NICE EFFICIENT PRESENTATIONS FOR ALL SMALL SIMPLE GROUPS AND THEIR COVERS

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Abstract. Prior to this paper all small simple groups were known to be efficient but the status of four of their covering groups was unknown. We produce nice efficient presentations for all of these groups, resolving the previously unknown cases. We give presentations that are better than available before in terms of length and in terms of computational properties. In many cases our presentations have minimal possible length. Our results are based on major amounts of computation. We make substantial use of systems for computational group theory and, in particular, of computer implementations of coset enumeration. To assist in reducing the number of relators we provide theorems which enable the amalgamation of power relations in certain presentations. We conclude with a selection of unsolved problems about efficient presentations for simple groups and their covers.

1. Introduction

For a finite group G the group H is a *stem extension* of G if there is $A \leq Z(H) \cap H'$ with $G \cong H/A$. A stem extension of maximal order is called a *covering group* of G and the maximal A in this case is the *Schur multiplier* of G denoted by $M(G)$. If G is perfect then G has a unique covering group which we denote by $\widehat{G}$. The *deficiency* of a finite presentation $\{X \mid R\}$ of G is $|R| - |X|$. The deficiency of G, $\text{def}(G)$, is the minimum of the deficiencies of all finite presentations of G. The group G is said to be *efficient* if $\text{def}(G) = \text{rank}(M(G))$.

No general methods are known to decide whether a given group is efficient and the problem is unsolvable [1]. Previous work has used a variety of techniques to try to find efficient presentations. In particular, considerable effort has been put into showing that simple groups and their covering groups are efficient. We give the names of simple groups in Atlas format [10]. A survey of results for simple groups of order up to one million is given by Campbell, Robertson and Williams [8]. Subsequent to this $L_3(5)$ has been shown to be efficient by Campbell, Havas, Hulpke and Robertson [3] and $\widehat{M}_{22}$, the covering group of the Mathieu group M_{22}, has been shown to be efficient by Havas and Ramsay [18].

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No comprehensive systematic approach has previously been taken to finding such efficient presentations. One basic method has been to start with a presentation on a minimal generating pair for these groups, see Campbell and Robertson [5] for the definition, and then to try combining relators. Intuition has suggested that presentations on a pair of generators will be more likely to lead to presentations which could be reduced to efficient ones than on larger generating sets. When this approach has failed other ad hoc methods have been used.

We report on systematic attempts to look for efficient presentations of small simple groups and their covering groups. By small we mean that the underlying simple group has order $\leq 10^5$. One systematic approach examines presentations on all distinct generating pairs for the groups in a search for short efficient presentations and for efficient presentations which lead to easy coset enumerations. A second approach involves looking at short presentations of perfect groups. A third approach generates special kinds of relators.

In the process of finding nice presentations we resolve all previously unresolved problems about the efficiency of small simple groups and their covers: we show that $\widehat{L}_3(4)$, $\widehat{A}_8$, $\widehat{S}_4(3)$ and $\widehat{M}_{12}$ are efficient by giving efficient presentations for them. To facilitate the construction of efficient presentations we give theorems which show how to combine certain kinds of power relations in presentations for simple groups and their stem extensions so that we obtain presentations for covering groups.

There is an undefined term in the title: "nice". What makes an efficient presentation a nice one? One view is that short presentations are nicer than longer ones. Also, since presentations may be used as input into group theoretic programs, we want them to behave well in that role. An important computer procedure for finitely presented groups is coset enumeration, and we assess the presentations produced in terms of their behaviour as targets of coset enumeration. Further, in view of our theorems for combining power relations, presentations which include relators showing the order of generators are nice. We list instances of nice efficient presentations found by our methods.

Since the groups $L_2(p)$ for prime $p \geq 5$ are covered by Sunday [24] and $\widehat{L}_2(p)$ by Campbell and Robertson [4], here we consider the other small simple groups. This is not meant to imply that the general presentations given in those papers provide the shortest presentations for the individual linear groups and their covering groups, but merely explains why we omit them here.

2. Methodology

The availability of systems for computational group theory (eg, GAP [14], Magma [2] and Magnus [22]) makes it quite easy to experiment with groups. Havas, Newman and O'Brien [15] have developed a Magma program which enables us to find all distinct generating sets for moderately sized permutation groups. (For the meaning of distinct in this context see [15].) Our first method is to use this program to find such distinct generating pairs for groups under consideration, and then to use the built-in algorithm of MAGMA to find a presentation of the group on each of these generating sets.

Presentations found this way tend to have a reasonably low number of relators, but are rarely efficient even for small groups. However often simply checking all efficient-sized subsets of the relators reveals efficient presentations. These checks are done by first quickly checking that a subset presents a perfect group (otherwise it does not present a group we are seeking). Note that here we might be looking for either the underlying simple group or some stem extension. If this test is passed we attempt to check that the presentation is correct by coset enumeration. We use the ACE enumerator (Havas and Ramsay [17]) either as available in GAP or MAGMA , or as a stand-alone program for some more difficult cases.

Most of our groups succumb to this straightforward search. For those that do not, we attempt to amalgamate relations, again preserving the property that the amalgamated presentation is that of a perfect group. Here we start with small, but not efficient, sets of relators. In particular, we give in Section 4 one theorem which shows how to combine certain kinds of power relations so that we obtain presentations for covering groups.

Our second source of nice presentations for small simple groups and their covers is provided by censuses of short presentations of perfect groups, extending work by Havas and Ramsay [18]. The extension includes 2-generator 2-relator presentations with length up to 24, 2-generator 3-relator presentations with length up to 26, 3-generator 3-relator presentations with length up to 20, and one-relator quotients of $C_l * C_m$ (the free product of an l-cycle and an m-cycle) for coprime l and m. (By a one-relator quotient of a particular group we mean a presentation obtained by adding one extra relator to a presentation for the specified group.)

