Search techniques, Fibonacci lengths and centro-polyhedral groups

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Abstract. For a finitely generated group $G = \langle A \rangle$ where $A = \{a_1, a_2, \ldots, a_n\}$ the sequence $x_i = a_{i+1}$, $0 \leq i \leq n-1$, $x_{i+n} = \prod_{j=1}^{n} x_{i+j-1}$, $i \geq 0$, is called the Fibonacci orbit of G with respect to the generating set A, denoted $F_A(G)$. If $F_A(G)$ is periodic, we call the length of the period of the sequence the *Fibonacci length* of G with respect to A, written $LEN_A(G)$. We examine the Fibonacci lengths of all generating pairs for certain centro-polyhedral groups. The problem requires a variety of approaches both exhaustive and random search.

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1 Introduction

The topic of the Fibonacci length of certain groups has been discussed by the authors over recent years, in particular at certain of the Fibonacci Association international conferences. Indeed one of the authors was a co-author of a paper on Fibonacci length at the Fibonaccci Association Pisa conference in 1988, see [6]. Some recent information by the authors has appeared in a variety of sources, see [3], [4] and [5]. It is the purpose of this note to bring together recent results on the Fibonacci length of a class of centro-polyhedral groups and to suggest that opportunities for further investigation still exist.

The *Fibonacci group* $F(r,n)$ is the group defined by the presentation

$$\langle\, a_1, a_2, \ldots, a_n \mid a_1a_2\ldots a_r = a_{r+1}, a_2a_3\ldots a_{r+1} = a_{r+2}, \ldots, a_{n-1}a_na_1\ldots a_{r-2} = a_{r-1}, a_na_1a_2\ldots a_{r-1} = a_r \,\rangle$$

where $r > 0, n > 0$ and all subscripts are assumed to be reduced modulo n. It is known that all finite and some infinite groups are homomorphic images of some Fibonacci groups, see [10] and [2]. In order to find which Fibonacci groups occur, the concept of Fibonacci length is used.

Let G be a finitely generated group, $G = \langle A\rangle$, where $A = (a_1, \ldots, a_n)$ an ordered n-tuple. Then we have:

Definition 1 The *Fibonacci orbit* of G with respect to the generating n-tuple A, written $F_A(G)$, is the sequence $x_1 = a_1, \ldots, x_n = a_n, x_{i+n} = \prod_{j=1}^{n} x_{i+j-1},\ i \geq 1$.

Definition 2 If $F_A(G)$ is periodic then the length of the period of the sequence is called the *Fibonacci length* of G with respect to the generating n-tuple A, written $LEN_A(G)$. If $F_A(G)$ is not periodic then we say that the group G has infinite Fibonacci length on the generating n-tuple A, written $LEN_A(G) = \infty$.

We will write $LEN(G)$ when it is clear which generating n-tuple is being used. It is also important to note that the Fibonacci length of a group depends on the chosen generating n-tuple.

Definition 3 The *sequence of Fibonacci words* is the infinite sequence generated by the following system

$$(\{x, y\}, \{x \mapsto y, y \mapsto xy\}, \{x\})$$

i.e. the sequence $(x, y, xy, yxy, xy^2xy, \ldots)$. The *sequence of tribonacci words* is the infinite sequence generated by the following system

$$(\{x, y, z\}, \{x \mapsto y, y \mapsto z, z \mapsto xyz\}, \{x\})$$

i.e. the sequence $(x, y, z, xyz, yzxyz, zxyzyzxyz, \ldots)$.

From the theory of group presentations and, in particular, von Dyck's Theorem, it is possible to prove the following theorem:

Theorem 1 *Let G be a group with generating n-tuple $(a_1, a_2, \ldots, a_n)$ and let $LEN_A(G) = m$ for finite m. Then G is an epimorphic image of the Fibonacci group $F(n, m)$.*

Thus given a group, G say, we can find a Fibonacci group of which G is an image. From von Dyck's Theorem, see for example [12], we may prove:

Theorem 2 *Let G be a group defined by the presentation $\langle\ X \mid R\ \rangle$. If $LEN_X(G) = n$ and H is a factor group of G on the same set of generating symbols, then $LEN_X(H) \mid LEN_X(G)$.*

We also require the definition of Wall number.

Definition 4 Let $k_{(a,b,c)}(n)$ denote the minimal period of the integer-valued recurrence relation $u_n = u_{n-1} + u_{n-2} + u_{n-3}, u_0 = a, u_1 = b, u_2 = c$ when each entry is reduced modulo n. The number $k_{(a,b,c)}(n)$ is called the *Wall number*.

