The Fibonacci lengths of binary polyhedral groups and related groups

C. M. Campbell and P. P. Campbell
Mathematical Institute, University of St Andrews,
North Haugh, St Andrews, KY16 9SS, Scotland.
E-mail: colinc@mcs.st-andrews.ac.uk, peterc@mcs.st-andrews.ac.uk

Abstract. For a finitely generated group $G = \langle A \rangle$ where $A = \{a_1, a_2, \ldots, a_n\}$ the sequence $x_i = a_{i+1}$, $0 \le i \le n-1$, $x_{i+n} = \prod_{j=1}^{n} x_{i+j-1}$, $i \ge 0$, is called the Fibonacci orbit of G with respect to the generating set A, denoted $F_A(G)$. If $F_A(G)$ is periodic we call the length of the period of the sequence the Fibonacci length of G with respect to A, written $LEN_A(G)$. In this report we examine the behaviour of the Fibonacci length of the finite polyhedral, binary polyhedral groups and related groups.

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1 Introduction

In [3] the Fibonacci length of a 2-generator group is defined, thus extending the idea of forming a sequence of group elements based on a Fibonacci-like recurrence relation first introduced by Wall in [21] where he considered the Fibonacci length of the cyclic groups C_n. The concept of Fibonacci length for more than two generators has also been considered, see for example [8] and [2]. Other work on Fibonacci length is discussed in, for example, [1], [9] and [22].

We have the following definitions of Fibonacci orbit and Fibonacci length for a finitely generated group $G = \langle A \rangle$, where $A = \{a_1, \ldots, a_n\}$:

Definition 1.1 The *Fibonacci orbit* of G with respect to the generating set A, written $F_A(G)$, is the sequence $x_1 = a_1, \ldots, x_{n-1} = a_{n-1}, x_{i+n} = \prod_{j=1}^{n} x_{i+j-1},\ i \ge 0$.

Definition 1.2 If $F_A(G)$ is periodic then the length of the period of the sequence is called the *Fibonacci length* of G with respect to the generating set A, written $LEN_A(G)$.

Notes

1. When it is clear which generating set is being investigated we will write $LEN(G)$ for $LEN_A(G)$.

2. If G is an epimorphic image of a Fibonacci group, then $F_A(G)$ is periodic. (A useful article on Fibonacci groups is given by R M Thomas in [20].)

3. From the definition it is clear that the Fibonacci length of a group depends on the chosen generating set and the order in which the assignments of $x_1, x_2, \ldots, x_n$ are made.

In [3] it was shown that, for any generating pair $\{a, b\} \in D_{2n}$, $LEN_{\{a,b\}}(D_{2n}) = 6$, where D_{2n} is the dihedral group of order $2n$. This result was generalised in [2] to powers of dihedral groups and it was shown that for $i > 1$ the Fibonacci length of D_{2n}^i on the natural generating set no longer remains constant but varies with i and n. Now the dihedral group D_{2n} is an example of a polyhedral group, being the group $(2, 2, n)$ as described below. We have the following definitions.

Definition 1.3 The *polyhedral group* (ℓ, m, n), for $\ell, m, n > 1$ is defined by the presentation

$$\langle\, x, y, z \mid x^\ell = y^m = z^n = xyz = 1 \,\rangle.$$

These groups are also called *triangle groups* and are also denoted by $T(\ell, m, n)$, see [12].

Definition 1.4 The *binary polyhedral group* $\langle \ell, m, n \rangle$, for $\ell, m, n > 1$ is defined by the presentation

$$\langle\, x, y, z \mid x^\ell = y^m = z^n = xyz \,\rangle.$$

For more information on these groups see [5].

Note When we remove the restriction that $\ell, m, n > 1$ we have a family of centro-polyhedral groups; see [5] for definitions and results.

The polyhedral groups are finite whenever $\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} > 1$, that is in the cases $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$. Coxeter in [6] investigated the groups $(2, m, n)$ whenever $(m - 2)(n - 2) < 4$. As Coxeter notes, these groups have the 'remarkable property' that the group $\langle 2, m, n \rangle$ is an extension of the cyclic group C_2 by the group $(2, m, n)$, whenever the group $(2, m, n)$ is finite.

In this paper we examine the Fibonacci length of these polyhedral and binary polyhedral groups and show the extent to which the 'remarkable' property of Coxeter is still present. We consider these polyhedral and binary polyhedral groups both as 2-generator and as 3-generator groups. We also show that groups related to the polyhedral groups involve 'tribonacci-like' sequences.

By $F(X)$ we will denote the free group on the set X, and $\overline{R}$ will denote the normal closure of X in $F(X)$. When rewriting words we use the standard convention found in [17], namely the use of underscores to highlight the subwords which are replaced in passing from one word to the next.

Definition 1.5 The *sequence of Fibonacci words* is the infinite sequence generated by the following system

$$(\{x, y\}, \{x \mapsto y, y \mapsto xy\}, \{x\})$$

i.e. the sequence $(x, y, xy, yxy, xy^2xy, \ldots)$. The *sequence of tribonacci words* is the infinite sequence generated by the following system

$$(\{x, y, z\}, \{x \mapsto y, y \mapsto z, z \mapsto xyz\}, \{x\}).$$

All computer calculations were carried out using the GAP computational algebra system, see [10]. Plots were produced using the Maple computer package, see [15].

2 Wall numbers

In the paper [21] the minimal period of the Fibonacci numbers modulo a given integer was discussed. This idea has natural generalizations in the theories of general linear recurrence relations [16], nonlinear recurrence relations [7], integer valued functions [18] and finitely generated groups [2]. Below we present some known results concerning the minimal period of a 3-step/tribonacci recurrence relation.

Definition 2.1 Let $k_{(a,b,c)}(n)$ denote the minimal period of the integer-valued recurrence relation $u_n = u_{n-1} + u_{n-2} + u_{n-3}, u_0 = a, u_1 = b, u_2 = c$ when each entry is reduced modulo n.

When it is clear that we are working with a 3-step tribonacci-like recurrence relation we will write $k(n)$ to denote $k_{(a,b,c)}(n)$.

