# On the efficiency of direct powers of $PGL(2,p)$

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## 1 Introduction

A finite group $G$ is said to be efficient if $G$ has a presentation $\langle\ X \mid R\ \rangle$ where $|R| = |X| + \text{rank}(M(G))$ where $M(G)$ is the Schur multiplier of $G$; see for example [14]. The efficiency of direct powers, $G^n$, of a finite group $G$ has been studied over a number of years, see for example [5], [8] and [14]. In [6] the problem of proving that $G^n$ is efficient for all $n \in \mathbb{N}$, in the case of an imperfect group, is shown to reduce to proving that $G^n$ is efficient for a finite number of values of $n$. Recently there has been interest in the semigroup efficiency of groups, see [2]. In particular the semigroup efficiency of direct powers of groups is considered in [1].

The purpose of this paper is to prove that $G(p)^n$ is efficient for all odd primes $p$ and all $n \in \mathbb{N}$, where $G(p)$ is given by the presentation

$$\langle\ a,\ b \mid a^2,\ b^p,\ (ab^2)^4,\ (abab^2)^3\ \rangle.$$

In [13] it was claimed that $G(p) \cong PGL(2,p)$ but in fact this is not always the case. However if 2 is primitive in $GF(p)$ then $G(p) \cong PGL(2,p)$ while if $G(p)$ is not $PGL(2,p)$ it is $C_2 \times PSL(2,p)$. See [9] for details.

To apply the results of [6] we note that $M(G(p)) \cong C_2$ and $G(p)/G(p)' \cong C_2$. Since $G(p)$ is proved efficient in [13], Corollary 3.3 (a) of [6] shows that we need only prove that $G(p)^2$ and $G(p)^3$ are efficient on a minimal generating set to obtain $G(p)^n$ is efficient for all $n \in \mathbb{N}$. That $G(p)^2$ is efficient on a minimal generating set is proved in Section 2, while the result for $G(p)^3$ is proved in Section 3. Finally note that $PGL(2,2) \cong S_3$ and so the efficiency of $PGL(2,2)^n$ follows from [5].

## 2 The efficiency of $G(p)^2$, $p$ an odd prime

We first find a presentation for $G(p)^2$ on two generators using the following easily proved lemma, see [11].

**Lemma 2.1** *If* $G$, $H$ *are groups presented by* $\langle\ X \mid R\ \rangle$, $\langle\ Y \mid S\ \rangle$ *respectively, then their direct product* $G \times H$ *has the presentation* $\langle\ X,\ Y \mid R,\ S,\ [X,Y]\ \rangle$, *where* $[X,Y]$ *denotes the set of commutators* $\{\ x^{-1}y^{-1}xy \mid x \in X,\ y \in Y\ \}$. ■

We now are able to give a presentation for $G(p)^2$ on two generators using

**Lemma 2.2** *A presentation for* $G(p)^2$ *is*

$$\begin{array}{lll}\langle\ x,\ y & \mid & x^{2p},\ y^{2p},\ x^{-4}(xy^2)^4,\ y^{-4}(yx^2)^4, \\ & & y^{-p}(x^pyx^py^2)^3,\ x^{-p}(y^pxy^px^2)^3,\ [x^p,y^p],\ [x^2,y^2]\ \rangle.\end{array} \tag{1}$$

**Proof.** Using Lemma 2.1 we write down a presentation for $G(p)^2$ in terms of the generating set $\{a,b,c,d\}$ where $a^2 = c^2 = 1$ and $b^p = d^p = 1$, and then use the transformation $x = ad$ and $y = bc$. ■

We can now start to remove redundant relations from (1).

**Lemma 2.3** *The relation* $x^{2p} = 1$ *is redundant in presentation* (1).

**Proof.** Since $(2,p) = 1$ there exist integers $\lambda$ and $\mu$ such that $1 = 2\lambda + p\mu$ and so using the last two relations of (1) we have that $x^{2p}y = yx^{2p}$, and so $x^{2p} \in Z(G(p)^2)$. It is also easily seen that $x^2 = 1$ in $G(p)/G(p)'$. Thus $x^{2p}$ is contained within the Schur multiplier, $C_2^3$, of $G(p)^2$, giving $x^{4p} = 1$.

