_2 a_1 b_2^{\pm 1} b_1^{\pm(2j^2-5^2)/5^2}, & j \equiv 5,\ 0 \bmod 10 \end{cases}$$

where the positive exponent is chosen for the first value of j *and the negative exponent is chosen for the second value of* j.

Proof. The assertion may be proved by induction on j. We first prove the anchor step of the induction. Since $a^i b a^i = b^{-1}$, i odd, we have

$$\begin{aligned}
&x_1 = a_1,\\
&x_2 = b_1,\\
&x_3 = a_2,\\
&x_4 = b_2,\\
&x_5 = a_2 a_1 b_2 b_1,\\
&x_6 = b_1 a_2 b_2 a_1 b_1 a_2 b_2 = a_1,\\
&x_7 = a_2 b_2 a_1 b_1 a_2 b_2 a_1 = a_1 b_1 a_1 = b_1^{-1},\\
&x_8 = b_2 a_1 b_1 a_2 b_2 a_1 b_1^{-1} = b_2 a_2 b_2 . a_1 b_1 a_1 b_1^{-1} = a_2 b_1^{-2},\\
&x_9 = a_1 b_1 a_2 b_2 a_1 b_1^{-1} a_2 b_1^{-2} = a_2 b_2 a_2 . a_1 b_1 a_1 b_1^{-3} = b_2^{-1} b_1^{-4},\\
&x_{10} = a_1 b_1^{-1} a_2 b_1^{-2} b_2^{-1} b_1^{-4} = a_2 a_1 b_2^{-1} b_1^{-7}.
\end{aligned}$$

Now let $k \equiv 0 \bmod 10$ and assume that the result holds for all values up to $k+5$, namely

$$\begin{aligned}
x_{k+1} &= a_1,\\
x_{k+2} &= b_1,\\
x_{k+3} &= a_2 b_1^{2((k+3)-3)/5} = a_2 b_1^{2k/5},\\
x_{k+4} &= b_2 b_1^{2((k+4)-4)((k+4)+1)/25} = b_2 b_1^{2k(k+5)/25},\\
x_{k+5} &= a_1 a_2 b_1^{(2(k+5)^2-25)/25} b_2 = a_2 a_1 b_2 b_1^{(2k^2+20k+25)/25}.
\end{aligned}$$

We now prove that the next ten entries have the required form and thus complete the induction.

$$\begin{aligned}
x_{k+6} &= b_1 a_2 b_1^{2k/5} b_2 b_1^{2k(k+5)/25} a_2 a_1 b_2 b_1^{(2k^2+20k+25)/25}\\
&= b_1^{1+2k/5+2k^2/25+10k/25} a_1 b_1^{(2k^2+20k+25)/25} a_2 b_2 a_2 b_2\\
&= a_1.
\end{aligned}$$

$$\begin{aligned}
x_{k+7} &= a_2 b_1^{2k/5} b_2 b_1^{2k(k+5)/25} a_2 a_1 b_2 b_1^{(2k^2+20k+25)/25} a_1 \\
&= b_1^{2k/5+2k^2/25+10k/25} a_1 b_1^{(2k^2+20k+25)/25} a_1 a_2 b_2 a_2 b_2 \\
&= b_1^{-1}.
\end{aligned}$$

$$\begin{aligned}
x_{k+8} &= b_2 b_1^{2k(k+5)/25} a_2 a_1 b_2 b_1^{(2k^2+20k+25)/25} a_1 b_1^{-1} \\
&= b_1^{2k^2/25+10k/25} a_1 b_1^{(2k^2+20k+25)/25} a_1 b_1^{-1} b_2 a_2 b_2 \\
&= a_2 b_1^{-2((k+8)-3)/5}.
\end{aligned}$$

$$\begin{aligned}
x_{k+9} &= a_2 a_1 b_2 b_1^{(2k^2+20k+25)/25} a_1 b_1^{-1} a_2 b_1^{-2((k+8)-3)/5} \\
&= a_1 b_1^{(2k^2+20k+25)/25} a_1 b_1^{-2k/5-3} a_2 b_2 a_2 \\
&= b_2^{-1} b_1^{-2((k+9)-4)((k+9)+1)/25}.
\end{aligned}$$

$$\begin{aligned}
x_{k+10} &= a_1 b_1^{-1} a_2 b_1^{-2((k+8)-3)/5} b_2^{-1} b_1^{-2((k+9)-4)((k+9)+1)/25} \\
&= a_1 b_1^{-2k/5-7-2k^2/25-15k/25} a_2 b_2^{-1} \\
&= a_2 a_1 b_2^{-1} b_1^{-(2(k+10)^2-25)/25}.
\end{aligned}$$

$$\begin{aligned}
x_{k+11} &= b_1^{-1} a_2 b_1^{-2((k+8)-3)/5} b_2^{-1} b_1^{-2((k+9)-4)((k+9)+1)/25} a_2 a_1 b_2^{-1} b_1^{-(2k^2+40k+175)/25} \\
&= b_1^{-10k/25-7-2k^2/25-6k/5} a_1 b_1^{-2k^2/25-8k/5-7} a_2 b_2^{-1} a_2 b_2^{-1} \\
&= a_1.
\end{aligned}$$