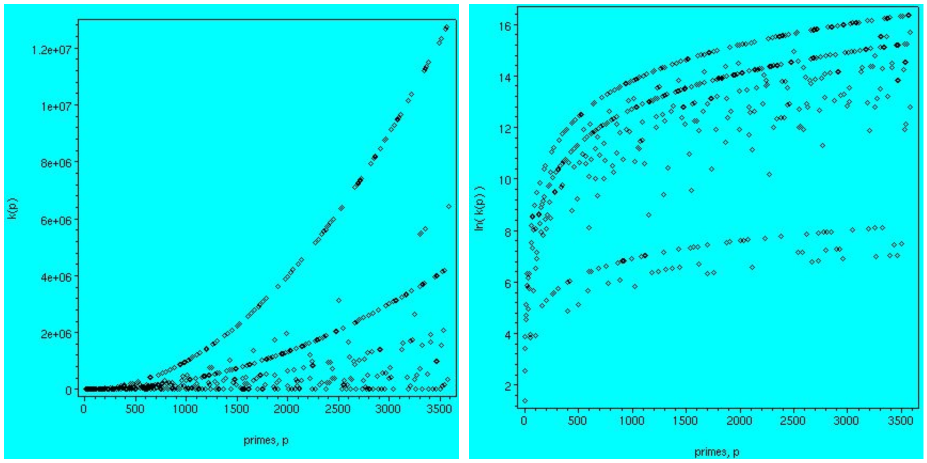
From the definition we may deduce:

Lemma 2.2 *For $a, b, c, x, y, z, n \in \mathbb{Z}$ with $n > 0$, a, b, c not all congruent to zero modulo n and x, y, z not all congruent to zero modulo n,*

$$k_{(a,b,c)}(n) = k_{(x,y,z)}(n).$$

Proof. The following is due to Robinson, see [16]. Let $U_n = [u_n, u_{n+1}, u_{n+2}]$ and

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$



(a) $k(p)$ against p (b) $\ln(k(p))$ against p

Then it follows that $U_n = U_0 A^n$. Since the integers modulo n form a finite set of equivalence classes, there exist integers m and r such that A^{m+r} is congruent, elementwise, to A^r modulo n. Since $\det A = 1$ is a unit modulo n, A^m is the 3×3 identity matrix. So $U_m \equiv U_0 \bmod n$, in the natural way. □

Corollary 2.3 *Let $a, b, c, x, y, z, m, n \in \mathbb{Z}$ with $m, n > 0$, a, b, c not all congruent to zero modulo n and x, y, z not all congruent to zero modulo n. Then we have*

$$k_{(a,b,c)}(n) | k_{(x,y,z)}(mn).$$

Proof. By Lemma 2.2 we have that $k_{(a,b,c)}(n) = k_{(x,y,z)}(n)$ and from [18] we find that $k_{(x,y,z)}(n) | k_{(x,y,z)}(mn)$. □

The Wall numbers, $k_{(a,b,c)}(n)$, have a very interesting structure. In the above figures we have in (a) a plot of $k(p)$ against p and in (b) a plot of $\ln(k(p))$ against p, p a prime, $2 \le p \le 2583$.

3 Epimorphic images

We first mention an important result from the theory of group presentations

Theorem 3.1 *If $G = \langle\, X \mid R \,\rangle$ and $H = \langle\, X \mid T \,\rangle$, where $R \subseteq T \subseteq F(X)$ and $F(X)$ is the free group on the set X, then there is an epimorphism $\phi : G \longrightarrow H$ fixing every $x \in X$ and such that $\ker \phi = \overline{T \setminus R}$. Conversely, every factor group of $G = \langle\, X \mid R \,\rangle$ has a presentation $\langle\, X \mid S \,\rangle$, where $S \subseteq R$.*

Proof. A proof of this can be found in any textbook about finitely presented groups; see for example [17]. □

This leads on to an easy to prove, but important, corollary:

Corollary 3.2 *Let G be a group defined by the presentation $\langle\ X \mid R\ \rangle$. If $LEN_X(G) = n$ and H is a factor group of G on the same set of generating symbols, then $LEN_X(H) \mid LEN_X(G)$.*

We may also deduce:

Corollary 3.3 *Let $G_n = \langle X \rangle$ be a family of groups such that the sequence $(|G_n|)$ is unbounded and monotonically increasing. Also let $LEN_X(G_n) = LEN_X(G_{n+1}) = m$, for $n \geq k$. Then the Fibonacci group $F(|X|, m)$ has infinite order.*

See [13] and [20] for information regarding Fibonacci groups.

4 The groups $\langle n, 2, 2\rangle$, $(n, 2, 2)$, $\langle 2, n, 2\rangle$ and $(2, n, 2)$

Theorem 4.1 *The group defined by the presentation $\langle x, y, z \mid x^n = y^2 = z^2 = xyz\rangle$ has Fibonacci length of 8 for any n.*

Proof. We prove this by direct calculation. We first note that in the group defined by $\langle x, y, z \mid x^n = y^2 = z^2 = xyz\rangle$, $|z| = 4$. We have the sequence

$$x, y, z, \underline{xyz} = z^2, yz\underline{z^2} = z^2yz, z\underline{z^2z^2}yz = zyz,$$
$$\underline{z^2z^2}y\underline{zz}yz = z^2\underline{y^2}z = \underline{z^4}z = z, z^2y\underline{zz}y\underline{zz} = z^2\underline{y^2} = 1,$$
$$zy\underline{zz} = z^2\underline{zy} = x, \underline{zx} = y, \underline{xy} = z, \ldots$$

□

Corollary 4.2 *Let $G = \langle x, y, z \mid x^n = y^2 = z^2 = xyz = 1\rangle$, $n > 2$. Then $LEN_{(x,y,z)}(G) = 8$.*

Proof. By Corollary 3.2 we need only show that $LEN_{(x,y,z)}(G) \neq 4$ or 2.

Trivially $LEN_{(x,y,z)}(G) \neq 2$.

Assume that $LEN_{(x,y,z)}(G) = 4$. Then we would have $yzxyz = x$ but in this case $x = yz\underline{xyz} = yzyzx$ or $1 = (yz)^2$ a contradiction to the order of yz. □

We note in passing that this last proof would also be sufficient to prove Theorem 4.1.