The third relation of (1) together with $x^{4p} = 1$ gives $x^{4(p-1)}(xy^2)^4 = 1$, but since $[x^2,y^2] = 1$ we have $x^{3(p-1)}xy^2xy^2xy^2x^{p-1}xy^2 = x^{3(p-1)}xy^2xy^2xy^2x^py^2 = 1$. Continuing in this way we finally obtain $(x^py^2)^4 = 1$. This relation gives $yx^py^2x^py = (yx^py^2x^py)^{-1}$. Substituting the last relation into the fifth relation we obtain $y^{p+1} = (yx^py^2x^py)^{-1}x^py^2x^p(yx^py^2x^py)$ and, on raising this to the power $p$ while using the facts that $[x^2,y^2] = 1$ and $y^{2p} = 1$, we get $1 = y^{p(p+1)} = x^{2p^2}$. Since $(2p^2, 4p) = 2p$ we have $x^{2p} = 1$. ■

**Lemma 2.4** *The two relations* $y^{2p} = 1$ *and* $y^{-4}(yx^2)^4 = 1$ *imply* $[x^2,y^2] = 1$.

**Proof.** From $y^4 = (yx^2)^4$ we have $[y^4, yx^2] = 1$, or $[y^4, x^2] = 1$. Since $(2,p) = 1$ there exist integers $\lambda$ and $\mu$ satisfying $2\lambda + p\mu = 1$. Thus $y^2 = y^{4\lambda+2p\mu} = y^{4\lambda}$ since $y^{2p} = 1$. So $x^2$ and $y^2$ commute. ■

Now our presentation for $G(p)^2$ is

$$\langle\ x,y \mid y^{2p}, x^{-4}(xy^2)^4, y^{-4}(yx^2)^4, y^{-p}(x^pyx^py^2)^3, x^{-p}(y^pxy^px^2)^3, [x^p,y^p]\ \rangle. \tag{2}$$

The deficiency of this presentation is four, which is one more than is theoretically possible. Now we examine the following presentation

$$\mathcal{P} = \langle\ x,y \mid x^{-4}(xy^2)^4, y^{-4}(yx^2)^4y^{2p}, y^{-p}(x^pyx^py^2)^3, x^{-p}(y^pxy^px^2)^3, x^{-p}y^{-p}x^py^{3p}\ \rangle.$$

First we need the following result if we are to use a standard multiplier argument, see for example [4].

**Lemma 2.5** *The factor group* $\langle\mathcal{P}\rangle/\langle\mathcal{P}\rangle'$ *is isomorphic to* $C_2^2$.

We now go about retrieving the relations that hold in (2) but are not explicit in $\mathcal{P}$ to show that $y^{2p} = 1$.

**Lemma 2.6** *In* $\langle\mathcal{P}\rangle$ *the element* $x^{2p}$ *is central, and* $x^{4p} = 1$.

**Proof.** From the third relation of $\mathcal{P}$ we have $[y^p, x^p y x^p] = 1$, and conjugating this by $x^p$ we deduce $[x^{-p}y^p x^p, yx^{2p}] = 1$. The fifth relation gives us $[y, x^{-p}y^p x^p] = 1$, so $[x^{-p}y^p x^p, x^{2p}] = 1$ i.e. $[y^p, x^{2p}] = 1$.