When it is clear that we are working with a 3-step tribonacci-like recurrence relation we will write $k(n)$ to denote $k_{(a,b,c)}(n)$.

In order to compare certain computational methods we will use polyhedral groups and the related centro-polyhedral groups. For more information about these groups see [7]. Polyhedral groups and centro-polyhedral groups are defined as follows:

Definition 5 The *polyhedral group* (ℓ, m, n), for $\ell, m, n \in \mathbb{Z}$ is defined by the presentation

$$\langle\ x, y, z \mid x^\ell = y^m = z^n = xyz = 1\ \rangle.$$

Definition 6 The *centro-polyhedral group* $\langle \ell, m, n \rangle$, for $\ell, m, n \in \mathbb{Z}$ is defined by the presentation

$$\langle\ x, y, z \mid x^\ell = y^m = z^n = xyz\ \rangle.$$

In [5] we reported on experiments designed to calculate all Fibonacci lengths on all n-tuples when $n = 2$ of certain finite polyhedral and centro-polyhedral groups. The task of calculating all Fibonacci lengths over all generating n-tuples when $n = 2$ and when $n = 3$ of a non-abelian group has only been attempted, to the authors' knowledge, in [6] where the Fibonacci lengths of D_{2n}, the dihedral groups of order $2n$, and Q_{2^n} were calculated due to a nice property of their automorphism groups and in [3], [4] and [5] where the above centro-polyhedral groups were considered. For related results see also [1] where Fibonacci lengths for p-groups are considered. The paper [5] started out as a straightforward generalisation of [3] but immediately ran into problems of computing time and resources requiring the various techniques described in the next section.

All computer calculations were carried out using a standard download of the computational algebra system GAP, see [9], together with the coset enumeration package ACE, see [8].

2 Methods

In order to calculate all Fibonacci lengths of a given group on all generating pairs the authors wrote several programs and compared the results and efficency of each program against the others. One of the main problems that occurred was that there is no known result that will predict, with any great accuracy, the number of distinct Fibonacci lengths that a given group might have. Our programs are denoted by the names full exhausive search, restrictive full exhausive search, restrictive search and random search. For more details see [5]. The two main methods used are full exhaustive search and random search.

1. **Full Exhausive Search**: Here we simply choose every pair of elements except for the obvious exceptions. If they generate the group then we calculate the Fibonacci length. This calculation is certain to complete given that the given group is finite but the time to complete can be prohibitively long.

2. **Random Search**: Here we search over a known number of randomly chosen generating pairs and calculate the Fibonacci lengths. This proved to be very fast to compute. The main problem was that nothing is known about the distribution of Fibonacci lengths within a group.

In the above computational methods the groups, given by finite presentations, were first converted into the isomorphic permutation groups.

3 The groups $\langle n,2,2\rangle$, $(n,2,2)$, $\langle 2,n,2\rangle$, $(2,n,2)$, $\langle 2,2,n\rangle$, $(2,2,n)$ and related groups

The proofs of the results in this section may be found in [3] and [4]. Note that here $n \in \mathbb{Z}$ rather than $n \in \mathbb{N}$ as in [3].

Theorem 3 *Let n be an integer, $|n| > 2$. Then $LEN(\langle 2,-2,n\rangle) = LEN(\langle -2,2,n\rangle)$.*

Theorem 4 *Consider the groups $\langle a,b,c\rangle$.*

1. *The groups defined by $\langle n,2,2\rangle$ and $\langle 2,n,2,\rangle$, $n \in \mathbb{Z}$, $|n| > 2$, have Fibonacci length 8.*
2. *The groups defined by $\langle 2,2,n\rangle$, $n \in \mathbb{Z}$, $|n| > 2$, have*

$$LEN_{(x,y,z)}(\langle 2,2,n\rangle) = \begin{cases} 4n, & n \equiv 0 \bmod 4, \\ 8n, & \text{otherwise.} \end{cases}$$

Corollary 5 *Let $G = \langle x,y,z \mid x^n = y^2 = z^2 = xyz = 1\rangle$, $|n| > 2$. Then $LEN_{(x,y,z)}(G) = 8$.*

Corollary 6 *Let $G = \langle x,y,z \mid x^2 = y^n = z^2 = xyz = 1\rangle$, where $|n| > 2$. Then $LEN_{(x,y,z)}(G) = 8$.*