Using analogous arguments we have:

Theorem 4.3 *The group defined by the presentation* $\langle x, y, z \mid x^2 = y^n = z^2 = xyz\rangle$ *has Fibonacci length of* 8 *for any* n.

Corollary 4.4 *Let* $G = \langle x, y, z \mid x^2 = y^n = z^2 = xyz = 1\rangle$, *where* $n > 2$. *Then* $LEN_{(x,y,z)}(G) = 8$.

By direct calculation it is easy to see that in the two generator case, that is for any generating pair, $LEN(\langle n, 2, 2\rangle) = 6, LEN((n, 2, 2)) = 6, LEN(\langle 2, n, 2\rangle) = 6$ and $LEN((2, n, 2)) = 6$. Thus in this case nothing remains of Coxeter's 'remarkable property'.

5 The groups $\langle 2, 2, n\rangle$ and $(2, 2, n)$

Due to a certain lack of symmetry in the sequence of tribonacci words, the Fibonacci length of the groups defined by the presentation $\langle x, y, z \mid x^2 = y^2 = z^n = xyz\rangle$ is a lot different from those already encountered in this paper.

Theorem 5.1 *Let* G_n, $n > 0$, *be the group defined by the presentation* $\langle x, y, z \mid x^2 = y^2 = z^n = xyz\rangle$. *Then*

$$LEN_{(x,y,z)}(G_n) = \begin{cases} 4n, & n \equiv 0 \bmod 4, \\ 8n, & \text{otherwise.} \end{cases}$$

Proof. Let G be the group defined by the presentation

$$\langle x, y, z \mid x^2 = y^2 = z^n = xyz\rangle.$$

Now consider the start of the Fibonacci orbit

$$\mathbf{x}, \mathbf{y}, \mathbf{z}, \underline{xyz} = \mathbf{z^n}, yz\underline{z^n} = \mathbf{z^n yz}, z\underline{z^n z^n}yz = z^{2n}z\underline{yz} = \mathbf{z^{2n}zx},$$
$$\underline{z^n z^n}yz\underline{z^{2n}}zx = z^{4n}\underline{yz}zx = \mathbf{z^{4n}xzx},$$
$$z^n yz\underline{z^{2n}}zx\underline{z^{4n}}xzx = z^{7n}\underline{yz}z\underline{xx}zx = \mathbf{z^{8n}xz^2x},$$
$$z^{2n}zxz^{4n}xzxz^{8n}xz^2x = z^{14n}z\underline{x^2}z\underline{x^2}z^2x = \mathbf{z^{16n}z^4x},$$
$$\underline{z^{4n}}xzx\underline{z^{8n}}xz^2x\underline{z^{16n}}z^4x = z^{28n}xz\underline{x^2}z^2xz^4x = z^{29n}x\underline{z^3xz^4}x = z^{29n}x^2zx = \mathbf{z^{30n}zx},$$
$$\underline{z^{8n}}xz^2x\underline{z^{16n}}z^4x\underline{z^{30n}}zx = z^{54n}x\underline{z^2xz^4}xzx = z^{54n}\underline{x^2}z^2xzx = z^{55n}\underline{z^2xz}x = z^{55n}z\underline{x^2} = \mathbf{z^{56n}z},$$
$$\underline{z^{16n}}z^4x\underline{z^{30n}}zx\underline{z^{56n}}z = z^{102n}\underline{z^4xz}xz = z^{102n}z^3\underline{x^2}z = \mathbf{z^{103n}z^4},$$
$$\underline{z^{30n}}zx\underline{z^{56n}}z\underline{z^{103n}}z^4 = z^{189n}\underline{zxz^5} = \mathbf{z^{189n}xz^4}, \ldots$$

Now we consider what happens to the orbit when we have a section of the form $\ldots, z^a x, zx, z, \ldots$. We also use the fact that $|z| = 2n$.

$$
\begin{aligned}
&\mathbf{z^a x},\\
&\mathbf{zx},\\
&\mathbf{z},\\
&\underline{z^a xz}xz = z^{a-1}\underline{x^2}z = \mathbf{z^n z^a},\\
&zxz\underline{z^n}z^a = z^n \underline{zxz^{a+1}} = \mathbf{z^n x z^a},\\
&z\underline{z^n}z^a\underline{z^n}xz^a = \underline{z^{a+1}xz^a} = \mathbf{zx},\\
&\underline{z^n}z^a\underline{z^n}xz^a zx = \underline{z^a xz^{a+1}}x = \mathbf{xzx},\\
&\underline{z^n}xz^a z\underline{xx}zx = \mathbf{xz^{a+2}x},\\
&z\underline{xx}z\underline{xx}z^{a+2}x = \mathbf{z^{a+4}x},\\
&xz\underline{xx}z^{a+2}xz^{a+4}x = z^n x\underline{z^{a+3}xz^{a+4}x} = \underline{z^n xx}zx = \mathbf{zx},\\
&x\underline{z^{a+2}xz^{a+4}}xzx = \underline{x^2 z^2 xz}x = z^n z\underline{x^2} = \mathbf{z}, \ldots
\end{aligned}
$$

So, as in [2], the fibonacci orbit can be said to form layers of length eight. Using the above, the orbit becomes:

$$
\begin{aligned}
&x_1 = \mathbf{x},\ x_2 = \mathbf{y},\ x_3 = \mathbf{z}, \ldots,\\
&x_9 = \mathbf{z^4 x},\ x_{10} = \underline{zx} = \mathbf{y},\ x_{11} = \mathbf{z}, \ldots,\\
&x_{17} = \mathbf{z^8 x},\ x_{18} = \underline{zx} = \mathbf{y},\ x_{19} = \mathbf{z}, \ldots,\\
&x_{8i+1} = \mathbf{z^{4i} x},\ x_{8i+2} = \underline{zx} = \mathbf{y},\ x_{8i+3} = \mathbf{z}, \ldots.
\end{aligned}
$$