The third method extends the idea of enumerating one-relator quotients of $C_l * C_m$ to building appropriate one relator quotients which present a simple group, or a stem extension, by computing and testing relators which hold. This is an easy modification of the process described by Campbell, Havas, Hulpke and Robertson [3] using the GAP program `PGRelFind` [13, 19].

When we find a suitable one-relator quotient of $C_l * C_m$, Theorem 4.1 enables us to build efficient presentations for the cover of the underlying simple group and for all quotients of that cover.

One problem we face in producing our table of new presentations is that once we are able to find an efficient presentation for a group then there are arbitrarily many. In Table 2 we give an instance of a presentation with shortest length that we have found. It is chosen such that, within the presentations for the group of that length which we have investigated, it has best coset enumeration performance. We measure coset enumeration performance by the total number of cosets used in a successful enumeration of the trivial subgroup using the `Hard` strategy of ACE, with the group generators given in alphabetical order.

We take the presentation as produced by our process and generally do not make efforts like those described in [16] to improve it. Thus we give presentations produced by MAGMA without modification, including various relators in what may seem less natural forms (in the sense that inversion or cyclic rotation may produce more usual forms). In contrast, presentations from censuses of short presentations arise with relators in canonic form, as described in [18]. Generally speaking, there are often longer presentations which enumerate better. We comment on each of the groups in Section 5.

As far as reliability of results is concerned we claim that all presentations given in this paper correctly define the groups. Each presentation which appears has been verified by both GAP and MAGMA programs to present the specified group.

3. Results

In this section we give nice efficient presentations for simple groups of order less than 10^5 and for their covering groups, excluding $L_2(p)$ for p prime. For convenience we adopt the convention of using upper-case letters to denote inverses in presentations so that, for example, $A = a^{-1}$. We denote the commutator $ABab$ by $[a, b]$. We give presentations which list sequences of relators and/or relations. For coset enumeration purposes the generators are always given in alphabetical order.

We put our results in context by comparing them with the previous shortest efficient presentations explicitly published and we provide citations for those earlier presentations. Sometimes those presentations imply the existence of shorter presentations. For example, where presentations include relators like a^2 and b^3 or like a^2b^3 it is possible to follow the ideas explained in [7] to give shorter presentations on $x = ab$ and $y = aB$ or $y = AB$.

The tables give the name of the simple group, relators and/or relations for the group, the total length of the relators in the corresponding presentation (freely and cyclically reducing relators as done by ACE), and the total number of cosets used in a successful coset enumeration for this presentation over the trivial subgroup using the `Hard` strategy of the ACE enumerator. In Table 1 we provide a citation and in Table 2 we provide a reference to the Subsection in which we give further information on the group.

In earlier work on presentations for simple groups, Đoković [12] pointed out the difficulty of finding shortest presentations in general. His length measure is somewhat more complicated than ours, but his sentiments still apply.

> *We mention that the problem of finding a shortest presentation for a given finite group G is quite open. We do not*

Table 1. Previous shortest efficient presentations

Name	Relators and/or relations	Length	Cosets	Ref.
A_6	$a^2 = (ab)^5, b^4, (ab^2)^5$	29	2609	[5]
$\widehat{A}_6$	$ab^3(ba)^{-4}, (ab^2ab^{-2})^2ab^2$	27	16278	[23]
$L_2(8)$	$xyXyxY, x^4(y^2xy)^2y$	19	855	[7]
A_7	$a^3 = (ab^2)^4, b^5, (b^2Aba)^2$	28	4144	[7]
$\widehat{A}_7$	$a^3 = (ab^2)^4, b^5 = (b^2Aba)^2$	24	111022	[7]
$L_2(16)$	$xyXyxY, xyx^4yxy^3x^3y^3$	23	7852	[7]
$L_3(3)$	$a^2B^3, BA(ba)^5(BA)^7(bA)^3(Ba)^2ba(Ba)^2(ba)^2$	51	148825	[5]
$U_3(3)$	$B^2ABa^3BA, b^2AB^2Ab^2aBa$	20	26722	[15]
$L_2(25)$	$a^2, b^3, (ab)^2(AB)^2(ab)^2(AB)^4(ab)^5(AB)^4$	43	8828	[7]
$\widehat{L}_2(25)$	$xyXyxY, x^2y^2x^2y^4x^5y^4$	25	22397	[7]
M_{11}	$aba^4b^3, babABaBabA$	19	13822	[21]
$L_2(27)$	$a^2, b^3, (ab)(AB)^2(ab)^3(AB)^2(ab)(AB)^8$	39	11269	[7]
$\widehat{L}_2(27)$	$xyXyxY, x(yx^5y)^2xy^7$	29	1759965	[7]
A_8	$a^2B^4, (ab)^{15}(ab^2)^4,$			
	$(ab)^6bab(aB)^2(ab)^2aB(ab)^7aB$	89	636449	[5]
$\widehat{A}_8$	not known to be efficient			
$L_3(4)$	$a^2 = (ab)^7, (ab^2)^5 = b^4, (b(ab)^3)^7,$			
	$b(ab)^3b(ba)^4b^2(ab)^3(ba)^3b^2aB(ab)^3(ba)^3b^2ab$	128	288596	[5]
$\widehat{L}_3(4)$	not known to be efficient			
$S_4(3)$	$a^2 = (ab)^9, [a, b]^4 = b^4,$			
	$(ab)^2baB(ab)^3aB(ab^2ab)^3b(aB)^3$	73	13658840	[5]
$\widehat{S}_4(3)$	not known to be efficient			
$Sz(8)$	$a^2 = (ab)^7, (ab^2)^{13}, [a, b]^{13} = b^4,$			
	$ab(ab^2(aB)^2)^2abaB(ab^2)^6$	145	2927643	[5]
$\widehat{Sz}(8)$	$a^2 = (ab)^5, (ab^2aB)^2(aB^2ab)^2a = (baBa)^5b^2$	53	227854383	[7]
$L_2(32)$	$xyXyxY, (xyx)^2y^3x^5y^3$	23	37177	[7]
$L_2(49)$	$a^2, b^3, (ab)(AB)^2((ab)^2(AB)^3(ab)^2)^2(AB)^2$	45	60069	[7]
$\widehat{L}_2(49)$	$xyXyxY, xy^2(x^2y^3x^2)^2y^2$	25	141922	[7]
$U_3(4)$	$a^2 = b^3, a^3b(aB)^2(ab)^2aBab(aB)^4abaB(BA)^8b$	50	602000	[5]
M_{12}	$a^2 = b^3, (ab)^{10} = [a, b]^6, ((ab)^4aBabaB)^3$	87	243246	[5]
$\widehat{M}_{12}$	not known to be efficient			