Now if we examine the second relation we get $[x^2, y^{2p-4}] = 1$ giving $[x^{2p}, y^{2p-4}] = 1$ so $[x^{2p}, y^4] = 1$. Since $[x^{2p}, y^p] = 1$, $[x^{2p}, y^4] = 1$, and $p = 4k \pm 1$ $(k \in \mathbb{Z})$ we have $[x^{2p}, y] = 1$, i.e. $x^{2p} \in Z(G(p)^2)$. Now applying the standard multiplier argument we have $x^{4p} = 1$. ■

**Lemma 2.7** *In* $\langle\mathcal{P}\rangle$ *the elements* $x^2$ *and* $y^2$ *commute.*

**Proof.** From Lemma 2.6 we have $[x^{2p}, y] = 1$, and on writing $p = 4\lambda \pm 1$, for $\lambda \in \mathbb{Z}$, we obtain $[x^{8\lambda\pm 2}, y] = 1$. Now the first relation of $\mathcal{P}$ tells us that $[y^2, x^4] = 1$, and so we have $[x^2, y^2] = 1$. ■

**Lemma 2.8** *In* $\mathcal{P}$, $[y^{2p}, x^i y^j x^k] = 1$, *where* $i$ *and* $k$ *are any odd integers and* $j \in \mathbb{Z}$.

**Proof.** The relation $x^{-p}y^{-p}x^p y^{3p} = 1$ implies the relations $[y, x^{-p}y^p x^p] = 1$ and $[y, x^{-p}y^{2p}x^p] = 1$. Now using Lemma 2.7 we have $x^{-p}y^{2p}x^p = x^{-p}x^{(p-1)}y^{2p}x = x^{-1}y^{2p}x$ and so $[y, x^{-1}y^{2p}x] = 1$ and hence $[y^{2p}, xyx^{-1}] = 1$. The last relation gives $[y^{2p}, xy^{any}x^{-1}] = 1$, and again using Lemma 2.7 we obtain the desired result. ■

**Lemma 2.9** *In* $\langle\mathcal{P}\rangle$ *the element* $y^{4p}$ *is central, and so* $y^{8p} = 1$.

**Proof.** From the fourth relation in the presentation we have $[y^p x y^p, x^p] = 1$, and on conjugating this by $y^p$ we obtain $[y^{-p}x^p y^p, xy^{2p}] = 1$ or $[y^{-p}x^p y^p, y^{-2p}x^{-1}] = 1$ so $y^{-p}x^p y^{-p}x^{-1} = y^{-2p}x^{-1}y^{-p}x^p y^p$. Postmultiply the last relation by $y^{2p}$ to obtain $y^{-p}x^p y^{-p}x^{-1}y^{2p} = y^{-2p}x^{-1}y^{-p}x^p y^{3p}$. By the fifth relation of $\mathcal{P}$ we may easily deduce that $[x^{-1}, y^{-p}x^p y^{3p}] = 1$, and using this we have $y^{-p}x^p y^{-p}x^{-1}y^{2p} = y^{-3p}x^p y^{3p}x^{-1}$. Using Lemma 2.8 the last relation becomes $y^{4p}x^p y^{-p}x^{-1} = x^p y^{3p}x^{-1}$ i.e. $[y^{4p}, x^p] = 1$. So, using this result together with $[y^2, x^2] = 1$, we see that $y^{4p}$ is central, and using the multiplier argument, we have $y^{8p} = 1$. ■

**Lemma 2.10** *The relation* $xy^{2p} = y^{-2p}x$ *holds in* $\langle\mathcal{P}\rangle$.

**Proof.** Using the fifth relation of $\mathcal{P}$ and the fact from Lemma 2.6 that $x^{4p} = 1$ we have $x^{-p} = x^{3p}$ and $x^{2p} \in Z(G(p)^2)$. Thus $x^p y^{-p} x^{-p} y^{3p} = 1$ and so $x^{-p} y^{3p} x^p y^{-p} = 1$. We now form the product $(y^{3p} x^{-p} y^{-p} x^p)(x^{-p} y^{3p} x^p y^{-p})$ to get $x^{-p} y^{2p} x^p y^{2p} = 1$, and the result follows from this and Lemma 2.7. ■