So we need an $i \in \mathbb{N}$ such that $4i = 2kn$ for $k \in \mathbb{N}$. If n is odd or of the form $2m + 2$ then the smallest positive value for i is n, giving a Fibonacci length of $8n$. If $n \equiv 0 \bmod 4$ then $i = n/2$ and hence the Fibonacci length is $4n$. □

Corollary 5.2 *Let G_n, $n > 0$, be the group defined by the presentation $\langle x, y, z \mid x^2 = y^2 = z^n = xyz = 1\rangle$. Then*

$$
LEN_{(x,y,z)}(G_n) = \begin{cases} 2n, & n \equiv 0 \bmod 4,\\ 4n, & n \equiv 2 \bmod 4,\\ 8n, & \text{otherwise.} \end{cases}
$$

Proof. The proof of this result is analogous to the previous result, except this time we have $z^n = 1$. □

Note that when $n \equiv 0 \bmod 4$ and when $n \equiv 2 \bmod 4$, the 'remarkable' property of Coxeter can be seen to be present in the Fibonacci length. In this case, the Fibonacci length of $\langle 2, 2, n\rangle$ is twice that of $(2, 2, n)$.

It is easy to show that for any generating pair $LEN(\langle 2, 2, n\rangle) = 6$ and $LEN((2, 2, n)) = 6$. This is the natural extension of the results for dihedral groups given in [3].

6 The groups $\langle 2, m, n\rangle$ and $(2, m, n)$, where $0 < (n-2)(m-2) < 4$

We now examine the remaining finite polyhedral and binary polyhedral groups. We have examined the Fibonacci length of each of the stated groups on all pairs of generators and on all generating triples. The results are summarized in the tables below (numbers in brackets indicate the total number of distinct generating pairs, or triples, with the Fibonacci length):

All groups isomorphic to	$LEN_{(x,y)}(G)$
$\langle 2,3,3\rangle$	16 (96), 48 (288)
$(2,3,3)$	16 (96)
$\langle 2,3,4\rangle$	18 (864)
$(2,3,4)$	18 (216)
$\langle 2,3,5\rangle$	12 (960), 14 (840), 42 (2520), 50 (1200), 150 (3600)
$(2,3,5)$	12 (240), 14 (840), 50 (1200)

Note that comparing the case $\langle 2,3,3\rangle$ and $(2,3,3)$ and the cases $\langle 2,3,5\rangle$ and $(2,3,5)$ the 'remarkable' property of Coxeter can be seen to be present in the Fibonacci length.

We now examine the Fibonacci length of generating triples.

All groups isomorphic to	$LEN_{(x,y,z)}(G)$
$\langle 2,3,3\rangle$	26 (1248), 52 (11232)
$(2,3,3)$	26 (312), 52 (1248)
$\langle 2,3,4\rangle$	8 (4992), 16 (6144), 24 (16128), 48 (36864) 56 (10752), 60 (5760)
$(2,3,4)$	8 (624), 16 (768), 24 (2592), 48 (4032), 56 (1344), 60 (720)
$\langle 2,3,5\rangle$	8 (36480), 10 (2400), 14 (6720), 20 (2400), 24 (23040), 25 (1200), 28 (6720), 40 (38400), 42 (6720), 45 (3600), 50 (1200), 84 (6720), 90 (3600), 92 (22080), 100 (2400), 154 (36960), 180 (6200), 204 (32640), 268 (257280), 308 (36960), 385 (18480), 716 (171840), 770 (18480), 880 (84480), 1188 (95040), 1540 (36960),1860 (297600), 3580 (343680)
$(2,3,5)$	8 (4560), 10 (600), 14 (1680), 24 (2880), 25 (600) 40 (4800), 42 (1680), 45 (1800), 92 (2760), 102 (4080) 154 (9240), 268 (32160), 385 (9240), 440 (10560) 594 (11880), 716 (21480), 930 (37200), 1790 (42960)

Again the 'remarkable property' is present in the $\langle 2,3,5\rangle$ case.

7 Extended triangle groups

In this section we examine families of groups that are closely associated to the polyhedral groups.

Definition 7.1 The extended triangle group $E(p,q,r)$, for $p,q,r>1$, is defined by the presentation

$$\langle x,y,z|x^2=y^2=z^2=(xy)^p=(yz)^q=(zx)^r=1\rangle.$$

The extended triangle groups are a very important class of groups closely linked to automorphism groups of regular maps, see [4]. The triangle groups (polyhedral groups), (p,q,r), are index two subgroups of extended triangle groups. To see this let $X=xy, Y=yz$ and $Z=zx$ in $E(p,q,r)$ and then use the obvious epimorphism.

In examining this family of groups we have

Lemma 7.2 *The Fibonacci length of* $E(\infty,2,\infty)$ *is* 8.

Proof. Let us calculate the members of the Fibonacci orbit:

$$\begin{aligned}
&x,y,z,xyz,yzxyz,zx\underline{yzyz}xyz=z\underline{xx}yz=zyz,\\
&x\underline{yzyz}xy\underline{zz}yz=\underline{xx}\underline{yy}z=z,\\
&yzxy\underline{zz}y\underline{zz}=yzx\underline{yy}=yzx,\\
&zy\underline{zz}yzx=z\underline{yy}zx=\underline{zz}x=x,\\
&zyz\underline{xx}=z\underline{yz}=\underline{zz}y=y,\\
&yz\underline{xx}y=y\underline{zy}=\underline{yy}z=z,\ldots
\end{aligned}$$

So the orbit has a Fibonacci length of 8. □

Corollary 7.3 *The Fibonacci length of* $E(p,2,r)$ *is* 8.