know the answer even in the case when G is a cyclic group of prime order.

Our census-based approach enables us to provide shortest presentations on fixed numbers of generators for many of the groups under consideration here. The proof is by adequately identifying all shorter possible presentations, extending the ideas in [18]. Indeed all presentations in Table 2 with length less than 20 are shortest possible on two generators.

Table 2. Our short presentations

Name	Relators	Length	Cosets	Order	Ref.
A_6	$a^4, b^5, abaBabaBAB$	19	1546	360	5.1
$\widehat{A}_6$	a^3bA^2b, ab^2AbAB^3aB	18	2210	2160	5.1
$L_2(8)$	$a^2bABAb, abAbab^2AB^2aB$	19	592	504	5.2
A_7	$a^5, ababaB^3, (a^2bAB)^2$	23	3253	2520	5.3
$\widehat{A}_7$	$a^4bAbAb, a^2bab^2A^2B^2$	19	43805	15120	5.3
$L_2(16)$	$a^3b^2A^2b^2, abAB^2AbaB$	18	21825	4080	5.4
$L_3(3)$	$a^3bAb^2AB, a^2b^2aB^2AbAb^2$	21	158892	5616	5.5
$U_3(3)$	$a^3bAbAb, a^2b^2AB^3Ab^2$	19	198076	6048	5.6
$L_2(25)$	$a^5, a^2b^2a^2b^2, ab^3Ab^3aB$	23	14917	7800	5.7
$\widehat{L}_2(25)$	$a^2bABABAb, a^3Bab^3aB$	19	16779	15600	5.7
M_{11}	$bA^3bAb^3, baBABAbaBa$	19	10428	7920	5.8
$L_2(27)$	$(ab)^2, a^7, a^2bAbaBAb^4$	23	10509	9828	5.9
$\widehat{L}_2(27)$	$a^6bAb, a^3BAb^4aB^2$	21	47253	19656	5.9
A_8	$(a^2b)^2, a^7, abAB^3AbAB^2$	24	427065	20160	5.10
$\widehat{A}_8$	$a^5bA^2b, a^2b^2ABab^2ABaB$	22	1252222	40320	5.10
$L_3(4)$	$a^5, b^5, (ab)^3, a^2B^2ABaB^2abAB$	29	30500	20160	5.11
$\widehat{L}_3(4)$	$a^4bAbAb, a^2bab^2aB^2ABA^2b$	23	30181644	967680	5.11
$S_4(3)$	$a^5, ab^2ab^2, a^2BaBabaB^2$	21	26561	25920	5.12
$\widehat{S}_4(3)$	$a^5b^4, abAbAbAB^2aB^2$	21	166020	51840	5.12
$Sz(8)$	$B^5, A^7, AB^2a^3ba^2B, BaBABA^2b^2a^2ba$	35	29420	29120	5.13
$\widehat{Sz}(8)$	$a^3bA^2b, ab^2ab^2ab^6aB$	22	346104	116480	5.13
$L_2(32)$	$a^2bABAb, abAbAbAbab^5$	21	35631	32736	5.14
$L_2(49)$	$a^4, b^5, a^2Ba^2BabababB$	21	223362	58800	5.15
$\widehat{L}_2(49)$	$BAB^2Ab^2AB, A^2Ba^3BA^2b$	19	214508	117600	5.15
$U_3(4)$	$a^4bAbAb, a^3BAb^2ABA^2B^2$	22	1557671	62400	5.16
M_{12}	$(Ba)^3, a^5b^6, ab^2aBa^2ba^2b^2$	29	119334	95040	5.17
$\widehat{M}_{12}$	$xyxYXY, x^2Y^3x^2YxYx^2Y^2xYxY^5x^5$	33	11717995	190080	5.17

4. An amalgamation theorem

Group presentations frequently include relators which specify the orders of the group generators. When we have suitable relators like this in a presentation for a simple group (or a stem extension) we can effectively amalgamate them and obtain a presentation for a stem extension of the original simple group. We start with an amalgamation theorem which handles the situation where we have three relators, two giving generator orders plus one other relator.

Theorem 4.1. *Let G be a finite simple group. Suppose that G, or some stem extension of G, can be presented as*

$$P = \{a, b \mid a^p = b^q = w(a, b) = 1\}.$$

Then the covering group of G, all stem extensions of G, and G itself, are efficient.

Proof. If we show that the covering group of G is efficient then the efficiency of G, and all stem extensions of G, follows by adding relations to kill factors in $M(G)$.

Let e_a and e_b be the exponent sums of a and b in $w(a,b)$ respectively. Note that $(e_a, p) = (e_b, q) = 1$ since, by assumption, the group presented by P is perfect.