**Theorem 2.11** *$\mathcal{P}$ is an efficient presentation for $G(p)^2$.*

**Proof.** By Lemmas 2.6 and 2.7 the first relation of $\mathcal{P}$ may be written as $(x^p y^2)^4 = 1$ which in turn can be rewritten as $(yx^p y^2 x^p y)^{-1} = yx^p y^2 x^p y$. Substituting this into the third relation we obtain $y^p = x^p (yx^p y^2 x^p y)^{-1} x^p y^2 x^p y x^p y^2$. Postmultiplying by $x^p y$ and noting that $x^p = x^{-3p}$ with $x^{-2p} \in Z(G(p)^2)$ gives $x^{-p} y^p x^p y = (yx^p y^2 x^p y)^{-1} x^{-p} y^2 x^p (yx^p y^2 x^p y) x^{2p}$. If we now use Lemma 2.10 then, after raising the last relation to the power $p$, we have $(x^{-p} y^p x^p y)^p = y^{-2p} x^{2p^2}$ which, on using the fifth relation of $\mathcal{P}$, becomes

$$y^{3p(p+1)} = x^{2p^2}. \tag{3}$$

We now examine separately the cases $p = 8\lambda + 1$, $p = 8\lambda - 1$, $p = 8\lambda + 3$ and $p = 8\lambda + 5$ ( $\lambda \in \mathbb{Z}$).

**Case I** $p = 8\lambda + 1$
Here (3) becomes $y^{3p(8\lambda+2)} = x^{2p(8\lambda+1)}$ which by Lemmas 2.6 and 2.9 gives $y^{6p} = x^{2p}$. If we now cube this relation and note the orders of the generators we observe that $y^{8p} = 1$ and so $x^{2p} = y^{2p}$. From the last relation and the fact that $x^{2p}$ is central the second relation of $\mathcal{P}$ becomes $(y^p x^2)^4 y^{2p} = 1$ or $xy^p x^2 y^p x = x^{-1} y^{-p} y^{-2p} x^{-2} y^{-p} x^{-1}$. If we now examine the fourth relation of $\mathcal{P}$ and if we use the last relation, the fact that $y^{4p} = 1$, $y^{2p} \in Z(G(p)^2)$ and Lemma 2.10, we get

$$y^{-p} x^p y^{-p} x = (x^{-1} y^p x^{-2} y^{-p} x^{-1} y^p) x^2 (y^{-p} x y^p x^2 y^{-p} x) y^{2p}. \tag{4}$$

Raise this to the power $p$ and use the last relation of $\mathcal{P}$, Lemmas 2.6 and 2.7, $y^{4p} = 1$ and $y^{2p} = x^{2p}$, to give $x^{p(p+1)} = x^{2p} y^{2p^2} = x^{2p^2+2p}$. This last relation reduces to $x^{p(p+1)} = 1$ which, implies $x^{2p} = 1$, and so $y^{2p} = 1$.

**Case II** $p = 8\lambda - 1$
We first examine the relation $y^{-4}(yx^2)^4 y^{2p} = 1$. Using Lemma 2.9 we deduce $y^{6p}(y^p x^2)^4 = 1$, i.e. $xy^p x^2 y^p x = x^{-1} y^p x^{-2} y^{-p} x^{-1}$. As we did in Case I we proceed to examine the last relation $x^p = (y^p x y^p x^2)^3$. Use Lemma 2.10 and multiply by $y^{2p}$ to obtain $y^{-p} x^p y^p x = (x^{-1} y^p x^{-2} y^{-p} x^{-1} y^p) x^2 (y^{-p} x y^p x^2 y^{-p} x)$. Again we raise this to the power $p$ and use Lemma 2.6 to give $(y^{-p} x^p y^p x)^p = x^{2p}$. Using the fifth relation of $\mathcal{P}$ we have $(x^p y^{-2p} x)^p = x^{2p}$. Now using Lemma 2.10 we get $(x^{p+1} y^{2p})^p = x^{2p}$, but since from Lemma 2.7 we have $[x^2, y^2] = 1$ we can write the last relation as $x^{p(p+1)} y^{2p^2} = x^{2p}$. On substituting $p = 8\lambda - 1$ into (3.1), and

using Lemmas 2.6 and 2.9 we get $x^{2p} = 1$. So $x^{p(p+1)}y^{2p^2} = x^{2p}$ is now $y^{2p^2} = 1$, but since $y^{8p} = 1$ we must have $y^{2p} = 1$.