We also have:

Lemma 7.4 *Let* $q>2$. *Then*

$$LEN_{(x,y,z)}(E(2,q,2))=\begin{cases}2n, & n\equiv 0 \bmod 4,\\ 4n, & n\equiv 2 \bmod 4,\\ 8n, & \text{otherwise.}\end{cases}$$

Proof. To prove this we again find a form for the Fibonacci orbit and then use this to place restrictions on the possible Fibonacci lengths. Firstly we state some consequences of the presentation of $E(2,q,2)$. It can be shown that x is central in $E(2,q,2)$, $(zy)^i y(zy)^i = y$ and $(zy)^i z(zy)^i = z$. We now state the form of the Fibonacci orbit (the proof is via a routine induction).

$$x_m = \begin{cases} (zy)^{(m-1)/2}x, & m \equiv 1 \bmod 8, \\ y(zy)^{(m-2)/2}, & m \equiv 2 \bmod 8, \\ z, & m \equiv 3 \bmod 8, \\ yzx, & m \equiv 4 \bmod 8, \\ y(zy)^{((m-5)/2)+1}zx, & m \equiv 5 \bmod 8, \\ (zy)^{((m-6)/2)+3}z, & m \equiv 6 \bmod 8, \\ z, & m \equiv 7 \bmod 8, \\ zyx, & m \equiv 0 \bmod 8. \end{cases}$$

Now the proof is over once we make the observations that the Fibonacci length, LEN say, must be congruent to 0 modulo 8 and $LEN/2$ must be divisible by q, the order of zy. □

These results highlight the fact that altering the generating set, even by such a simple set of Tietze transformations, can change the Fibonacci length.

8 The groups defined by (A) where $A = \{-2, 2, n\}$ or $A = \{2, 2, -n\}$.

We also examined the groups $\langle a, b, c\rangle$ where $a, b, c \in \mathbb{Z}$ (rather than $a, b, c > 1$). This family also has a 'remarkable property', see [5] and the table at the end of this section.

It is easy to see that $\langle P\rangle$, where P is a permutation of the set $\{2, -2, n\}$ or $\{2, 2, -n\}$ is finite. Our investigations led us to the following results; in all cases we have proved the results for $n \geq 3$.

Theorem 8.1 *Let n be an integer, $n \geq 3$. Denote $LEN(\langle 2, -2, n\rangle)$ by LEN. Then LEN is the smallest non-trivial integer such that*

$$\begin{aligned} 4n &\mid LEN, \\ k_{(0,2,3)}(4(n-1)) &\mid LEN. \end{aligned}$$

Proof. By using the modified Todd-Coxeter procedure, or otherwise, from the group defined by the presentation of $\langle 2, -2, n\rangle$, $|x| = |y| = 4(n-1)$ and $|z| = 2n(n-1)$.

From the presentation for $\langle 2, -2, n\rangle$ we can deduce the following:

$$\begin{aligned} z &= y^{-1}x, \\ \underline{z}xz &= y^{-1}\underline{xx}z = y^{-1}\underline{z}x^2, \\ &= y^{-1}y^{-1}xx^2 = y^{-2}x\underline{x^2} = y^{-4}x, \\ &= (\underline{y^{-2}})^2 x = z^{2n}x. \end{aligned}$$

By an elementary induction argument it can be shown that

$$z^j x z^j \;=\; z^{2jn} x.$$

This last result is useful as the exponent sum of x is constant and the exponent sum of z changes by a factor of n. This will let us produce a simple standard form for the Fibonacci orbit.

The Fibonacci orbit starts:

$$\begin{array}{l} x, y, z, \underline{xyz} = x^2, \underline{yz}x^2 = xx^2, \\ zxx^4, xzxx^8, xz^2xx^{16}, \\ x\underline{z^3xz^3}zxx^{50} = z^{6n}zxx^{52}, \\ z^{6n}x\underline{z^2xz^2}z\underline{zxz}xx^{92} = z^{12n}zx^{96}, \ldots . \end{array}$$

Thus after the first two terms we can write the Fibonacci orbit using only x and z. In fact it is easy to see that the Fibonacci orbit conforms to the following pattern:

$$x_m = \begin{cases} z^{(a_m-(m-1)/2)n} z^{(m-1)/4} x x^{b_m-1}, & m \equiv 1 \bmod 8, \\ z^{(a_m-1)n} z x x^{b_m-1}, & m \equiv 2 \bmod 8, \\ z^{(a_m-1)n} z x^{b_m}, & m \equiv 3 \bmod 8, \\ z^{(a_m-(m-4)/2)n} z^{(m-4)/2} x^{b_m}, & m \equiv 4 \bmod 8, \\ z^{(a_m-(m-5)/2)n} x z^{(m-5)/2} x^{b_m-1}, & m \equiv 5 \bmod 8, \\ z^{(a_m-1)n} z x x^{b_m-1}, & m \equiv 6 \bmod 8, \\ z^{(a_m-1)n} x z x x^{b_m-2}, & m \equiv 7 \bmod 8, \\ z^{(a_m-(m-8)/4+2)n} x z^{(m-8)/4+2} x x^{b_m-2}, & m \equiv 0 \bmod 8. \end{cases}$$

where

$$\begin{array}{rcl} a_m &=& a_{m-3} + a_{m-2} + a_{m-1},\ a_3 = 1, a_4 = 0, a_5 = 0; \\ b_m &=& b_{m-3} + b_{m-2} + b_{m-1},\ b_3 = 0, b_4 = 2, b_5 = 3. \end{array}$$

Letting $LEN = LEN(\langle 2, -2, n \rangle)$ we have:

$$\begin{array}{rcl} x_{LEN+3} &=& z^{(a_{LEN+3}-1)n} z x^{b_{LEN+3}}, \\ x_{LEN+4} &=& z^{(a_{LEN+4}-((LEN+4)-4)/2)n} z^{((LEN+4)-4)/2} x^{b_{LEN+4}}, \\ x_{LEN+5} &=& z^{(a_{LEN+5}-((LEN+5)-5)/2)n} x z^{((LEN+5)-5)/2} x^{b_{LEN+5}-1}. \end{array}$$

So we need $k(|x|)|LEN$ that is $k(4(n-1))|LEN$, where $k(m)$ is the 3-step Wall number of the positive integer m. Using Lemma 2.2 and Corollary 2.3 the first of the above equalities gives

$$x_{LEN+3} \;=\; z^{(1-1)n} z x^0 = z.$$

The second equality gives

$$\begin{aligned} x_{LEN+4} &= z^{(0-(LEN/2))n} z^{(LEN)/2} x^2, \\ &= z^{(LEN/2)(1-n)} x^2. \end{aligned}$$