Assume first that $pe_b + qe_a = 1$. Then consider the group H with presentation

$$\{a, b \mid a^p b^{-q} = w(a,b) = 1\}.$$

Since H is perfect, $H/Z(H) \cong G$, and H has trivial multiplier (since it has a balanced presentation), it follows that H is the covering group of G.

Now consider the case where $pe_b + qe_a \neq 1$. Certainly $(p,q) = 1$ since $\langle P \rangle$ is perfect, so there exist m, n with $pm + qn = 1$. Choose s and t so that $se_a \equiv 1 \mod p$ and $te_b \equiv 1 \mod q$ and consider the transformation $a \to a^{sn}, b \to b^{tm}$. Then $w \to \widetilde{w}$ with a-exponent sum congruent to n mod p and b-exponent sum congruent to $m \mod q$ so, inserting powers of a^p and b^q into $\widetilde{w}$ if necessary, we can transform P to $\widetilde{P}$ where

$$\widetilde{P} = \{a, b \mid a^p = b^q = \widetilde{w}(a,b) = 1\}.$$

Now $\langle P \rangle = \langle \widetilde{P} \rangle$ but $\widetilde{w}$ has the property that it satisfies $e_a = n$ and $e_b = m$ and so in $\widetilde{P}$ we have $pe_b + qe_a = 1$ as required. □

Theorem 4.1 has already been applied by Havas, Newman and O'Brien [15] to obtain special kinds of presentations for $U_3(3)$.

Corollary 4.2. *Let G be a finite simple group. Suppose that G, or some stem extension of G, can be presented as*

$$P = \{a, b \mid u(a,b)^p = v(a,b)^q = w(a,b) = 1\}.$$

Suppose also that $u(a,b)$ and $v(a,b)$ generate the free group on a and b. Then the covering group of G, all stem extensions of G, and G itself, are efficient.

Proof. Let $r = u(a,b)$ and $s = v(a,b)$. Then G can be presented as

$$\{a, b, r, s \mid u(a,b)^p = v(a,b)^q = w(a,b) = 1,\ r = u(a,b),\ s = v(a,b)\}.$$

But since $u(a,b)$ and $v(a,b)$ generate the free group on a and b we can write $a = U(r,s)$ and $b = V(r,s)$. Add these relations to the presentation for G and then use them to eliminate a and b. The relations $r = u(a,b)$ and $s = v(a,b)$ vanish when we substitute $a = U(r,s)$ and $b = V(r,s)$ since $u(a,b)$ and $v(a,b)$ are free generators.

We thus have a presentation for G of the form

$$P = \{r, s \mid r^p = s^q = W(r,s) = 1\}$$

and we can apply Theorem 4.1. □

A natural extension of Theorem 4.1 gives methods for amalgamating relations given a presentation for (a stem extension of) a simple group with more relations, such as

$$P = \{a, b \mid a^p = b^q = w_1(a, b) = \ldots = w_n(a, b) = 1\}.$$

We point out that our primary focus is on presentations which are efficient in terms of deficiency. This does not always coincide with best presentations for other purposes. In particular, for deficiency-zero groups, the deficiency-one presentation

$$\{a, b \mid a^p = b^q = w(a, b) = 1\}$$

is likely to be much more useful for practical computation than the efficient presentation produced by Theorem 4.1,

$$\left\{a, b \mid a^p b^{-q} = \widetilde{w}(a, b) = 1\right\}.$$

For example, these deficiency-one presentations are better for coset enumeration than the corresponding efficient presentations. Likewise a presentation explicitly involving an involutory generator (a, say, with the presentation including the relator a^2) is generally better for coset enumeration than presentations without such a relator because coset enumeration programs usually save space and time by utilizing the fact that $a = a^{-1}$. In our commentary on each group in the next Section we provide some information on such presentations.

5. Commentary

Our methodology produces a very large number of efficient presentations for most of the groups under consideration. Then simple modifications to these lead to many more presentations which are efficient. In our tables we have given the shortest efficient presentation which arose as one of our generated presentations and which enumerates with the least total cosets over the trivial subgroup. We emphasise that we use this purely as a measure of coset enumeration performance and do not suggest that enumerations over the trivial subgroup are the best way to compute with the presentation to gain other information about the group. We also give further information for each simple group.

5.1. A_6**.** For A_6 we investigated 33 distinct generating pairs and found 8 efficient presentations amongst presentations on these generating sets. Four of these were one-relator quotients of $C_4 * C_5$ with length 19, like the tabulated presentation which arises in both a census of one-relator quotients of $C_4 * C_5$ and in a census of short 2-generator 3-relator presentations. Two other presentations on the distinct generating pairs were one-relator quotients of $C_2 * C_5$. A shortest instance of one of these,

$$\{a, b \mid (ab)^2, a^5, ab^2AbAb^2aB^2\}$$

(one longer than the tabulated presentation), uses 501 total cosets, which is much better than our tabulated presentation.

We can use Theorem 4.1 to construct an efficient presentation for $\widehat{A}_6$ from any of the one-relator quotients of $C_l * C_m$. We also find 49 efficient presentations from the 978 distinct generating pairs of the group. We tabulate a presentation from a census of short two-relator presentations for perfect groups. It is one of two length-18 canonic presentations for the group and can be obtained by amalgamating the power relations in a presentation for A_6 which is a one-relator quotient of $C_2 * C_5$. It follows from the analysis of shorter presentations in [18] that this is a shortest possible two-relator presentation for $\widehat{A}_6$.