$$\begin{aligned}
x_{k+12} &= a_2 b_1^{-2((k+8)-3)/5} b_2^{-1} b_1^{-2((k+9)-4)((k+9)+1)/25} a_2 a_1 b_2^{-1} b_1^{-(2k^2+40k+175)/25} a_1 \\
&= b_1^{-2k/5-2-2k^2/25-6k/5-4} a_1 b_1^{-2k^2/25-8k/5-7} a_1 a_2 b_2^{-1} a_2 b_2^{-1} \\
&= b_1.
\end{aligned}$$

$$\begin{aligned}
x_{k+13} &= b_2^{-1} b_1^{-2((k+9)-4)((k+9)+1)/25} a_2 a_1 b_2^{-1} b_1^{-(2k^2+40k+175)/25} a_1 b_1 \\
&= b_1^{-2k^2/25-6k/5-4} a_1 b_1^{-2k^2/25-8k/5-7} a_1 b_1 b_2^{-1} a_2 b_2^{-1} \\
&= a_2 b_1^{2((k+13)-3)/5}.
\end{aligned}$$

$$\begin{aligned}
x_{k+14} &= a_2 a_1 b_2^{-1} b_1^{-(2k^2+40k+175)/25} a_1 b_1 a_2 b_1^{2(k+10)/5} \\
&= a_1 b_1^{-(2k^2+40k+175)/25} a_1 b_1^{1+2(k+10)/5} a_2 b_2^{-1} a_2 \\
&= b_2 b_1^{2((k+14)-4)((k+14)+1)/25}.
\end{aligned}$$

$$\begin{aligned}
x_{k+15} &= a_1 b_1 a_2 b_1^{2(k+10)/5} b_2 b_1^{2((k+14)-4)((k+14)+1)/25} \\
&= a_1 b_1^{1+2(k+10)/5+2((k+14)-4)((k+14)+1)/25} a_2 b_2 \\
&= a_2 a_1 b_2 b_1^{(2(k+15)^2-25)/25}.
\end{aligned}$$

□

In the same manner as the previous case we examine the Fibonacci sequence of D_∞^3 i.e.

$$x_1 = a_1, x_2 = b_1, x_3 = a_2, x_4 = b_2, x_5 = a_3, x_6 = b_3, x_n = \prod_{j=1}^{6} x_{n-7+j}, \ (n \geq 7).$$

Lemma 2.2 *Every element of the Fibonacci orbit $\{x_j\}$ of $D_\infty^3 = \langle a_1, b_1, a_2, b_2, a_3, b_3 \rangle$ may be represented by:*

$$x_j = \begin{cases} a_1, & j \equiv 1, -6 \bmod 14 \\ b_1^{\pm 1}, & j \equiv 2, -5 \bmod 14 \\ a_2 b_1^{\pm 2(j-3)/7}, & j \equiv 3, -4 \bmod 14 \\ b_2^{\pm 1} b_1^{\pm 2(j-4)(j+3)/7^2}, & j \equiv 4, -3 \bmod 14 \\ a_3 b_2^{\pm 2(j-5)/7} b_1^{\pm 4(j-5)(j+2)(j+9)/(3\times 7^3)}, & j \equiv 5, -2 \bmod 14 \\ b_3^{\pm 1} b_2^{\pm 2(j+1)(j-6)/7^2} b_1^{\pm 2(j-6)(j+1)(j+8)(j+15)/(3\times 7^4)}, & j \equiv 6, -1 \bmod 14 \\ a_3 a_2 a_1 b_3^{\pm 1} b_2^{\pm (2j^2-7^2)/7^2} b_1^{\pm (2(j/7)^4+8(j/7)^3+4(j/7)^2-8(j/7)-3)/3}, & j \equiv 7, \ 0 \bmod 14 \end{cases}$$

where the positive exponent is chosen for the first value of j and the negative exponent is chosen for the second value of j.

Proof. As in Lemma 2.1, it is sufficient to compute $x_1, x_2, \ldots, x_{14}$ and the result follows by induction on j. □

We now seek to generalize the above to obtain a normal form for the Fibonacci orbit of D_∞^i, $i \geq 2$. We will need the following result in our calculations.

Lemma 2.3 *For $n \geq 3$ the following polynomial identity holds:*

$$2 + \sum_{j=3}^{n-1} \frac{2^{j-2}}{(j-2)!} m(m+1)(m+2)\ldots(m+j-3)$$

$$+ \sum_{j=3}^{n} \frac{2^{j-2}}{(j-2)!} (m-1)m(m+1)\ldots(m+j-4)$$

$$= \frac{2^{n-2}}{(n-2)!} m(m+1)(m+2)\ldots(m+n-3)$$

with the convention that when $n = 3$, the first term in the first summation on the left hand side is zero.