So we will also need $2n|LEN/2$ if this last line is to equal x^2 (the order of z is $2n(n-1)$). With these results the final equality gives

$$\begin{aligned} x_{LEN+5} &= z^{(0-LEN/2)n} x z^{LEN/2} x^2 \\ &= x z^{(LEN/2)(1-n)} x^2 = x^3. \end{aligned}$$

So all we need is LEN to be the smallest number satisfying

$$\begin{aligned} 4n &\mid LEN, \\ k_{(0,2,3)}(4(n-1)) &\mid LEN. \end{aligned}$$

□

Analogously we have:

Theorem 8.2 *Let n be an integer, $n \geq 3$. Then $LEN(\langle 2, -2, n\rangle) = LEN(\langle -2, 2, n\rangle)$.*

Proof. In $\langle -2, 2, n\rangle$ we have $y^2 = yzx$ or $y = zx$. So

$$\begin{aligned} zy\underline{z} &= z\underline{yy}x^{-1} = \underline{z}x^{-1}y^2 \\ &= yx^{-1}x^{-1}\underline{y^2} = yx^{-4} = yz^{2n} \end{aligned}$$

So by a routine induction

$$z^j y z^j = y z^{2jn}.$$

Using a similar method to that used for the $\langle -2, 2, n\rangle$ case, it is possible to show that the Fibonacci orbit follows the following pattern:

$$x_m = \begin{cases} z^{(a_m-(m-1)/2+1)n} z^{(m-1)/2-1} y y^{b_m-1}, & n \equiv 1 \bmod 8, \\ z^{a_m n} y y^{b_m-1}, & n \equiv 2 \bmod 8, \\ z^{(a_m-1)n} z y^{b_m}, & n \equiv 3 \bmod 8, \\ z^{(a_m-(m-4)/2)n} z^{(m-4)/2} y^{b_m}, & n \equiv 4 \bmod 8, \\ z^{(a_m-(m-5)/2-1)n} y z^{(m-5)/2+1} y^{b_m-1}, & n \equiv 5 \bmod 8, \\ z^{a_m n} y y^{b_m-1}, & n \equiv 6 \bmod 8, \\ z^{(a_m-1)n} y z y y^{b_m-2}, & n \equiv 7 \bmod 8, \\ z^{(a_m-(m-4)/2)n} y z^{(m-4)/2} y y^{b_m-2}, & n \equiv 0 \bmod 8, \end{cases}$$

where

$$\begin{aligned} a_m &= a_{m-3} + a_{m-2} + a_{m-1},\ a_2 = 0, a_3 = 1, a_4 = 0; \\ b_m &= b_{m-3} + b_{m-2} + b_{m-1},\ b_2 = 1, b_3 = 0, b_4 = 2. \end{aligned}$$

From the above we require LEN to be the smallest integer such that

$$\begin{array}{rcl} y &=& z^{a_{LEN+2}n}yy^{b_{LEN+2}-1} = z^{a_{LEN+2}n}y^{b_{LEN+2}}, \\ z &=& z^{(a_{LEN+3}-1)n}zy^{b_{LEN+3}}, \\ y^2 &=& z^{(a_{LEN+4}-(LEN+4-4)/2)n}z^{(LEN+4-4)/2}y^{b_{LEN+4}} = z^{a_{LEN+4}n}z^{(LEN/2)(1-n)}y^{b_{LEN+4}}. \end{array}$$

This means that $k(4(n-1))|LEN$ and $4n|LEN$. □

Theorem 8.3 *Consider the groups $\langle a, b, c\rangle$. Then*

1. *The groups defined by $\langle -n, 2, 2\rangle$ and $\langle 2, -n, 2, \rangle$, have Fibonacci length* 8.
2. *The groups defined by $\langle 2, 2, -n\rangle$, have*

$$LEN_{(x,y,z)}(\langle 2, 2, -n\rangle) \begin{cases} 4n, & n \equiv 0 \bmod 4, \\ 8n, & \text{otherwise.} \end{cases}$$

Proof. We first show that $\langle -n, 2, 2\rangle$ has a Fibonacci length of 8. Recall that z^2 is central, $xy = z$ and $|y| = |z| = 4$.

$$\begin{array}{l} x, y, z, \underline{xyz} = z^2, z^2yz, z^4zyz, z^{12}z, z^{24}, \\ \underline{z^{42}}zy = z^3y = z^{-1}y^2y^{-1} = x, \\ \underline{z^{80}}y = y, \\ \underline{z^{148}}z = z, \ldots \end{array}$$

Now consider $\langle 2, -n, 2, \rangle$. We note that both x^2 and z^2 are central, $yz = x$ and $|x| = |z| = 4$. The Fibonacci orbit becomes:

$$\begin{array}{l} x, y, z, \underline{xyz} = z^2, xz^2, z^4zx, z^8xzx, z^{20}, \\ \underline{z^{36}}x = x, \\ \underline{z^{66}}xz = \underline{z^2}xz = xz^{-1} = y, \\ \underline{z^{124}}z = z, \ldots. \end{array}$$

To prove the second part of the Theorem we need only see that in $\langle 2, 2, -n\rangle$ the order of z is $2n$ and the orders of x and y are 4. Thus $\langle 2, 2, -n\rangle \cong \langle 2, 2, n\rangle$ and the proof now follows from the result for $\langle 2, 2, n\rangle$. □

Theorem 8.4 *The Fibonacci length of $\langle -2, n, 2\rangle$ is $k(4(n-1))$.*

Proof. Consider the groups defined by the presentation $\langle -2, n, 2\rangle$.

In the group defined by this presentation both x^{-2} and z^2 are central, $|x| = |z| = 4(n-1)$ and $x^{-3} = yz$, $n \geq 6$.

Consider the recurrence relations defined by the following:

$$\begin{aligned} c_m &= c_{m-3} + c_{m-2} + c_{m-1},\ c_3 = 0, c_4 = 0, c_5 = 3; \\ d_m &= d_{m-3} + d_{m-2} + d_{m-1},\ d_3 = 1, d_4 = 2, d_5 = 2. \end{aligned}$$

Then a routine induction will suffice to show that the number of x^{-1}'s and z's in the mth entry of the Fibonacci orbit is given by c_m and d_m respectively.