The best efficient canonic presentation for A_6 with respect to total cosets that we have found is

$$\{a, b \mid a^4, ab^2aB^3, abaBABaBAB\},$$

which uses a total of 392 cosets. Note that the deficiency-two presentation

$$\{a, b \mid (ab)^2, a^5, b^5, (aB)^4\}$$

(already investigated as $(5, 5 \mid 2, 4)$ by Coxeter [11]) allows enumeration of the cosets of the trivial subgroup using a total of 360 cosets, that is, without the definition of any redundant cosets.

5.2. $L_2(8)$. For $L_2(8)$ we investigated 85 distinct generating pairs but did not find any efficient presentations directly. However among the three-relator subsets of presentations we found 15 one-relator quotients of $C_2 * C_9$, 13 of $C_2 * C_7$ and one of $C_2 * C_3$ which present the group. Presentations for $L_2(8)$ also arise as one-relator quotients of $C_3 * C_7$. Theorem 4.1 enables us to construct an efficient presentation for $L_2(8)$ from any of these. We tabulate a presentation from a census of short two-relator presentations for perfect groups. The shortest such presentations that we have found have length 19 and we found 16 different canonic ones. The group arises implicitly as a one-relator quotient of $C_2 * C_3$ via one of the length-19 canonic presentations:

$$\{x, y \mid xyxYXY, x^4Y^2xY^3xY^2\}$$

which uses 849 cosets and is the canonic variant of the Table 1 presentation. Setting $x = ab$ and $y = aB$ in the second relator of this presentation to produce a relator w_3 gives

$$\{a, b \mid a^2, b^3, w_3\}$$

as an explicit presentation for $L_2(8)$ as a one-relator quotient of $C_2 * C_3$ where w_3 has length 26, which improves on the shortest extra relator with length 30 found for one specific generating set in [13]. If we count only relator length then shorter efficient presentations exist, for example on three generators and with length 16,

$$\{a, b, c \mid a^2bAc, abcAB, b^2CBCC\}$$

which requires 552 cosets.

Note that various deficiency-one presentations, including

$\{a, b \mid (ab)^2, a^7, a^3bAb^3Ab\}$, $\{a, b \mid a^2, b^7, abaBab^3ab^3aB\}$ and $\{a, b \mid a^2, b^9, abab^4abaBaB\}$

allow enumeration of the cosets of the trivial group using a total of 504 cosets, that is, without the definition of any redundant cosets.

5.3. A_7. For A_7 we investigated 505 distinct generating pairs and found 330 three-relator presentations for preimages of A_7 which defined finite perfect groups. Amongst these were many efficient presentations for A_7 itself and also one-relator quotients of $C_2 * C_7$, $C_3 * C_4$, $C_3 * C_5$ and $C_3 * C_7$ which presented A_7 or a stem extension. We can use Theorem 4.1 to construct an efficient presentation for $\widehat{A}_7$ from any of these one-relator quotients of $C_l * C_m$.

The tabulated presentation for A_7 is from a census of short 2-generator 3-relator presentations. For $\widehat{A}_7$ we tabulate a presentation from a census of short two-relator presentations for perfect groups which is the unique shortest canonic two-relator presentation for this group. A 2-longer canonic presentation enumerates better:

$$\{a, b \mid a^3bAbAb, a^2BabAB^2Ab^2AB\}$$

uses 21125 cosets.

5.4. $L_2(16)$. For $L_2(16)$ we investigated 524 distinct generating pairs and, in contrast to the situation with $L_2(8)$, we found 15 efficient presentations directly. We also found one-relator quotients of $C_2 * C_3$, $C_2 * C_5$ and $C_2 * C_{15}$ which present the group. Theorem 4.1 enables us to construct an efficient presentation for $L_2(16)$ from any of these. We tabulate a length-18 presentation from a census of short two-relator presentations for perfect groups which is the unique shortest canonic two-relator presentation for this group. A canonic length-19 presentation,

$$\{a, b \mid a^2bABAb, ab^2Ab^2aB^5\},$$

enumerates much better, using 4575 cosets. The group arises implicitly as a one-relator quotient of $C_2 * C_3$ via the length-23 presentation:

$$\{x, y \mid xyxYXY, x^3YxY^4xYx^3Y^3\}$$

which uses 7515 cosets and is the canonic variant of the Table 1 presentation.

5.5. $L_3(3)$. For $L_3(3)$ we investigated 1275 distinct generating pairs and found twelve efficient presentations directly. We also found one-relator quotients of $C_2 * C_3$, $C_3 * C_4$ and $C_3 * C_8$ which present the group. Theorem 4.1 enables us to construct an efficient presentation for $L_3(3)$ from any of these. We tabulate a length-21 presentation from a census of short two-relator presentations for perfect groups. Three-generator efficient presentations with length 20 exist;

$$\{a, b, c \mid a^2bAb, abcbCaC, acAcb^3C\}$$

is the best canonic one found for total cosets, 92576. The best canonic 2-generator presentation for coset enumeration that we found has length 24,

$$\{a, b \mid a^4bAb^2AB^2, a^4B^2ab^5\},$$

and uses 27778 cosets.

5.6. $U_3(3)$. This group is considered in detail by Havas, Newman and O'Brien [15] in the context of efficient semigroup presentations. For $U_3(3)$ we investigated 1442 distinct generating pairs and found two efficient presentations amongst presentations on these generating sets. We also found one-relator quotients of $C_3 * C_4$, $C_3 * C_7$ and $C_3 * C_8$ which present the group. Theorem 4.1 enables us to construct an efficient presentation for $U_3(3)$ from any of these. We tabulate a presentation from a census of short two-relator presentations for perfect groups, the unique canonic presentation with length 19. This may be obtained by amalgamating the power relations in a one-relator quotient of $C_3 * C_4$. A canonic presentation with length 20,

$$\{a, b \mid a^2bA^2b^2AB, a^3b^2AB^2A^2B\},$$

enumerates better, using 15583 cosets. A 3-generator, 3-relator presentation with length 19,

$$\{a, b, c \mid a^2bAb, abcBcAC, aB^2CbAC\},$$

uses fewer cosets, 10673. A canonic presentation with length 21,

$$\{a, b \mid a^2ba^2B^2, a^3bABABAba^3B\},$$

is better again, 8878 cosets.