Proof. We use induction on n. When $n = 3$ the result holds since $2 + 0 + \frac{2}{1!}(m-1) = 2m$, i.e. $\frac{2}{1!}m = 2m$. When $n = 4$ the result also holds since

$$2 + \frac{2}{1!}m + \frac{2}{1!}(m-1) + \frac{2^2}{2!}(m-1)m = 2m(m+1), \ \text{i.e.}\ \frac{2^2}{2!}m(m+1) = 2m(m+1).$$

Now assume that the result holds for all values less than n. Then

$$2+\sum_{j=3}^{n-1}\frac{2^{j-2}}{(j-2)!}m(m+1)\dots(m+j-3)+\sum_{j=3}^{n}\frac{2^{j-2}}{(j-2)!}(m-1)m\dots(m+j-4)$$

$$=2+\sum_{j=3}^{n-2}\frac{2^{j-2}}{(j-2)!}m(m+1)\dots(m+j-3)+\sum_{j=3}^{n-1}\frac{2^{j-2}}{(j-2)!}(m-1)m\dots(m+j-4)$$
$$+\frac{2^{n-3}}{(n-3)!}m(m+1)\dots(m+n-4)+\frac{2^{n-2}}{(n-2)!}(m-1)m\dots(m+n-4).$$

By the inductive hypothesis this is equal to

$$\frac{2^{n-3}}{(n-3)!}m(m+1)\dots(m+n-4)+\frac{2^{n-3}}{(n-3)!}m(m+1)\dots(m+n-4)$$
$$+\frac{2^{n-2}}{(n-2)!}(m-1)m\dots(m+n-4)$$
$$=\frac{2^{n-2}}{(n-3)!}m(m+1)\dots(m+n-4)+\frac{2^{n-2}}{(n-2)!}(m-1)m\dots(m+n-4)$$
$$=\frac{2^{n-2}}{(n-2)!}m(m+1)\dots(m+n-3)$$

as required. □

Now we make the following observations about the Fibonacci orbits $\{x_j\}$ of the groups $D_\infty^i=\langle a_1,b_1,a_2,b_2,\dots,a_i,b_i\rangle$ (where $i\in\mathbb{N}$).

Lemma 2.4 *The exponent of b_l in x_j is zero if $j\equiv 1,2,\dots,2l-1 \bmod (2i+1)$, where $l\in\{1,2,\dots,i\}$.*

Proof. This is easy to see if we look at the 'structure' of the entries of the Fibonacci orbit $\{x_j\}$ of D_∞^i. For $j\in\{1,2,\dots,2i\}$ we have

$$x_j=\begin{cases} a_{(j+1)/2}, & \text{if } j \text{ is odd,}\\ b_{j/2}, & \text{if } j \text{ is even,}\end{cases}$$

and

$$x_{2i+1} = \prod_{d=1}^{2i} x_d.$$

Now it is easy to see that the next term of the Fibonacci orbit to contain an a_l will be

$$x_{2l+2i} = \prod_{d=2l}^{2l+2i-1} x_d.$$

Likewise the next b_l occurs in

$$x_{2l+2i+1} = \prod_{d=2l+1}^{2l+2i} x_d.$$

The final stage of the induction follows by using an argument analogous to that above. □

Remark It can easily be shown that the exponents of b_l can be calculated once one knows the exponents of b_1 since the exponents of b_l 'lag' behind the exponents of b_1. This holds because all b_l's initially have exponent one and from Lemma 2.4 above it can be seen when the exponents of b_l are nonzero. As an example of this 'lag' see the D^3_∞ case where, when $j \equiv 3 \bmod 14$, the power of b_1 is $2(j-3)/7$, and when $j \equiv 5 \bmod 14$, the power of b_2 is $2(j-5)/7$.

These results are best illustrated by looking at the D^3_∞ case (given separately in Lemma 2.2). It can be used as an example in the following proof to elucidate concepts. In this case the orbit is

$$\begin{aligned} Z = \; & (a_1, b_1, a_2, b_2, a_3, b_3, a_3a_2a_1b_3b_2b_1, \\ & a_1, b_1^{-1}, a_2b_1^{-2}, b_2^{-1}b_1^{-4}, a_3b_2^{-2}b_1^{-4}, b_3^{-1}b_2^{-4}b_1^{-8}, a_3a_2a_1b_3^{-1}b_2^{-7}b_1^{-19}, \\ & a_1, b_1, \ldots). \end{aligned}$$

So the Fibonacci orbit behaves as if it is in 'layers' of length $2i+1$ where in alternate layers the exponents of b_l are all positive and all negative. Since one layer's 'structure' depends only on the previous layer, a proof by induction will only need the first layer to be proved to anchor the induction. Also we note here that it is unnecessary to know the general form of the $k(2i+1)th$ entry ($k \in \mathbb{N}$) in the Fibonacci orbit since this will always be the product of all the previous terms in its layer and so this entry has a known shape, namely $a_ia_{i-1}\ldots a_1b_i^{w_i}b_{i-1}^{w_{i-1}}\ldots b_1^{w_1}$ where w_l is the sum of the exponents of b_l in the layer.

We are now ready to give the main result of this section.

