Presentation for the partial dual inverse symmetric monoid

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Abstract

We give a monoid presentation in terms of generators and defining relations for the partial analogue of the finite dual inverse symmetric monoid.

1 Introduction and Statement of Main Result

sec:intro

It were known for quite a long time some presentations for certain classical semigroups. Say, for the *symmetric group* $\mathcal{S}_n$, H. S. M. Coxeter found a presentation with generators $\sigma_1, \dots, \sigma_{n-1}$, subject to the following relations:

$$\sigma_i^2 = 1 \qquad 1 \le i \le n-1 \qquad (1)$$ s1

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad |i-j| > 1 \qquad (2)$$ s2

$$\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j \qquad |i-j| = 1. \qquad (3)$$ s3

Later on, in 1960's, A. Aĭsenštat [1] found a presentation for the *symmetric inverse semigroup*. Following these results, in 1995, N. Ruškuc [11] finds a presentation for the *special* and *general linear semigroups*. Since then, a number of mathematicians became interested in finding presentations for various transformation semigroups and their generalisations, the so-called *Brauer-type semigroups*. So, in [3], J. East provides a presentation for the *singular part* of the symmetric inverse monoid; and in [10] V. Mazorchuk and the second author find a presentation for the singular part of the *Brauer semigroup*. A presentation for the Brauer semigroup itself is established by V. Mazorchuk and the first author in [8]. For further results and comments about this topic we refer the reader to [4] and [5].

Among the Brauer-type semigroups there exists a distinguished one, the *dual symmetric inverse semigroup* $\mathcal{I}_n^*$ (see Sec. 2 for the definition). It has

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a rich combinatorial nature which could be studied by two methods: purely combinatorial and geometric, see [6] and [9] respectively. Surprisingly enough, despite this, it is a difficult problem to write a "nice" presentation for $\mathcal{I}_n^*$. Some possible approaches towards finding such a presentation and the arising difficulties are discussed in [2].

The aim of the present paper is to find a presentation for the *partial dual inverse symmetric monoid* $\mathcal{PI}_n^*$ (which we define in Sec. 2). This monoid arose in the work of the authors [7] as a generalisation of $\mathcal{I}_n^*$, and it also has a quite rich combinatorial nature. Bearing in mind that $\mathcal{PI}_n^*$ looks as a "more complicated" oversemigroup of $\mathcal{I}_n^*$, we hope that the ideas and technique we involve in this paper could be utilised for finding a presentation for $\mathcal{I}_n^*$.

Now we will in several stages introduce certain generators and relations on them which will constitute our presentation for $\mathcal{PI}_n^*$. Some of them have been already defined in (1)–(3). Now introduce the letters $\lambda_1, \dots, \lambda_{n-1}$ and $\rho_1, \dots, \rho_{n-1}$, and put the following relations on them:

$$\lambda_i\lambda_j = \lambda_j\lambda_i \qquad \rho_i\rho_j = \rho_j\rho_i \qquad \lambda_i\rho_j = \rho_j\lambda_i \qquad |i-j|>1 \tag{4}$$ lr

$$\lambda_i\rho_i\lambda_i = \lambda_i \qquad \rho_i\lambda_i\rho_i = \rho_i. \tag{5}$$ lrl

Our following relations bind λ-s with σ-s, and ρ-s with σ-s, where it is a single occurrence of λ or ρ:

$$\sigma_i\lambda_j = \lambda_j\sigma_i \qquad \sigma_i\rho_j = \rho_j\sigma_i \qquad |i-j|>1 \tag{6}$$ sl

$$\sigma_i\lambda_j\sigma_i = \sigma_j\lambda_i\sigma_j \qquad \sigma_i\rho_j\sigma_i = \sigma_j\rho_i\sigma_j \qquad |i-j|=1 \tag{7}$$ sls

$$\lambda_i\sigma_i = \lambda_i \qquad \sigma_i\rho_i = \rho_i. \tag{8}$$ ls

Now we put some relations which bind λ-s, ρ-s and σ-s with multiple occurrence of λ or ρ:

$$\lambda_i\lambda_{i+1} = \lambda_i\lambda_{i+1}\sigma_i = \sigma_{i+1}\lambda_i\lambda_{i+1} \tag{9}$$ m1

$$\rho_{i+1}\rho_i = \sigma_i\rho_{i+1}\rho_i = \rho_{i+1}\rho_i\sigma_{i+1} \tag{10}$$ m2

$$\lambda_i^2 = \rho_i\sigma_i\lambda_i = \rho_i^2 \quad \text{and} \tag{11}$$ m3

$$\lambda_i\rho_{i+1} = \sigma_i\rho_{i+1}\sigma_i\lambda_i \qquad \lambda_{i+1}\rho_i = \rho_i\sigma_i\lambda_{i+1}\sigma_i \tag{12}$$ m4

$$\lambda_{i+1}\lambda_i = \sigma_{i+1}\lambda_{i+1}\lambda_i \qquad \rho_i\rho_{i+1} = \rho_i\rho_{i+1}\sigma_{i+1}. \tag{13}$$ m5