5.7. $L_2(25)$. As the number of generating sets goes up it becomes very time-consuming to look at all three-relator subsets for each generating set. $L_2(25)$ has 1016 distinct generating pairs. We investigated only a sample of three-relator subsets and found many efficient presentations for $L_2(25)$, including one-relator quotients of $C_2 * C_3$, $C_2 * C_5$ and $C_2 * C_{13}$. We can use Theorem 4.1 to construct an efficient presentation for $\widehat{L}_2(25)$ from any of these one-relator quotients of $C_l * C_m$. For $L_2(25)$ we tabulate a presentation from a census of short 2-generator 3-relator presentations. For its cover we tabulate a presentation from a census of short two-relator presentations for perfect groups which is the unique shortest canonic two-relator presentation for this group.

5.8. M_{11}. For M_{11} we investigated 3297 distinct generating pairs and found many efficient presentations directly, including two with what turns out to be minimal length. The group also arises readily as one-relator quotients of $C_2 * C_5$, $C_4 * C_5$, $C_3 * C_8$, $C_5 * C_6$ and $C_5 * C_8$.

The census of short two-relator presentations for perfect groups reveals exactly one canonic two-relator presentation for the group with length 19 and nothing shorter. This presentation must be shortest by an extension of the argument in [18] and Kenne [21] already found a form of it.

We tabulate one which enumerates better. In this case, since there is only one canonic presentation of this length, we find the best coset enumeration for a shortest presentation using the method of [17] applied to all different 2-generator length-19 presentations for the group.

5.9. $L_2(27)$. $L_2(27)$ has 864 distinct generating pairs. We investigated only a sample of three-relator subsets and found many efficient presentations. We also found one-relator quotients of $C_2 * C_3$, $C_2 * C_7$ and $C_3 * C_7$. We can use Theorem 4.1 to construct an efficient presentation for $\widehat{L}_2(27)$ from any of these one-relator quotients of $C_l * C_m$. For $L_2(27)$ we tabulate a presentation from a census of short 2-generator 3-relator presentations. For its cover we tabulate a presentation from a census of short two-relator presentations for perfect groups. Notice the enormous improvement in coset enumeration performance of our presentation for $\widehat{L}_2(27)$ compared with the Table 1 presentation.

5.10. A_8. For A_8 we investigated 3868 distinct generating pairs. We found a number of efficient presentations for A_8, including some, like the one tabulated (from a census of short 2-generator 3-relator presentations), which are one-relator quotients of $C_2 * C_7$. This implicitly solves the first of the previously unsolved problems by providing a base for constructing an efficient presentation for $\widehat{A}_8$ via Theorem 4.1 and Corollary 4.2. A longer presentation for A_8 (from the distinct generating pairs),

$$\{a, b \mid B^7, babA^3Ba^2b^2, B^2AbA^2b^2a^2B\},$$

uses only 22363 cosets. Presentations for A_8 also arise as one-relator quotients of $C_4 * C_7$. For $\widehat{A}_8$ we tabulate a presentation from a census of short two-relator presentations for perfect groups. This presentation can also be obtained by simply amalgamating the power relators in a presentation for A_8 as a one-relator quotient of $C_2 * C_7$, namely

$$\{a, b \mid (A^2b)^2, a^7, a^2b^2ABab^2ABaB\}.$$

The best coset enumeration that we have found for a canonic, 2-generator, efficient presentation for $\widehat{A}_8$ uses 63560 cosets:

$$\{a, b \mid a^2bAbaBA^2B^2, a^2bAbAbabAbab\}.$$

If we count only relator length then shorter efficient presentations exist, for example on three generators and with length 19,

$$\{a, b, c \mid a^2bAb, bcbcBC, abcAbcAc\}$$

which requires 52934 cosets.

5.11. $L_3(4)$. For $L_3(4)$ we investigated 779 distinct generating pairs. This is a moderate enough number. However a problem with this group is that it is the smallest simple group with multiplier of rank 2. This implies that efficient presentations have four relations and there are very many four-relator subsets of the presentations on the distinct generating pairs. We investigated all of them and found many efficient presentations. In the same way that $C_l * C_m$ is a good basis for three-relator presentations, the group $(l,m,n) = \{a,b \mid a^l, b^m, (ab)^n\}$ is a good basis for four-relator presentations. Included in the efficient presentations of $L_3(4)$ that we found were a substantial number of variants of one-relator quotients of $(5,5,3)$, but no one-relator quotients of any other (l,m,n). Having observed this we listed one-relator quotients of $(5,5,3)$ and found four canonic extra relators with length 13 which yield $L_3(4)$ and we list the one with best coset enumeration behaviour.

Unfortunately none of the efficient presentations for $L_3(4)$ that we found this way initially enabled us to construct an efficient presentation for $\widehat{L}_3(4)$ by amalgamating relators. So we looked at three-relator preimages of $L_3(4)$ hoping to find a stem extension which we could use as a starting point. A presentation satisfying Theorem 4.1 would have been ideal. We did not find any such presentation; however we did find eleven variants of

$$\{a,b \mid a^5, (ABAb)^3, w_3\}$$

which present a stem extension, $12.L_3(4)$. When we simply amalgamate the power relators we obtain as a first efficient presentation for $\widehat{L}_3(4)$:

$$\{a,b \mid a^5(ABAb)^3, a^2B^2A^2baBAb\}.$$

This presentation is difficult for coset enumeration, using 145807531 cosets. This practical construction is an instance of the following easy theorem which can in fact be viewed as a precursor to Theorem 4.1 and Corollary 4.2.