Proposition 2.5 *The exponents of* b_1 *in the Fibonacci orbit* $\{x_j\}$ *of* D^i_∞ *are given in the table below*

$$\begin{array}{ll} 0, & j \equiv 1,\ 2i+2 \bmod (4i+2) \\ \pm 1, & j \equiv 2,\ 2i+3 \bmod (4i+2) \\ \pm A_n(\prod_{d=0}^{n-3}(j-n+d(2i+1)))/(2i+1)^{n-2}, & j \equiv n,\ 2i+1+n \bmod (4i+2) \\ \sum_{d=j-2i}^{j-1} z_d, & j \equiv 2i+1,\ 0 \bmod (4i+2) \end{array}$$

where $A_n = 2^{n-2}/(n-2)!$, z_r *is the exponent of* b_1 *in* x_r, $n \equiv j \bmod (2i+1)$ *so* $3 \le n \le 2i$ *and the positive forms of elements in the orbit are chosen if* $1 \le j \bmod (4i+2) \le 2i+1$*; otherwise choose the (second) negative forms.*

Proof. Let the exponent of b_1 in the x_r entry of the Fibonacci orbit be z_r. Assume that the result is true for all values of j such that $1 \le j < k$. There are several cases to examine:

Case 1. $k \equiv 1, 2i+2 \bmod (4i+2)$
In this case we have

$$\underline{b_1^{z_{k-2i}+\ldots+z_{k-2}} a_1 b_1^{z_{k-2i}+\ldots+z_{k-2}}} = a_1.$$

Case 2. $k \equiv 2, 2i+3 \bmod (4i+2)$
Here we have

$$\underline{b_1^{z_{k-2i}+\ldots+z_{k-3}} a_1 b_1^{z_{k-2i-2}+\ldots+z_{k-3}}} a_1 = \underline{a_1 b_1^{0+z_{k-2i-1}} a_1} = b_1^{-z_{k-2i-1}} = b_1^{\pm 1}.$$

Case 3. $k \equiv n, 2i+n+1 \bmod (4i+2)$, $3 \le n \le 2i+1$

When we are trying to calculate the exponent of b_1 for the kth entry in the Fibonacci orbit we need only concentrate on the terms in a_1 and b_1. The exponent of b_1 and a_1 in the kth entry in the Fibonacci orbit is

$$(\prod_{l=k-2i}^{x-1} b_1^{z_l})(a_1 \prod_{l=x-2i}^{x-1} b_1^{z_l}) a_1 (\prod_{l=x+1}^{k-1} b_1^{z_l})$$

where $x = (2i+1)\lfloor k/(2i+1) \rfloor$. (Note the first bracket is from the layer below that of x_k, the second bracket is the last entry in the lower layer). The above sum can be simplified using the group relations as follows

$$\underline{(\prod_{l=k-2i}^{x-1} b_1^{z_l})(a_1 \prod_{l=x-2i}^{x-1} b_1^{z_l})} a_1 (\prod_{l=x+1}^{k-1} b_1^{z_l}) = \underline{a_1 (\prod_{l=x-2i}^{k-2i-1} b_1^{z_l})} a_1 (\prod_{l=x+1}^{k-1} b_1^{z_l}),$$

$$= (\prod_{l=x-2i}^{k-2i-1} b_1^{-z_l})(\prod_{l=x+1}^{k-1} b_1^{z_1}).$$

Thus the exponent of b_1 is given by

$$\sum_{l=x+1}^{k-1} z_l - \sum_{l=x-2i}^{k-2i-1} z_l$$

$$\begin{aligned}
= \quad & \pm(0+1+A_3((x+3)-3)/(2i+1)+\dots \\
& +A_{k-x-1}\left[\prod_{d=0}^{k-x-4}((k-1)-(k-x-1)+d(2i+1))\right]/(2i+1)^{k-x-3}) \\
& -(\mp(0+1+A_3((x-2i+2)-3)/(2i+1)+\dots \\
& +A_{k-x}\left[\prod_{d=0}^{k-x-3}((k-2i-1)-(k-x)+d(2i+1))\right]/(2i+1)^{k-x-2})
\end{aligned}$$

So we want to show that

$$2+\sum_{l=3}^{n-1}\frac{2^{l-2}}{(l-2)!}m(m+1)\dots(m+l-3)+\sum_{l=3}^{n}\frac{2^{l-2}}{(l-2)!}(m-1)m(m+1)\dots(m+l-4)$$

$$=\frac{2^{n-2}}{(n-2)!}m(m+1)\dots(m+n-3)$$

where $n \equiv k \bmod (2i+1)$, $m = \lfloor k/(2i+1) \rfloor$ and $2 < n < 2i+1$ and $\sum_{l=3}^{n-1}(2^{l-2}m(m+1)\dots(m+l-3)/(l-2)!)$ is zero if $n=3$. Now the result follows by using Lemma 2.3.

Case 4. $k \equiv 2i+1, 0 \bmod (4i+2)$

Here there is nothing to prove.

□

3 The Fibonacci Length of D_{2m}^i, $i \geq 2$

In order to give an expression for the Fibonacci length of D_{2m}^i, $i \geq 2$, we require the following definitions and lemmas.

Definition In D_{2m}^i let $MinLEN(D_{2m}^i) = 2m(2i+1)/(4,m)$.

Lemma 3.1 *In D^i_{2m}, $i \geq 2$, $MinLEN(D^i_{2m})$ divides $LEN(D^i_{2m})$, and the quotient $LEN(D^i_{2m})/MinLEN(D^i_{2m})$ only involves odd prime divisors of m.*

Proof. We first note that $MinLEN(D^i_{2m})$ is the smallest possible Fibonacci length since we must have $(4i+2) \mid LEN(D^i_{2m})$ and, from the third entry in the Fibonacci orbit of D^i_{2m}, we also have $m \mid (2LEN(D^i_{2m})/(2i+1))$. Thus $MinLEN(D^i_{2m})$ divides $LEN(D^i_{2m})$.