Theorem 7 *The Fibonacci length of $\langle -2,n,2\rangle$, $\langle 2,n,-2\rangle$, $\langle n,2,-2\rangle$ and $\langle n,-2,2\rangle$ is $k(4(n-1))$, where $k(4(n-1))$ is the appropriate Wall number.*

Theorem 8 *The Fibonacci length of $(-2,n,2), (2,n,2), (2,n,-2), (n,2,2)$, $(n,-2,2)$ and $(n,2,-2)$ is 8.*

The Fibonacci length of $(-2,2,n), (2,-2,n)$ and $(2,2,n)$ is

$$\begin{cases} 2n, & n \equiv 0 \bmod 4, \\ 4n, & n \equiv 2 \bmod 4, \\ 8n, & \text{otherwise.} \end{cases}$$

Placing all our results from this section into a table together with the group order (with the obvious meaning when n is negative) and an associated quantity gives:

Presentation $\mathcal{P}$	Fibonacci length	$\|\langle\mathcal{P}\rangle\|$	$\|\langle\mathcal{P}\rangle\|/2n$
$\langle -2, 2, n\rangle$	$\min\{x : x > 1,\ 4n\|x \text{ and } k(4(n-1))\|x\}$	$4n(n-1)$	$2(n-1)$
$\langle 2, -2, n\rangle$	$\min\{x : x > 1,\ 4n\|x \text{ and } k(4(n-1))\|x\}$	$4n(n-1)$	$2(n-1)$
$\langle 2, 2, n\rangle$	$4n$ if $n \equiv 0 \bmod 4$, $8n$ otherwise	$4n$	2
$\langle -2, n, 2\rangle$	$k(4(n-1))$	$4n(n-1)$	$2(n-1)$
$\langle 2, n, -2\rangle$	$k(4(n-1))$	$4n(n-1)$	$2(n-1)$
$\langle 2, n, 2\rangle$	8	$4n$	2
$\langle n, -2, 2\rangle$	$k(4(n-1))$	$4n(n-1)$	$2(n-1)$
$\langle n, 2, -2\rangle$	$k(4(n-1))$	$4n(n-1)$	$2(n-1)$
$\langle n, 2, 2\rangle$	8	$4n$	2

Note that, in the above table, if the group has order $4n(n-1)$ then two of the group generators have order $4(n-1)$ and the third generator has order $2n(n-1)$ whereas, if the group order is $4n$, then two of the generators have order 4 and the other generator has order $2n$. The next table gives the corresponding results for polyhedral groups.

Presentation $\mathcal{P}$	Fibonacci length	$\|\langle\mathcal{P}\rangle\|$
$(-2, 2, n)$	$2n$ if $n \equiv 0 \bmod 4, 4n$ if $n \equiv 2 \bmod 4, 8n$ otherwise	$2n$
$(2, -2, n)$	$2n$ if $n \equiv 0 \bmod 4, 4n$ if $n \equiv 2 \bmod 4, 8n$ otherwise	$2n$
$(2, 2, n)$	$2n$ if $n \equiv 0 \bmod 4, 4n$ if $n \equiv 2 \bmod 4, 8n$ otherwise	$2n$
$(-2, n, 2)$	8	$2n$
$(2, n, -2)$	8	$2n$
$(2, n, 2)$	8	$2n$
$(n, -2, 2)$	8	$2n$
$(n, 2, -2)$	8	$2n$
$(n, 2, 2)$	8	$2n$

4 The groups $\langle 2, m, n\rangle$ and $(2, m, n)$

In deciding which groups to examine we chose to use the centro-polyhedral groups of the form $\langle \pm 2, X\rangle$ or $(\pm 2, X)$, where $X = \{\pm 3, \pm 3\}, \{\pm 3, \pm 4\}$ or $\{\pm 3, \pm 5\}$. Using the Full Exhausive Search method as in Section 2 we were able to find all Fibonacci lengths for the following groups and the distribution of the Fibonacci lengths within each group. In this section we are concerned with generating *pairs*.