Theorem 5.1. *Let G be a finite simple group. Suppose that G, or some stem extension of G, can be presented as*

$$\{a,b \mid u(a,b)^p = v(a,b)^q = w(a,b) = 1\}.$$

In addition suppose $\widetilde{G}$ presented by

$$\left\{a,b \mid u(a,b)^{kp} v(a,b)^{lq} = w(a,b) = 1\right\}$$

is perfect and generated by $u(a,b)$ and $v(a,b)$. Then $\widetilde{G}$ is the covering group of G.

The structure of our first efficient presentation for $\widehat{L}_3(4)$ suggests that it might be profitable to study one-relator quotients of $C_3 * C_5$ more carefully. We did so and uncovered various useful presentations which we had ignored in our censuses because the coset enumerations attempted during the census process failed due to space limitations. These include as presentations for $12.L_3(4)$:

$$P_1 = \{a, b \mid a^3, b^5, ababAbAB^2Ab^2ab^2aB\}$$
$$P_2 = \{a, b \mid a^3, b^5, ab^3aB^2AB^2ABAbabab^2\}.$$

Over the trivial subgroup ACE uses 41128739 and 60689170 cosets, respectively. Application of Theorem 4.1 to P_1 yields a variant of P_2. The power relations of P_2 amalgamate simply to give

$$\{a, b \mid a^3b^5, ab^3aB^2AB^2ABAbabab^2\}$$

as an efficient presentation for $\widehat{L}_3(4)$. This presentation uses a total of 315894198 over the trivial subgroup, worse than our first presentation.

Finally, by looking carefully at one-relator quotients of $\{a, b \mid a^5, (ab)^3\}$ we discover that

$$\{a, b \mid a^5, (ab)^3, a^2bA^2bAb^2AB^2aB\}$$

is a presentation for $12.L_3(4)$ leading to

$$\{a, b \mid A^5(ab)^3, a^2bA^2bAb^2AB^2aB\}$$

as a presentation for $\widehat{L}_3(4)$. This presentation has length 23 and its canonic version is in our tabulation, using a more modest 30181644 cosets over the trivial subgroup. In retrospect we see that the difficulty of the coset enumeration meant that the presentation was initially classified as an overflow in our census process.

5.12. $S_4(3)$. For $S_4(3)$ we investigated 5993 distinct generating pairs. Presentations on these include 10 which are already efficient and 105 with 4 relators. From the 105 deficiency-two presentations we can build another 67 efficient presentations. Seven of these efficient presentations include relators $u(a,b)^4$ and $v(a,b)^5$ where $u(a,b)$ and $v(a,b)$ generate the free group on a and b. Using these and Corollary 4.2 we obtain efficient presentations for $\widehat{S}_4(3)$, for which efficient presentations were not previously known. For $S_4(3)$ we tabulate a presentation from a census of short 2-generator 3-relator presentations. For its cover we tabulate a presentation from a census of short two-relator presentations for perfect groups. Adding the relator a^5 to the presentation for $\widehat{S}_4(3)$ and simplifying gives a presentation for $S_4(3)$ as a one-relator quotient of $C_4 * C_5$ which has the same length as the tabulated presentation but enumerates somewhat worse, 44575 cosets.

5.13. $Sz(8)$. For $Sz(8)$ we investigated 4577 distinct generating pairs. This is the second smallest simple group with multiplier of rank 2. We found very many efficient presentations but, in contrast to the situation with $L_3(4)$, none as variants of one-relator quotients of (l, m, n). However they did arise as two-relator quotients of $C_5 * C_7$ and $C_7 * C_7$. We tabulate a two-relator quotient of $C_5 * C_7$ and mention that a two-relator quotient of $C_7 * C_7$,

$$\{a, b \mid A^7, B^7, BaBaBAb^2AB, a^2bA^3BabAB\},$$

has the same length and enumerates almost as well (29859 cosets). There are very many efficient presentations for this group which enumerate well.

By simply looking at two-relator preimages of the presentations on the distinct generating pairs we find many efficient presentations for $\widehat{Sz}(8)$. We tabulate one found in a census of short two-relator presentations for perfect groups. A 3-generator, 3-relator presentation with length 18,

$$\{a, b, c \mid a^2bcB, abaCAc, a^2BCAbc\},$$

uses fewer cosets, 131283.

5.14. $L_2(32)$. For $L_2(32)$ we investigated 3351 distinct generating pairs and, not unlike the situation with $L_2(16)$, we found many efficient presentations directly. We also found one-relator quotients of $C_2 * C_3$ and $C_2 * C_{11}$ which present the group. We tabulate a length-21 presentation from a census of short two-relator presentations for perfect groups.

5.15. $L_2(49)$. For $L_2(49)$ we investigated 7553 distinct generating pairs and we found many efficient presentations directly. We also found one-relator quotients of $C_2 * C_3$, $C_2 * C_5$, $C_3 * C_4$ and $C_4 * C_5$ which present the group. We can use Theorem 4.1 to construct an efficient presentation for $\widehat{L}_2(49)$ from any of these one-relator quotients of $C_l * C_m$. For $L_2(49)$ we tabulate a presentation from a census of short 2-generator 3-relator presentations. For its cover we tabulate a presentation obtained from looking at presentations on distinct generating pairs. Its canonic version,

$$\{a, b \mid a^2ba^2bA^2b, ab^2Ab^2aB^3\},$$

is the unique shortest canonic two-relator presentation for this group but enumerates a little worse, 265430 cosets.