Now let $l = MinLEN(D^i_{2m})$. By Proposition 2.5 for $3 \leq n \leq 2i$ we have

$$z_{l+n} = \frac{2^{n-2}l(l+(2i+1))\dots(l+(n-3)(2i+1))}{(n-2)!(2i+1)^{n-2}}.$$

If $\bar{l} = l/(2i+1) = 2m/(4,m)$ the above becomes

$$z_{l+n} = \frac{2^{n-2}\bar{l}(\bar{l}+1)\dots(\bar{l}+(n-3))}{(n-2)!}.$$

Now z_{l+n} is obviously an integer. Since z_{l+n} is the power of b_1 in the Fibonacci orbit we require that m divides $z_{LEN(D^i_{2m})+n}$, $3 \leq n \leq 2i$. It may occur that $m \not| z_{l+n}$ because powers of primes from the factorization of m that also occur in the numerator of z_{l+n} may be factored out by the $(n-2)!$ denominator. Let these 'missing' primes be $p_j^{\alpha_j} \dots p_r^{\alpha_r}$. Now multiplying l by a factor q less than $p_j^{\alpha_j} \dots p_r^{\alpha_r}$ will not be sufficient. For m will not divide z_{ql+n} since the denominator $(n-2)!$ will be the same as in the z_{l+n} case and the numerator will still be a product of 2^{n-2} and $n-2$ consecutive integers but this time starting at $2qm/(4,m)$. Thus $(n-2)!$ will still factor out $p_j^{\alpha_j} \dots p_r^{\alpha_r}$ and since q is less than $p_j^{\alpha_j} \dots p_r^{\alpha_r}$ we still have $m \not| z_{ql+n}$. If we let $q = p_j^{\alpha_j} \dots p_r^{\alpha_r}$ then we will have $m | z_{ql+n}$. □

Definition Let $\pi_{m,i} \in \mathbb{N}$ be defined as satisfying the equation $LEN(D^i_{2m}) = \pi_{m,i} MinLEN(D^i_{2m})$.

We now give a property of $\pi_{m,i}$ that will be used in a later lemma.

Lemma 3.2 *For any fixed m the sequence $(\pi_{m,i})_{i=2}^{\infty}$ is monotonically increasing.*

Proof. Let $l = LEN(D^i_{2m}) = 2m\pi_{m,i}(2i+1)/(4,m)$ and $\bar{l} = l/(2i+1)$. From Proposition 2.5 for $3 \leq n \leq 2i$ we have

$$\begin{aligned} z_{l+n} &= \frac{2^{n-2}\bar{l}(\bar{l}+1)\dots(\bar{l}+(n-3))}{(n-2)!} \\ &= \frac{2^{n-2}(2m\pi_{m,i}/(4,m))((2m\pi_{m,i}/(4,m))+1)\dots((2m\pi_{m,i}/(4,m))+(n-3))}{(n-2)!}. \end{aligned}$$

As i increases, and m remains the same, the number of entries in the layers of the Fibonacci orbit increases as does $(2i-2)!$. This gives the possibility that the number and size of primes in the prime decomposition of $(2i-2)!$ will increase and as a consequence so will $\pi_{m,i}$. Note if $\pi_{m,i+1} \not> \pi_{m,i}$ then since $(2i)! > (2i-2)!$, $\pi_{m,i+1} = \pi_{m,i}$. □

Now we show a multiplicative property of $\pi_{m,i}$.

Lemma 3.3 *For $m, n \in \mathbb{N}$, $(m,n) = 1$ and $i \geq 2$, $\pi_{mn,i} = \pi_{m,i}\pi_{n,i}$.*

Proof. This proof is similar to that of the previous lemma. Let $l = MinLEN(D^i_{2(mn)})$. As in Lemma 3.1 we look at

$$z_{l+k} = \frac{2^{k-2}l(l+(2i+1))\ldots(l+(k-3)(2i+1))}{(k-2)!(2i+1)^{k-2}},$$

where $3 \leq k \leq 2i$. Now $\pi_{mn,i}$ is found by seeing which primes in the prime decomposition of mn are factored out by the $(k-2)!$ denominator. Any primes that are factored out by the denominator are multiplied back so that $mn|z_{l+k}$. Now since $(m,n) = 1$ and the layers of the Fibonacci orbit are the same length the result follows immediately. □

Lemma 3.4 *If p is an odd prime and $\alpha \in \mathbb{N}$ then $\pi_{p^\alpha,i} = \pi_{p,i}$. Also for any $\beta \in \mathbb{N}$, $\pi_{2^\beta,i} = 1$.*

Proof. Here we use induction on i.