**Case III** $p = 8\lambda + 3$
Now relation (3) together with Lemmas 2.6 and 2.9 gives $y^{4p} = x^{2p}$. The previous argument proved the general relation $x^{p(p+1)}y^{2p^2} = x^{2p}$. Now this is equivalent to $x^{p(p+1)} = x^{2p}y^{-2p} = y^{4p}y^{-2p^2} = y^{2p(2-p)}$. Since $p = 8\lambda + 3$ this last relation becomes $y^{2p} = 1$.

**Case IV** $p = 8\lambda + 5$
An analogous argument to that used in Case I will suffice to prove $y^{2p} = 1$.

Combining Cases I - IV we therefore have that $G(p)^2$ is efficient on a minimal generating set. ■

# 3 The efficiency of $G(p)^3$, $p$ an odd prime

In this section we prove that $G(p)^3$ has an efficient presentation on a minimal generating set. After failing to find a minimal presentation using the methods of Section 2 we tried a different approach. The key idea of this section comes from Lemma 3.1 of [4].

**Lemma 3.1** (Lemma 3.1 of [4]) *Let $G$ be a group and let $a$, $b$, $c \in G$ satisfy the relations*

$$a(ab^{-1})^2 = 1, c^{\gamma} = (c^k ab^{-1})^6$$

*where $\gamma = \pm 1$ and $k$ is an integer. Then $\langle a, b, c\rangle$ is cyclic and the relations $b^2 = a^3 = c^{(6k-\gamma)}$ hold in $G$.*

For technical reasons we found it easier to work with presentations for $G(p) \times G(p) \times G(p)$ rather than $G(p) \times G(p)^2$. We first find a presentation for $G(p)^3$ using methods similar to those of [4].

**Lemma 3.2** *If $I$ is the group given by the presentation*

$$\begin{aligned} \langle\, a,\ b,\ \mid\ & a^2 = b^p = (abab^2)^3 = s, \\ & (ab^2)^4 = t: \text{ where } s \text{ and } t \text{ are central involutions } \rangle \end{aligned}$$

*for $p \geq 5$ then $s = t$.*

**Proof.** The relations $b^p a^2 = 1$ and $(abab^2)^3 b^p = 1$ can easily be seen to hold in $I$. Now $(ab^2)^4 = t$ is equivalent to $bab^2ab = tb^{-1}a^{-1}b^{-2}a^{-1}b^{-1}$ since $t$ is central. Substitute the last relation into $(abab^2)^3 b^p = 1$ to give $(b^{-1}a^{-1}b^{-2}a^{-1}b^{-1})ab^2abab^2a = b^{-p}t$. Now $a^2$ is a central involution and so we have

$$(b^{-1}a^{-1}b^{-2}a^{-1}b^{-1}a^{-1})b^2(abab^2ab) = a^{-2}b^{1-p}t = bt \tag{5}$$

since $b^pa^2 = 1$. Raising (5) to the power $p$ gives $b^{2p} = b^pt$ and so $b^p = t$. Thus $s = t$. ∎

**Lemma 3.3** *Let $J(p)$, $p$ a prime $\geq 5$, be the group defined by the presentation*

$$\begin{array}{rcl}\langle\, a,\ b,\ u,\ v,\ x,\ y & \mid & (xyxy^2)(xyxy^2a^{-1})^2,\ v^{\varepsilon}(v^{(p-\varepsilon)/6}xyxy^2a^{-1})^6,\\ & & (uvuv^2)(uvuv^2x^{-1})^2,\ b^{\varepsilon}(b^{(p-\varepsilon)/6}uvuv^2x^{-1})^6,\\ & & (abab^2)(abab^2u^{-1})^2,\ y^{\varepsilon}(y^{(p-\varepsilon)/6}abab^2u^{-1})^6,\\ & & (xyxy^2)(xyxy^2u^{-1})^2,\ b^{\varepsilon}(b^{(p-\varepsilon)/6}xyxy^2u^{-1})^6,\\ & & (ab^2)^{-4}(xy^2)^4(uv^2)^4,\\ & & [a,y],\ [v,x],\ [a,u],\ [a,x],\\ & & [u,x],\ [b,v],\ [b,y],\ [v,y]\,\rangle,\end{array}$$