To simplify the next relations and to make further text more readable we introduce now a new, somewhat redundant, series of letters $e_1, \dots, e_n$, given by $e_i = \lambda_{i-1}\rho_{i-1}$ for $i \geq 2$ and $e_1 = \sigma_1 e_2 \sigma_1$. Put some relations which relate e-s to λ-s, ρ-s and σ-s:

$$e_i\sigma_j = \sigma_j e_i \qquad e_i\lambda_j = \lambda_j e_i \qquad e_i\rho_j = \rho_j e_i \quad j \neq i, i-1 \tag{14}$$ h1

$$e_i\sigma_i = \sigma_i e_{i+1} \qquad \sigma_i e_i = e_{i+1}\sigma_i \tag{15}$$ h2

$$\rho_i\lambda_{i+1} = \rho_i e_{i+2} \qquad \rho_{i+1}\lambda_i = e_{i+2}\lambda_i \tag{16}$$ h3

$$\lambda_{i+1}\lambda_i = \lambda_i e_{i+2} \qquad \rho_i\rho_{i+1} = e_{i+2}\rho_i \tag{17}$$ h4

$$e_{i+1}\lambda_i = \lambda_i \qquad \rho_i e_{i+1} = \rho_i \tag{18}$$ h5

$$e_i^2 = e_i \qquad e_i e_{i+1} = e_{i+1}e_i = \lambda_i^2 \tag{19}$$ h6

$$\lambda_i\rho_{i+1} = \sigma_{i+1}\rho_i\lambda_i e_{i+2} \qquad \lambda_{i+1}\rho_i = e_{i+2}\rho_i\lambda_i\sigma_{i+1}. \tag{20}$$ h7

Our last relations are

$$e_i\lambda_i = \lambda_i e_{i+1} = \lambda_i e_i = e_i e_{i+1} \tag{21}$$ z1

$$\rho_i e_i = e_{i+1}\rho_i = e_i\rho_i = e_i e_{i+1}. \tag{22}$$ z2

We are ready to state our

Main Theorem. *The abstract monoid $\mathfrak{M}$, generated by $(\sigma_i)_{i\leq n-1}$, $(\lambda_i)_{i\leq n-1}$, $(\rho_i)_{i\leq n-1}$ and $(e_i)_{i\leq n}$, subject to the relations* (1)–(22)*, is isomorphic to $\mathcal{PI}_n^*$.*

The remainder of the paper is devoted to the proof of this theorem. Our strategy is as follows. In Sec. 2 we define the monoid $\mathcal{PI}_X^*$. In Sec. 3 we study the monoid $\mathfrak{M}$ and provide some auxiliary relations which are certain consequences of (1)–(22). Using them, in Sec. 4, we develop some rewriting technique for the elements of $\mathfrak{M}$ presented as words over its generators. Relying on this technique, we further construct certain canonical forms for words from $\mathfrak{M}$. Then we define a certain equivalence of these canonical forms and show that the equivalent canonical forms are equal in $\mathfrak{M}$. Finally, in Section 5 we prove that if the images, under the natural surjective homomorphism from $\mathfrak{M}$ onto $\mathcal{PI}_n^*$, of two canonical forms are equal in $\mathcal{PI}_n^*$, then the canonical forms are equivalent. This will complete the proof of the theorem.

2 Definition of the Monoid $\mathcal{PI}_n^*$

sec:def-PI

In this section we recall a definition of $\mathcal{PI}_n^*$, introduced in [7]. For this we will need to define $\mathcal{I}_n^*$ first, which, in turn, requires some auxiliary notation.