When $i = 2$, letting $l = MinLEN(D^i_{2p}) = 2p5/(4,p) = 10p$ and using Proposition 2.5 we obtain

$$\begin{aligned} z_{l+3} &= 2(10p), \\ z_{l+4} &= 2(10p)(10p+1). \end{aligned}$$

Both z_{l+3} and z_{l+4} are divisible by p and so $l = MinLEN(D^i_{2p}) = LEN(D^i_{2p})$ and $\pi_{p,2} = 1$. Using an analogous argument we see that $\pi_{p^\alpha,2} = 1$.

Now assume that $\pi_{p^\alpha,i} = \pi_{p,i}$ for all $i \leq \mu$. We examine the case $i = \mu + 1$. By Lemma 3.2 there are three possibilities:

Case 1: $\pi_{p^\alpha,\mu+1} = \pi_{p^\alpha,\mu}$ and $\pi_{p,\mu+1} = \pi_{p,\mu}$

Here there is nothing to prove.

Case 2: $\pi_{p^\alpha,\mu+1} > \pi_{p^\alpha,\mu}$

Let $l = LEN(D^{\mu}_{2p^\alpha})$, $3 \le n \le 2\mu$, $\bar{l} = l/(2\mu+1) = 2p^\alpha \pi_{p^\alpha,\mu}/(4, p^\alpha) = 2p^\alpha \pi_{p^\alpha,\mu}$ so the power of b_1 in the Fibonacci orbit is given by

$$z_{l+n} = \frac{2^{n-2} 2p^\alpha \pi_{p^\alpha,\mu}(2p^\alpha \pi_{p^\alpha,\mu} + 1) \dots (2p^\alpha \pi_{p^\alpha,\mu} + n - 3)}{(n-2)!}.$$

We must have $p^\alpha | z_{l+n}$ (since $|b_1| = p^\alpha$).

Now consider what happens in the $D^{\mu+1}_{2p^\alpha}$ case. Keeping l to be the Fibonacci length of $D^{\mu}_{2p^\alpha}$ but letting $3 \le n \le 2\mu + 2$, as above we obtain

$$z_{l+n} = \frac{2^{n-2} 2p^\alpha \pi_{p^\alpha,\mu}(2p^\alpha \pi_{p^\alpha,\mu} + 1) \dots (2p^\alpha \pi_{p^\alpha,\mu} + n - 3)}{(n-2)!},$$

and since we require $\pi_{p^\alpha,\mu+1} > \pi_{p^\alpha,\mu}$ we have $p^\alpha \not| z_{l+n}$ for some n, $3 \le n \le 2\mu + 2$. For n in the range $3 \le n \le 2\mu$ we have $p^\alpha | z_{l+n}$ (this is just the previous μ case), so p^α does not divide one or both of z_{l+n}, $n = 2\mu + 1$ or $2\mu + 2$. This means that $(2\mu)!$ contains a power of p, p^γ say. Thus $\pi_{p^\alpha,\mu+1} = p^\gamma \pi_{p^\alpha,\mu}$.

We now examine the powers of b_1 in the Fibonacci orbit of $D^{\mu+1}_{2p}$ and letting $l = LEN(D^{\mu}_{2p})$, $3 \le n \le 2\mu + 2$ gives

$$z_{l+n} = \frac{2^{n-2} 2p\pi_{p,\mu}(2p\pi_{p,\mu} + 1) \dots (2p\pi_{p,\mu} + n - 3)}{(n-2)!}.$$

For $3 \le n \le 2\mu$, $p | z_{l+n}$ but for either $n = 2\mu + 1$ or $2\mu + 2$ we introduce a factor of $(2\mu - 1)(2\mu)$ in the denominator, as in the $D^{\mu+1}_{2p^\alpha}$ case, which contains p^γ. So $\pi_{p,\mu+1} = p^\gamma \pi_{p,\mu}$.

By the inductive hypothesis we know that $\pi_{p^\alpha,\mu} = \pi_{p,\mu}$. Thus $\pi_{p,\mu+1} = p^\gamma \pi_{p,\mu} = p^\gamma \pi_{p^\alpha,\mu} = \pi_{p^\alpha,\mu+1}$.

Case 3: $\pi_{p,\mu+1} > \pi_{p,\mu}$

Using an analogous argument as the above we see that in this case $\pi_{p^\alpha,\mu+1} = \pi_{p,\mu+1}$.

The second statement of the lemma holds since if we examine the powers of b_1 in the Fibonacci orbit of $D^i_{2(2^\beta)}$ we have, for $l = MinLEN(D^i_{2(2^\beta)}) = 2(2i+1)(2^\beta/(4, 2^\beta))$, $i \ge 2$, $3 \le n \le 2i$,

$$z_{l+n} = \frac{2^{n-2} 2(2^\beta/(4, 2^\beta))(2(2^\beta/(4, 2^\beta)) + 1) \dots (2(2^\beta/(4, 2^\beta)) + n - 3)}{(n-2)!}.$$

Now this is always divisible by 2^β. □

Definition For $i \ge 2$, let $m = 2^{\alpha_0} p_1^{\alpha_1} \dots p_k^{\alpha_k}$ where p_l are distinct odd primes, and let $t_l = \lfloor \log_{p_l}(2i - 3) \rfloor$. Define $\Phi_{m,i} = p_1^{t_1} \dots p_k^{t_k}$, and for all $\alpha \ge 1$, $\Phi_{2^\alpha,i} = 1$.