*where $p \equiv \varepsilon$ mod(3) and $\varepsilon \in \{-1, 1\}$. Then $J(p)$ is isomorphic to $G(p)^3$.*

**Proof.** The proof is similar to the proof of Theorem 3.2 in [4]. Let $H = \langle a, b\rangle$, $K = \langle x, y\rangle$, and $L = \langle u, v\rangle$. By Lemma 3.1, and the fact that any missing commutators have been added in the final eight relations, we have $[H, K] = [H, L] = [K, L] = 1$ and the following relations holding in $J(p)$:

$$a^2 = b^p = (abab^2)^3 = u^2 = v^p = (uvuv^2)^3 = x^2 = y^p = (xyxy^2)^3.$$

Let $D = \langle a^2, (ab^2)^4, (xy^2)^4\rangle$. Obviously $J(p)/D \cong G(p)^3$ and we also have $D \leqslant Z(J(p))$ and so by Lemma 4.1 of [14], $D$ is a homomorphic image of $M(G(p)^3) \cong C_2^6$. By Lemma 3.2 we have $a^2 = (ab^2)^4$, $u^2 = (uv^2)^4$, $x^2 = (xy^2)^4$ but we also have $a^2 = u^2 = x^2$ and so by the ninth relation of the presentation for $J(p)$ we have $a^2 = 1$ and so $D$ is the trivial group, and hence $J(p) \cong G(p)^3$. ∎

We can now proceed to the main result of this section

**Theorem 3.4** *For $p$ a prime $\geq 5$, $G(p)^3$ has an efficient presentation on a minimal generating set.*

**Proof.** Let $\varepsilon$ be defined as in Lemma 3.3. We use the transformations $\alpha = v^{(p-\varepsilon)/6}xyxy^2a^{-1}$, $\beta = b^{(p-\varepsilon)/6}uvuv^2x^{-1}$, and $\gamma = y^{(p-\varepsilon)/6}abab^2u^{-1}$ to obtain $v = \alpha^{-6\varepsilon}$. Thus $xyxy^2a^{-1} = \alpha^{\varepsilon p}$, and the first relation of the presentation for $J(p)$ gives $xyxy^2 = \alpha^{-2\varepsilon p}$ and so $a = \alpha^{-3\varepsilon p}$. The other generators of $J(p)$ are obtained using a similar argument. We now transform the presentation of $J(p)$ to get

$$\begin{array}{rcl}\mathcal{M}(p) = \langle\, \alpha,\ \beta,\ \gamma & \mid & \alpha^{-2\varepsilon p}(\alpha^{-2\varepsilon p}\gamma^{3\varepsilon p})^2,\ \beta^{-6}(\beta^{1-\varepsilon p}\alpha^{-2\varepsilon p}\gamma^{3\varepsilon p})^6,\\ & & (\alpha^{-3\varepsilon p}\beta^{-12\varepsilon})^{-4}(\beta^{-3\varepsilon p}\gamma^{-12\varepsilon})^4(\gamma^{-3\varepsilon p}\alpha^{-12\varepsilon})^4,\\ & & [\alpha^{-3\varepsilon p}, \gamma^{-6\varepsilon}],\ [\alpha^{-6\varepsilon}, \beta^{-3\varepsilon p}],\ [\alpha^{-3\varepsilon p}, \gamma^{-3\varepsilon p}],\\ & & [\alpha^{-3\varepsilon p}, \beta^{-3\varepsilon p}],\ [\beta^{-3\varepsilon p}, \gamma^{-3\varepsilon p}],\ [\alpha^{-6\varepsilon}, \beta^{-6\varepsilon}],\\ & & [\beta^{-6\varepsilon}, \gamma^{-6\varepsilon}],\ [\alpha^{-6\varepsilon}, \gamma^{-6\varepsilon}]\,\rangle.\end{array}$$