Other longer presentations enumerate better. For $L_2(49)$

$$P = \{x, y \mid (xY^2)^2, (xY)^3, xy^2(x^2y^3x^2)^2y^2\}$$

uses 59769 cosets. For its cover, the Table 1 presentation (which is based on the same long relator as in P) is better; the presentation

$$\{a, b \mid a^3bA^2BA^2b, a^4BaB^3aB\}$$

is better again, using 126569 cosets. A 3-generator, 3-relator presentation with length 17,

$$\{a, b, c \mid a^2bAb, acACC, b^3cB^2c\},$$

uses 432168 cosets.

5.16. $U_3(4)$. For $U_3(4)$ we investigated 7778 distinct generating pairs. In a partial search of two-relator preimages on these generating sets we detected only proper preimages. However among the three-relator preimages we found a number of one-relator quotients of $C_3 * C_5$ which present the group. We can use Theorem 4.1 to construct an efficient presention from any of them. We tabulate, from a census of short two-relator presentations for perfect groups, the length-22 canonic presentation which enumerates best.

A length-23 canonic presentation enumerates substantially better:

$$\{a, b \mid a^3bA^2BA^2b, a^3b^2a^2B^3AB^2\}$$

uses 416708 cosets. We found a one-relator quotient of $C_2 * C_3$:

$$\{a, b \mid (bA)^3, (bAb)^2, a^7b^2a^4bab^2a^2b^2\},$$

which requires 98318 cosets. Applying Corollary 4.2 and simplifying we obtain

$$\{r, s \mid s^3, r^2, (sr)^7(Sr)^2(sr)^4Srsr(Sr)^2(sr)^2(Sr)^2\}.$$

Then applying Theorem 4.1 we can obtain as an example efficient presentation

$$\{r, s \mid s^3r^2, (sR)^7(Sr)^2(sr)^4SrsR(Sr)^2(sr)^2(Sr)^2\}$$

which uses 477214 cosets.

5.17. M_{12}. M_{12} is the largest simple group in our catalogue. It has 19801 distinct generating pairs and $\widehat{M}_{12}$ has 77979 distinct generating pairs. We investigated all the distinct generating sets for M_{12} and found many efficient presentations for the group. Unfortunately none was of a form suitable to apply Theorem 4.1 or Theorem 5.1. We tabulate the shortest instance found with best total cosets. The length-39 presentation

$$\{a, b \mid bABaBABa^2b^2aB^2ab, ABa^3BA^2, aBaBaBA^2b^2Ab^2ab\}$$

enumerates quite well, using 106282 cosets.

We investigated some of the distinct generating sets for the cover and failed to find any efficient presentations or any presentations to which Theorem 4.1 or Theorem 5.1 could be applied.

The group $\widehat{M}_{12}$ is not only the last group in our catalogue but was the last for which we were able to find an efficient presentation. It did not arise in our censuses. In the end it succumbed to our third method; we studied longer one-relator quotients of $C_2{*}C_3$. On mapping $C_2{*}C_3$ onto a representation of M_{12} satisfying presentation 13.1 of [9], we found various words with length from 54 up which mapped onto the identity. Furthermore, in many cases we could show the words sufficed to produce $\widehat{M}_{12}$ as a one-relator quotient of $C_2 * C_3$.

Starting with $\{a, b \mid a^2, b^3\}$ and defining $x = ab$ and $y = aB$ we found, inter alia, that the word

$$w = x^2y^3x^2yxyx^2y^2xyxy^5x^5$$

maps to the identity in the representation of M_{12}. We investigated the group presented by $\{a, b \mid a^2, b^3, w\}$ and found (by coset enumeration) that it is $\widehat{M}_{12}$. Theorem 4.1 then delivers us many efficient presentations for $\widehat{M}_{12}$ including:

$$\{a, b \mid a^2b^3, (Ab)^2(aB)^3(ab)^2(aB)(ab)(aB)(ab)^2(aB)^2(ab)(aB)(Ab)(AB)^5(Ab)^5\},$$

which was chosen somewhat arbitrarily from among the options. We can enumerate the cosets of the trivial subgroup for this presentation using a total of 27890300 cosets. From this we construct the shorter presentation which we tabulate.

6. Review

In retrospect we observe that many deficiency-zero presentations that we list can be viewed as results of applications of Theorem 4.1, Corollary 4.2 or Theorem 5.1. These include most of the deficiency-zero presentations in both Table 1 and Table 2.

Observe that in our Table 2 presentations all enumerations over the trivial subgroup can be completed using a total of less than 31 million cosets. This is in sharp contrast to many earlier efficient presentations for which coset enumerations are much harder. Perhaps the worst example is the first published presentation for $L_2(49)$, referred to in [8] and appearing in [6], which is theoretically correct but fails to complete enumerations over cyclic subgroups even when allowed to define more than 2×10^9 cosets.

Suffice it to say, our methods enable us to produce efficient presentations which are both short and computationally useful.

7. Open Problems

We list below some standard open problems and some which have arisen during the work described in this paper.

(1) Is every simple group efficient? If not, which is the smallest inefficient simple group? Only one simple group with order less than one million is a candidate, $S_4(4)$. In particular, is $L_2(2^n)$ efficient for all n? Note that this has a positive solution for $n = 2, 3, 4, 5, 6$.

(2) Does the covering group of every finite simple group have a balanced presentation?

(3) Is A_n efficient for all n? This has a positive solution for $n \leq 9$. A much weaker question even appears to be open. Is there a 2-generator presentation for A_n with k relators where k is independent of n?

(4) Is there a group with 2 generators which has an efficient presentation on 3 generators but not on 2 generators?

(5) Do there exist groups with efficient presentations on one generating set but not on another. In particular do the small simple groups have efficient presentations on every generating pair?

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