We note here that $\Phi_{m,i}$ is obviously multiplicative in that for $(m,n) = 1$, $\Phi_{mn,i} = \Phi_{m,i}\Phi_{n,i}$.

Lemma 3.5 *For all primes p, for $\alpha \in \mathbb{N}$ and $i \geq 2$, $\Phi_{p^\alpha,i} = \Phi_{p,i}$.*

Proof. The definition of $\Phi_{m,i}$ is equivalent to finding for $m = 2^{\alpha_0}p_1^{\alpha_1}\dots p_k^{\alpha_k}$, where p_l are distinct odd primes, the largest natural number t_l such that $(p_l^{t_l} + 3)/2 \leq i$ and letting $\Phi_{m,i} = \prod_{p_l|m} p_l^{t_l}$. Since the power of a prime in the prime decomposition of m is not used in computing $\Phi_{m,i}$, the lemma holds. □

Proposition 3.6 *For D^i_{2p}, $i \geq 2$ and p a prime, $\pi_{p,i} = \Phi_{p,i}$.*

Proof. Let p be a fixed odd prime. We use induction on i, so $\pi_{p,k} = \Phi_{p,k}$ for $k < i$.

- $i = 2$.

It follows by Lemma 3.4 that $\pi_{p,2} = 1$ and by definition that $\Phi_{p,2} = 1$.

We now prove the result for i. First we note that $\pi_{p,i}$ forms a monotonic increasing sequence by Lemma 3.2. There are two cases to consider:

1. $\pi_{p,i-1} = \pi_{p,i}$;
2. $\pi_{p,i-1} < \pi_{p,i}$.

- Case 1, $\pi_{p,i-1} = \pi_{p,i}$.

There are two possibilities for the value of $\Phi_{p,i}$; either $\Phi_{p,i-1} = \Phi_{p,i}$ or $\Phi_{p,i-1} < \Phi_{p,i}$. We will assume the latter case and reach a contradiction and so obtain $\Phi_{p,i} = \Phi_{p,i-1} = \pi_{p,i-1} = \pi_{p,i}$, the desired result.

Assume that $\Phi_{p,i} > \Phi_{p,i-1}$. If $\Phi_{p,i-1} = p^c$, where $c = \lfloor \log_p(2(i-1)-3)\rfloor$, then $\Phi_{p,i} = p^{c+1}$, where $c+1 = \lfloor \log_p(2i-3)\rfloor$. We know that $\pi_{p,i-1} = \pi_{p,i}$ so, using Proposition 2.5 and Lemma 3.1, the prime p divides

$$z_{l+(2i-1)} = \frac{2^{2i-3}\bar{l}(\bar{l}+1)\dots(\bar{l}+(2i-1)-3)}{((2i-1)-2)!},$$

where $\bar{l} = LEN(D^i_{2p})/(2i+1) = 2p\pi_{p,i}/(4,p)$. But $p^c = \pi_{p,i-1}$, so $\bar{l} = 2p.p^c = 2p^{c+1}$ giving

$$\begin{aligned} z_{l+(2i-1)} &= \frac{2^{2i-3}(2p^{c+1})(2p^{c+1}+1)\dots(2p^{c+1}+2i-4)}{(2i-3)!} \\ &= 2^{2i-3}\left(\frac{2p^{c+1}}{2i-3}\right)\left(\frac{2p^{c+1}+1}{1}\right)\left(\frac{2p^{c+1}+2}{2}\right)\dots\left(\frac{2p^{c+1}+2i-4}{2i-4}\right). \end{aligned}$$

Note that the bth bracketed term, $2 \le b \le 2i-3$, in the above product may be written as $\binom{2p^{c+1}+p^i m_b}{p^i m_b}$, where $(m_b, p) = 1$ and $0 \le i \le c+1$ (this last inequality follows from the definition of c). Carrying out the obvious simplification we obtain $\binom{2p^{c+1-i}+m_b}{m_b}$. Now notice that $(\prod_{b=2}^{2i-3} m_b)$ is coprime to p. Recall that $p|z_{l+(2i-1)}$ and, by assumption, $\lfloor \log_p(2i-3) \rfloor = c+1$ or $2i-3 = p^{c+1}$. From this last equation it follows that $p \not| z_{l+(2i-1)}$, a contradiction. Hence $\Phi_{p,i} = p^c = \Phi_{p,i-1} = \pi_{p,i-1} = \pi_{p,i}$.

- Case 2, $\pi_{p,i-1} < \pi_{p,i}$.

Let $l = 2p\pi_{p,i-1}$. So by Lemma 3.1 we have $l = 2p^k$ for some integer k. Using Proposition 2.5 for $3 \le n \le 2i$ gives

$$\begin{aligned} z_{l+n} &= \frac{2^{n-2}2p^k(2p^k+1)\dots(2p^k+n-3)}{(n-2)!} \\ &= 2^{n-2}\binom{2p^k}{n-2}\binom{2p^k+1}{1}\binom{2p^k+2}{2}\dots\binom{2p^k+n-3}{n-3}. \end{aligned}$$

Since $\pi_{p,i-1} < \pi_{p,i}$ we must have $p \not| z_{l+2i-1}$ or $p \not| z_{l+2i}$. Using an argument similar to that given above we see that $p^k = (2i-1)-2$ or $p^k = (2i)-2$. Hence $k = \lfloor \log_p(2i-3) \rfloor$. Finally by Lemma 3.1 we have $\pi_{p,i} = p^{k-1}.p = p^k$, and so $p^{k-1} = \pi_{p,i-1} = \Phi_{p,i-1}$ and $p^k = \Phi_{p,i} = \pi_{p,i}$.