For $n \in \mathbb{N}$ put $\mathbf{n} = \{1, \dots, n\}$. Consider a set $\mathbf{n}' = \{i'\}_{i\in\mathbf{n}}$ disjoint with $\mathbf{n}$ and a bijection $' : \mathbf{n} \to \mathbf{n}'$ sending $i \in \mathbf{n}$ to $i' \in \mathbf{n}'$. Denote the inverse bijection by the same symbol, that is $(i')' = i$ for all $i \in \mathbf{n} \cup \mathbf{n}'$. Further, we shall say that a subset A of $\mathbf{n} \cup \mathbf{n}'$ is a *line* provided that $A \cap \mathbf{n} \neq \varnothing$ and $A \cap \mathbf{n}' \neq \varnothing$; we shall also say that A is a *point* provided that $|A| = 1$.

As sets, $\mathcal{I}_n^*$ is a collection all decompositions of $\mathbf{n} \cup \mathbf{n}'$ into lines, and $\mathcal{PI}_n^*$ is a collection of all decompositions of $\mathbf{n} \cup \mathbf{n}'$ into lines and points. Obviously, $\mathcal{I}_n^* \subset \mathcal{PI}_n^*$. Now we shall define the multiplication in $\mathcal{I}_n^*$. For this we require some way how to interpret the elements of $\mathcal{PI}_n^*$:

Let $a \in \mathcal{PI}_n^*$ and $x, y \in \mathbf{n} \cup \mathbf{n}'$. Set $x \equiv_a y$ provided that x and y are of the same block of a. The mapping $a \mapsto \equiv_a$ is a bijection between the elements of $\mathcal{PI}_n^*$ and the equivalence relations on $\mathbf{n} \cup \mathbf{n}'$ whose classes are either points or lines. Under this bijection the set $\mathcal{I}_n^*$ maps onto the set of those equivalence relations on $\mathbf{n} \cup \mathbf{n}'$ whose classes are lines.

Now take two arbitrary elements $a, b \in \mathcal{I}_n^*$ and define a new equivalence relation, $\equiv$, on $\mathbf{n} \cup \mathbf{n}'$ as follows:

- for $x, y \in \mathbf{n}$ we have $x \equiv y$ if and only if $x \equiv_a y$ or there is a sequence, $c_1, \dots, c_{2s}$, $s \geq 1$, of elements of $\mathbf{n}$, such that $x \equiv_a c_1'$, $c_1 \equiv_b c_2$, $c_2' \equiv_a c_3', \dots, c_{2s-1} \equiv_b c_{2s}$, and $c_{2s}' \equiv_a y$;
- for $x, y \in \mathbf{n}$ we have $x' \equiv y'$ if and only if $x' \equiv_b y'$ or there is a sequence, $c_1, \dots, c_{2s}$, $s \geq 1$, of elements of $\mathbf{n}$, such that $x' \equiv_b c_1$, $c_1' \equiv_a c_2'$, $c_2 \equiv_b c_3, \dots, c_{2s-1}' \equiv_a c_{2s}'$, and $c_{2s} \equiv_b y'$;

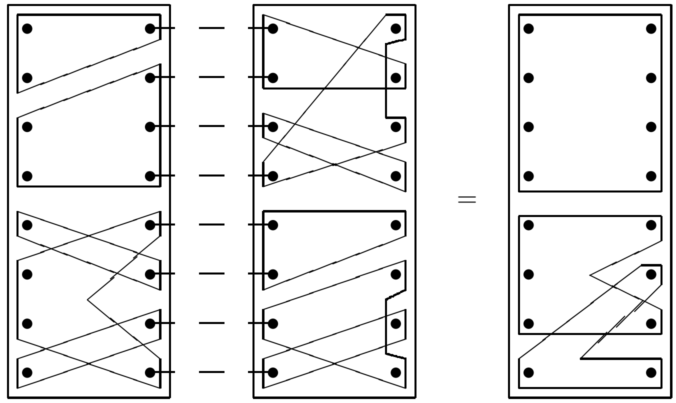

Figure 1: Elements of $\mathcal{I}_8^*$ and their multiplication. fig:ip

- for $x, y \in \mathbf{n}$ we have $x \equiv y'$ if and only if $y' \equiv x$ if and only if there is a sequence, $c_1, \ldots, c_{2s-1}$, $s \geq 1$, of elements of $\mathbf{n}$, such that $x \equiv_a c_1'$, $c_1 \equiv_b c_2$, $c_2' \equiv_a c_3', \ldots, c_{2s-2}' \equiv_a c_{2s-1}'$, and $c_{2s-1} \equiv_b y'$.

See [9] for the proof that $\equiv$ is an equivalence relation on $\mathbf{n} \cup \mathbf{n}'$. We set the decomposition of $\mathbf{n} \cup \mathbf{n}'$ into $\equiv$-classes to be the product $a \cdot b$ of a and b in $\mathcal{I}_n^*$. With respect to this multiplication, $\mathcal{I}_n^*$ is a semigroup.