$\mathcal{P}$	$\lvert\langle\mathcal{P}\rangle\rvert$	$LEN_{(a,b)}(\langle\mathcal{P}\rangle)$
Centro-polyhedral groups		
$\langle x,y,z \mid x^2=y^3=z^3=xyz\rangle$	24	16 (96), 48 (288)
$\langle x,y,z \mid x^{-2}=y^3=z^3=xyz\rangle$	120	16 (384), 48 (1152), 80 (1920), 240 (5760)
$\langle x,y,z \mid x^2=y^{-3}=z^3=xyz\rangle$	72	48 (3456)
$\langle x,y,z \mid x^{-2}=y^{-3}=z^3=xyz\rangle$	216	144 (31104)
$\langle x,y,z \mid x^2=y^{-3}=z^{-3}=xyz\rangle$	168	16 (4608), 48 (13824)
$\langle x,y,z \mid x^{-2}=y^{-3}=z^{-3}=xyz\rangle$	312	112 (16128), 336 (48384)
The polyhedral group $(2,3,3)$		
$\langle x,y,z \mid x^2=y^3=z^3=xyz=1\rangle$	12	16 (96)

$\mathcal{P}$	$\lvert\langle\mathcal{P}\rangle\rvert$	$LEN_{(a,b)}(\langle\mathcal{P}\rangle)$
Centro-polyhedral groups		
$\langle x,y,z \mid x^2=y^3=z^4=xyz\rangle$	48	18 (864)
$\langle x,y,z \mid x^{-2}=y^3=z^4=xyz\rangle$	528	90 (103680)
$\langle x,y,z \mid x^2=y^{-3}=z^4=xyz\rangle$	336	144 (41472)
$\langle x,y,z \mid x^{-2}=y^{-3}=z^4=xyz\rangle$	912	18 (311040)
$\langle x,y,z \mid x^2=y^3=z^{-4}=xyz\rangle$	240	36 (3456), 108 (17280)
$\langle x,y,z \mid x^{-2}=y^3=z^{-4}=xyz\rangle$	816	36 (248832)
$\langle x,y,z \mid x^2=y^{-3}=z^{-4}=xyz\rangle$	624	252 (145152)
$\langle x,y,z \mid x^{-2}=y^{-3}=z^{-4}=xyz\rangle$	1200	180 (86400), 900 (432000)
The polyhedral group $(2,3,4)$		
$\langle x,y,z \mid x^2=y^3=z^4=xyz=1\rangle$	24	18 (216)

$\mathcal{P}$	$\lvert\langle\mathcal{P}\rangle\rvert$	$LEN_{(a,b)}(\langle\mathcal{P}\rangle)$
Centro-polyhedral groups		
$\langle x,y,z \mid x^2=y^3=z^5=xyz\rangle$	120	12 (960), 14 (840), 42 (2520), 50 (1200), 150 (3600)
$\langle x,y,z \mid x^2=y^3=z^{-5}=xyz\rangle$	1320	50 (144000), 60 (115200), 70 (100800), 150 (432000), 210 (302400)
$\langle x,y,z \mid x^2=y^{-3}=z^5=xyz\rangle$	2280	36 (345600), 126 (1209600), 450 (1728000)
$\langle x,y,z \mid x^2=y^{-3}=z^{-5}=xyz\rangle$	3720	60 (921600), 150 (4608000), 210 (3225600)
The polyhedral group $(2,3,5)$		
$\langle x,y,z \mid x^2=y^3=z^5=xyz=1\rangle$	60	12 (240), 14 (840), 50 (1200)

The number in brackets indicates the total number of distinct pairs with the stated Fibonacci length.

It turned out that the above groups proved particularly amenable to the random search method. On each run of the random search method a complete list of Fibonacci lengths was calculated, while the full exhausive search method took many times longer to obtain the same results. It is also interesting to note that looking again at Section 3.3 of [6] the random method was able, on one run, to find all Fibonacci lengths on generating pairs.

For certain of the above groups we have also considered the Fibonacci length of generating triples. For more details on this and the various search techniques see [4]. In this case the random search method proved particularly helpful.

5 Further Work

It is to be hoped that many questions remain to be cosidered. Questions one might ask include:

1. In [2] the Fibonacci length of powers of dihedral groups is considered. What about powers of the other centro-polyhedral groups considered in this paper?

2. Can one predict the distribution of Fibonacci lengths within a particular group? Why is the random search method so efficient in obtaining the complete list of Fibonacci lengths?

3. What general theories can be obtained regarding the Fibonacci lengths of a general group? For example does there exist a decision process to determine whether, or not, a given group has finite Fibonacci length?

4. Resolve Wall's conjecture, see [13]. The conjecture is that $k(p^2) = pk(p)$, where p is an odd prime. (This holds in all known cases. The other possibility would be $k(p^2) = k(p)$.)

5. To find the Fibonacci lengths of infinite groups it would be useful to have a program that would use the Knuth-Bendix method to find Fibonacci lengths, see [11].

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