Here the start of the orbit is

$$x, y, z, z^2, x^{-2}x^{-1}z^2, x^{-2}zx^{-1}z^4, x^{-4}zx^{-1}z^8, x^{-12}z^{16}, \ldots .$$

For $m > 6$ we can see that the orbit will separate into some natural layers and each layer will be of the form

$$x_m = \begin{cases} x^{-(c_m-1)}x^{-1}z^{d_m}, & m \equiv 1 \bmod 8, \\ x^{-(c_m-1)}x^{-1}zz^{d_m-1}, & m \equiv 2 \bmod 8, \\ x^{-c_m}zz^{d_m-1}, & m \equiv 3 \bmod 8, \\ x^{-c_m}z^{d_m}, & m \equiv 4 \bmod 8, \\ x^{-(c_m-1)}x^{-1}z^{d_m}, & m \equiv 5 \bmod 8, \\ x^{-(c_m-1)}zx^{-1}z^{d_m-1}, & m \equiv 6 \bmod 8, \\ x^{-(c_m-2)}x^{-1}zx^{-1}z^{d_m}, & m \equiv 7 \bmod 8, \\ x^{-c_m}z^{d_m}, & m \equiv 0 \bmod 8. \end{cases}$$

Now the proof is finished when we note that the orbit will repeat when $x_{k+3} = z, x_{k+4} = z^2$ and $x_{k+5} = x^{-3}z^2$, where k represents the Fibonacci length. Examining this statement in more detail gives

$$\begin{aligned} x^{-c_{LEN+3}}zz^{d_{LEN+3}-1} &= x^{-c_{LEN+3}}z^{d_{LEN+3}} = z, \\ x^{-c_{LEN+4}}z^{d_{LEN+4}} &= z^2, \\ x^{-(c_{LEN+5}-1)}x^{-1}z^{d_{LEN+5}} &= x^{-c_{LEN+5}}z^{d_{LEN+5}} = x^{-3}z^2. \end{aligned}$$

The smallest non-trivial integer satisfying the above conditions occurs when $LEN = k(4(n-1))$. □

Theorem 8.5 *The Fibonacci length of* $\langle 2, n, -2\rangle$ *is* $k(4(n-1))$.

Proof. Let x_i be the ith entry in the Fibonacci orbit and let k be the Fibonacci length of $\langle 2, n, -2\rangle$. Note that since x^2 and z^2 are central and $x = yz$ then, for $n \geq 6$, an elementary induction shows that the Fibonacci orbit is of the form:

$$x_n = \begin{cases} z^{a_n}xx^{b_n-1}, & n \equiv 1 \bmod 8, \\ z^{a_n-1}xzx^{b_n-1}, & n \equiv 2 \bmod 8, \\ z^{a_n-1}zx^{b_n}, & n \equiv 3 \bmod 8, \\ z^{a_n}x^{b_n}, & n \equiv 4 \bmod 8, \\ z^{a_n}xx^{b_n-1}, & n \equiv 5 \bmod 8, \\ z^{a_n-1}zxx^{b_n-1}, & n \equiv 6 \bmod 8, \\ z^{a_n-1}xzxx^{b_n-2}, & n \equiv 7 \bmod 8, \\ z^{a_n}x^{b_n}, & n \equiv 0 \bmod 8. \end{cases}$$

In the above we have written members of the orbit so that all powers of x and z are either one, or an even number.

It is easy to see that in the nth entry of the Fibonacci orbit, the number of z's present is equal to a_n and the number of x's present is given by b_n, where

$$\begin{aligned} a_n &= a_{n-3} + a_{n-2} + a_{n-1}, \ a_3 = 1, a_4 = 0, a_5 = 0; \\ b_n &= b_{n-3} + b_{n-2} + b_{n-1}, \ b_3 = 0, b_4 = 2, b_5 = 3. \end{aligned}$$

Let k represent the Fibonacci length of $\langle 2, n, -2 \rangle$. Then k is the smallest non-trivial integer such that

$$\begin{aligned} z &= z^{a_{k+3}-1} z x^{b_{k+3}} = z^{a_{k+3}} x^{b_{k+3}}, \\ x^2 &= z^{a_{k+4}} x^{b_{k+4}}, \\ x^3 &= z^{a_{k+5}} x x^{b_{k+5}-1} = z^{a_{k+5}} x^{b_{k+5}}. \end{aligned}$$

By definition the smallest positive integer satisfying the above conditions is $k(4(n-1))$. □

Theorem 8.6 *The Fibonacci length of $\langle n, 2, -2 \rangle$ is $k(4(n-1))$.*

Proof. As before we will first collect some important consequences of the presentation, namely z^2 and y^2 are central and $|x| = 2n(n-1)$ and $|y| = |z| = 4(n-1)$.

Now the Fibonacci orbit starts with

$$x, y, z, y^2, yzy^2, zyzy^4, z^2zy^{10}, z^6y^{18}, \ldots.$$

It is easy to see that the number of y's in the ith member of the Fibonacci orbit is given by h_i and the number of z's is given by e_i, where

$$\begin{aligned} e_m &= e_{m-3} + e_{m-2} + e_{m-1}, \ e_2 = 0, e_3 = 1, e_4 = 0; \\ h_m &= h_{m-3} + h_{m-2} + h_{m-1}, \ h_2 = 1, h_3 = 0, h_4 = 2. \end{aligned}$$

A routine induction show that the Fibonacci orbit has the following form, $m > 6$:

$$x_m = \begin{cases} z^{e_m-1} zyy^{h_m-1}, & m \equiv 1 \bmod 8, \\ z^{e_m} yy^{h_m-1}, & m \equiv 2 \bmod 8, \\ z^{e_m-1} zy^{h_m}, & m \equiv 3 \bmod 8, \\ z^{e_m} y^{h_m}, & m \equiv 4 \bmod 8, \\ z^{e_m-1} yzy^{h_m-1}, & m \equiv 5 \bmod 8, \\ z^{e_m-2} zyzy^{h_m-1}, & m \equiv 6 \bmod 8, \\ z^{e_m-1} zy^{h_m}, & m \equiv 7 \bmod 8, \\ z^{e_m} y^{h_m}, & m \equiv 0 \bmod 8. \end{cases}$$