Now the third commutator relation $[\alpha^{-3\varepsilon p}, \gamma^{-3\varepsilon p}] = 1$, on writing $p = 2\lambda + 1$, can be written as $[\alpha^{-3\varepsilon p}, \gamma^{-6\varepsilon\lambda-3\varepsilon}] = 1$. However the first commutator relation implies that $[\alpha^{-3\varepsilon p}, \gamma^{-3\varepsilon}] = 1$. This last relation implies the first and third commutator relations, and in turn is implied by the first and third commutator relations of $\mathcal{M}(p)$ and so we can replace two of the relations in $\mathcal{M}(p)$ with a single relation. Using the last argument we can replace the second and fourth relations of $\mathcal{M}(p)$ with a single relation. Now our presentation is

$$\begin{array}{rl}\langle\, \alpha,\ \beta,\ \gamma \ \mid & \alpha^{-2\varepsilon p}(\alpha^{-2\varepsilon p}\gamma^{3\varepsilon p})^2,\ \beta^{-6}(\beta^{1-\varepsilon p}\alpha^{-2\varepsilon p}\gamma^{3\varepsilon p})^6, \\ & (\alpha^{-3\varepsilon p}\beta^{-12\varepsilon})^{-4}(\beta^{-3\varepsilon p}\gamma^{-12\varepsilon})^4(\gamma^{-3\varepsilon p}\alpha^{-12\varepsilon})^4, \\ & [\alpha^{-3\varepsilon p}, \gamma^{-3\varepsilon}],\ [\alpha^{-3\varepsilon}, \beta^{-3\varepsilon p}], \\ & [\beta^{-3\varepsilon p}, \gamma^{-3\varepsilon p}],\ [\alpha^{-6\varepsilon}, \beta^{-6\varepsilon}], \\ & [\beta^{-6\varepsilon}, \gamma^{-6\varepsilon}],\ [\alpha^{-6\varepsilon}, \gamma^{-6\varepsilon}]\ \rangle. \end{array}$$

This presentation has deficiency six, and so is an efficient presentation for $G(p)^3$, $p$ a prime $\geq 5$, on a minimal generating set. ■

We now give a presentation for $G(3)^3$ to complete the requirement that $G(p)$ be efficient for all odd primes.

**Lemma 3.5** *$G(3)^3$ has the following as an efficient presentation on a minimal generating set:*

$$\begin{array}{rl}\langle\, \alpha,\ \beta,\ \gamma \ \mid & \alpha^{-4}(\alpha\beta^2)^4\beta^6,\ \beta^{-4}(\beta\gamma^2)^4\gamma^6\alpha^6,\ \gamma^{-4}(\gamma\alpha^2)^4, \\ & \beta^{-3}(\alpha^3\beta\alpha^3\beta^2)^3, \gamma^{-3}(\beta^3\gamma\beta^3\gamma^2)^3,\ \alpha^{-3}(\gamma^3\alpha\gamma^3\alpha^2)^3, \\ & [\alpha^3, \gamma]\alpha^6,\ [\beta^3, \alpha],\ [\gamma^3, \beta]\ \rangle. \end{array}$$

**Proof.** This may be verified using a Todd-Coxeter program. We used the ACE share package in GAP [7]. ■

We are now in a position to prove the main result of this paper.

**Theorem 3.6** *Direct powers of the group $G(p)$ ($p$ an odd prime) are efficient. In particular when 2 is primitive in $GF(p)$ then direct powers of $PGL(2, p)$ are efficient.*

**Proof.** This follows from Theorem 2.11, Theorem 3.4 and Lemma 3.5 as discussed in the introduction. ■

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