□

Theorem 3.7 *For $i \ge 2$, $LEN(D^i_{2m}) = \Phi_{m,i} MinLEN(D^i_{2m})$.*

Proof. The theorem follows from Lemmas 3.3, 3.4, 3.5 and Proposition 3.6. □

Corollary 3.8

$$LEN(D^2_{2m}) = 10m/(4, m)$$

Proof. In this case, for $m = 2^{\alpha_0} p_1^{\alpha_1} \dots p_k^{\alpha_k}$ where p_l are distinct odd primes, we have $t_l = \lfloor \log_{p_l}(1) \rfloor = 0$ giving $\Phi_{m,2} = 1$. So by Theorem 3.7 we have $LEN(D^2_{2m}) = MinLEN(D^2_{2m}) = 10m/(4, m)$. □

Corollary 3.9

$$LEN(D^3_{2m}) = 14m\Phi_{m,3}/(4, m)$$

where $\Phi_{m,3} = 3$ if $m \equiv 0 \bmod 3$, and $\Phi_{m,3} = 1$ otherwise.

Proof. Using the same notation as in the proof of the previous corollary we obtain $t_l = \lfloor \log_{p_l}(3) \rfloor$ which is equal to 1 if $p_l = 3$ and 0 otherwise. □

4 Other dihedral group generators

We now examine the case of other presentations of D^2_{2m} when m is odd.

Theorem 4.1 *For m odd, D^2_{2m} is defined by the presentation*

$$\langle\ x,\ y \mid x^{2m} = 1,\ (x^m y)^2 = 1,\ y^{2m} = (xy)^2\ \rangle.$$

In this case $LEN_{\{x,y\}}(D^2_{2m}) = 6$.

Proof. That the presentation defines D^2_{2m} is shown in [4]. We first show that the following relations hold in the presentation for D^2_{2m}, m odd:

$$xy^2xy^2xyxy^2xy = x$$

and

$$yxyxy^2xyxy^2xy^2xyxy^2xy = y.$$

In [4] it is shown that the following relations also hold in the presentation for D^2_{2m}:

$$\text{(i) } y^{2m} = 1,\ \text{(ii) } (xy)^2 = 1,\ \text{(iii) } xy^{m-1}x^{-1} = y^{-m+1},\ \text{(iv) } y^{-m}xy^m = x^{-1}.$$

Now

$$1 = xy^{m-1}x^{-1}xy^{m+1}x^{-1} \text{ by (i)}$$

and

$$1 = \underline{xy^{m-1}x^{-1}}xy^{m+1}x^{-1} = y^{-m+1}xy^{m+1}x^{-1} \text{ by (iii)}.$$

We now have

$$1 = y\underline{y^{-m}xy^m}yx^{-1} = yx^{-1}yx^{-1} \text{ by (iv)}.$$

Now using (ii) we see that

$$1 = yx^{-1}y\underline{x^{-1}} = yx^{-1}y^2xy = y\underline{x^{-1}}yxyxy^2xy = y^2xy^2xyxy^2xy.$$

Thus $xy^2xy^2xyxy^2xy = x$ in D^2_{2m} , m odd.

The second result follows since

$$yxyxy^2xy\underline{xy^2xy^2xyxy^2xy} = y\underline{xyxy}yxyx = y\underline{yxyx} = y.$$

We now recall the well-known result that every finite 2-generator group $G = \langle a, b\rangle$ is a homomorphic image of some Fibonacci group $F(2, n)$, where $n =$

$LEN_{\{a,b\}}(G)$. In the preceding paragraph we have shown that $LEN(D^2_{2m}) \leq 6$ for all m , where D^2_{2m} is defined by the presentation

$$\langle\ x, y \quad | \quad x^{2m} = 1, (x^m y)^2 = 1, y^{2m} = (xy)^2\ \rangle,$$

m odd; the proof is complete on noting that $6 = \min\{n\ :\ |F(2,n)| \text{ is infinite}\}$. □

Relating this result to that of Corollary 3.8 we see that the generating set for D^2_{2m}, m odd, used in this section always gives a constant Fibonacci length of 6 while the generating set used in the previous section gives a Fibonacci length of $10m/(4,m)$, which is always greater than 6.

Note The presentation given in Theorem 4.1 for D^2_{2m}, m odd, is efficient. For odd values of m, D^3_{2m} has an efficient presentation

$$\langle\ a, x, z \quad | \quad x^{2m} = z^{2m} = (xz^m)^{2m} = a^2 = [x,a] = [a, z^m] = [x, z^{m-1}] = 1,$$
$$(x^{m-1}z^m)^2 = (az^{m-1})^2 = 1\ \rangle.$$

In this case our computations, using the GAP computer algebra package [8], imply that the Fibonacci length is equal to $8m$. This remains as a conjecture.

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