So the Fibonacci length of $\langle n, 2, -2\rangle$ is divisible by 8 and we need $\{e_i \bmod 4(n-1)\}$ and $\{h_i \bmod 4(n-1)\}$ to repeat. Examining the form of the second, third and fourth entry above yields:

$$\begin{aligned} z^{e_{LEN+2}} y y^{h_{LEN+2}-1} &= z^{e_{LEN+2}} y^{h_{LEN+2}}, \\ z^{e_{LEN+3}-1} z y^{h_{LEN+3}} &= z^{e_{LEN+3}} y^{h_{LEN+3}}, \\ & \quad z^{e_{LEN+4}} y^{h_{LEN+4}}. \end{aligned}$$

So we need the smallest nontrivial integer LEN such that

$$\begin{aligned} z^{e_{LEN+2}} y^{h_{LEN+2}} &= y, \\ z^{e_{LEN+3}} y^{h_{LEN+3}} &= z, \\ z^{e_{LEN+4}} y^{h_{LEN+4}} &= y^2. \end{aligned}$$

This is equivalent to the condition stated in the theorem. □

Theorem 8.7 *The Fibonacci length of $\langle n, -2, 2\rangle$ is $k(4(n-1))$.*

Proof. Again we use the facts that y^2 and z^2 are central, $|x| = 2n(n-1)$ and $|y| = |z| = 4(n-1)$. The first few terms of the Fibonacci orbit are

$$x, y, z, z^2, yzz^2, zyzz^4, y^2zz^{10}, y^4z^{20}, y^6zyz^{36}, \ldots .$$

By a routine induction it is easy to see that the Fibonacci orbit follows the following pattern for $m > 6$:

$$x_m = \begin{cases} y^{j_m-1}zyz^{\ell_m-1}, & m \equiv 1 \bmod 8, \\ y^{j_m-1}yz^{\ell_m}, & m \equiv 2 \bmod 8, \\ y^{j_m}zz^{\ell_m-1}, & m \equiv 3 \bmod 8, \\ y^{j_m}z^{\ell_m}, & m \equiv 4 \bmod 8, \\ y^{j_m-1}yzz^{\ell_m-1}, & m \equiv 5 \bmod 8, \\ y^{j_m-1}zyzz^{\ell_m-2}, & m \equiv 6 \bmod 8, \\ y^{j_m}zz^{\ell_m-1}, & m \equiv 7 \bmod 8, \\ y^{j_m}z^{\ell_m}, & m \equiv 0 \bmod 8. \end{cases}$$

where

$$\begin{aligned} j_m &= j_{m-3} + j_{m-2} + j_{m-1},\ j_2 = 1, j_3 = 0, j_4 = 0; \\ \ell_n &= \ell_{m-3} + \ell_{m-2} + \ell_{m-1},\ \ell_2 = 0, \ell_3 = 1, \ell_4 = 2. \end{aligned}$$

Let the Fibonacci orbit be denoted by LEN. Then we require LEN to be the smallest non-trivial integer modulo $4(n-1)$ to satisfy

$$\begin{aligned} y &= y^{j_{LEN+2}} z^{\ell_{LEN+2}}, \\ z &= y^{j_{LEN+3}} z^{\ell_{LEN+3}}, \\ z^2 &= y^{j_{LEN+4}} z^{\ell_{LEN+4}}. \end{aligned}$$

Thus the Fibonacci orbit will be the length of the shortest period of j_m modulo $4(n-1)$. □

Placing all our results from this section into a table together with group order and an associated quantity gives:

Presentation $\mathcal{P}$	Fibonacci length	$\lvert\langle\mathcal{P}\rangle\rvert$	$\lvert\langle\mathcal{P}\rangle\rvert/2n$
$\langle -2, 2, n\rangle$	$\min\{x : x > 1,\ 4n \mid x \text{ and } k(4(n-1)) \mid x\}$	$4n(n-1)$	$2(n-1)$
$\langle 2, -2, n\rangle$	$\min\{x : x > 1,\ 4n \mid x \text{ and } k(4(n-1)) \mid x\}$	$4n(n-1)$	$2(n-1)$
$\langle 2, 2, -n\rangle$	$4n$ if $n \equiv 0 \bmod 4$, $8n$ otherwise	$4n$	2
$\langle -2, n, 2\rangle$	$k(4(n-1))$	$4n(n-1)$	$2(n-1)$
$\langle 2, -n, 2\rangle$	8	$4n$	2
$\langle 2, n, -2\rangle$	$k(4(n-1))$	$4n(n-1)$	$2(n-1)$
$\langle -n, 2, 2\rangle$	8	$4n$	2
$\langle n, -2, 2\rangle$	$k(4(n-1))$	$4n(n-1)$	$2(n-1)$
$\langle n, 2, -2\rangle$	$k(4(n-1))$	$4n(n-1)$	$2(n-1)$

Note that, in the above table, if the group has order $4n(n-1)$ then two of the group generators have order $4(n-1)$ and the third generator $2n(n-1)$ whereas, if the group order is $4n$, then two of the generators have order 4 and the other generator order $2n$. Column 4 mirrors the 'remarkable property' of Coxeter, see [6].

9 Further questions

There are many open questions in this area. Below are a few of them:

1. Resolve Wall's conjecture; see [21] and [19].

2. Does there exist an formula for calculating the nth Wall number? If so then find it.

3. Is having a finite Fibonacci length in general a decidable property?

4. We would like to consider infinite groups which may have finite Fibonacci length. To investigate this it would be necessary to implement an efficient method for calculating the Fibonacci length of an infinite group. This would possibly rely on using the Knuth-Bendix procedure; see [14] and [11].

5. Families of groups have been found whose Fibonacci lengths are constant (D_{2n}), Wall numbers (C_n), linearly increasing (this paper), logarithmically increasing (see [2]). Do other families of groups exist (not defined using Fibonacci words as non-redundant relators) that produce Fibonacci lengths that increase quadratically? exponentially? etc? If so describe them.

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