Finally move to the definition of multiplication in $\mathcal{PI}_n^*$. For this we denote by $\mathcal{I}_{n+1}$ the subset of $\mathcal{I}_{n+1}^*$ consisting of those decompositions of $(\mathbf{n+1}) \cup (\mathbf{n+1})'$ into lines such that both $n+1$ and $(n+1)'$ belong to the same line. The set $\mathcal{I}_{n+1}$ is a subsemigroup of $\mathcal{I}_{n+1}^*$.

Take $a \in \mathcal{PI}_n^*$ and denote by $\varphi(a)$ the element of $\mathcal{I}_{n+1}$, consisting of all lines of a and of one additional block, whose elements are $n+1$, $(n+1)'$ and all the points of a. It was noticed in [7] (and is easy to see) that the mapping φ is a bijection from the set $\mathcal{PI}_n^*$ onto the set $\mathcal{I}_{n+1}$. Now we are prepared to define the (associative) multiplication on $\mathcal{PI}_n^*$:

$$a \cdot b = \varphi^{-1}\big(\varphi(a) \cdot \varphi(b)\big).$$

The above defined multiplication in the monoid $\mathcal{PI}_n^*$ has a natural realisation as a "superposition of diagrams". We interpret the elements of $\mathcal{PI}_n^*$ as diagrams with vertices on the left hand side indexed by $\mathbf{n}$ and vertices on the right hand side indexed by $\mathbf{n}'$. To multiply two such diagrams α and β, one places β to the right of α such that the corresponding right vertices of α and left vertices of β are identified, which uniquely determines the diagram of the product decomposition $\alpha\beta$. This is illustrated on Fig. 1 and 2.

3 Some Auxiliary Results About the Monoid $\mathfrak{M}$

sec:aux

In this section we collect some auxiliary claims about the monoid $\mathfrak{M}$ which we will further use in Sec. 4.

For the first note that, since each of the relations (4)–(22) contains in both its sides some of the letters λ, ρ or e, the submonoid generated by all σ_i-s, is isomorphic to $\mathcal{S}_n$. From now on identify this submonoid with $\mathcal{S}_n$. In what

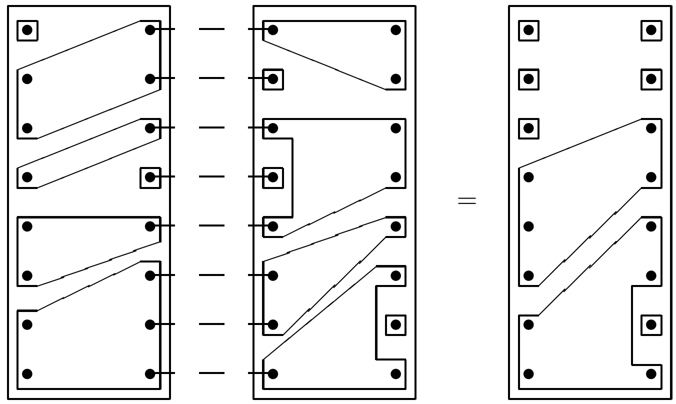

Figure 2: Elements of $\mathcal{PI}_8^*$ and their multiplication. fig:PI

follows, when talking about an action of $\mathcal{S}_n$ on $\mathfrak{M}$, we will mean the action by inner automorphisms: for $\pi \in \mathcal{S}_n$ and $\mu \in \mathfrak{M}$, $\mu^\pi = \pi^{-1}\mu\pi$.

Our next local goal is to understand which elements from $\mathcal{S}_n$ stabilise λ_i-s, ρ_i-s and e_i-s. For this we need some more notation: for $1 \leq i < j \leq n$ define

$$\sigma_{i,j} = \begin{cases} \sigma_i & \text{if } j = i+1 \\ \sigma_i\sigma_{i+1}\cdots\sigma_{j-2}\sigma_{j-1}\sigma_{j-2}\cdots\sigma_i & \text{if } j > i+1. \end{cases}$$

We also extend this definition putting $\sigma_{j,i} = \sigma_{i,j}$, and notice that $\sigma_{i,j}^2 = 1$ for all appropriate i, j.

lem:stabilizer **Lemma 3.1.** *(i) If $2 \leq i \leq n-2$, then the elements $\sigma_1, \dots, \sigma_{i-2}, \sigma_{i+2}, \dots, \sigma_n$ and $\sigma_{i-1,i+2}$ stabilise both λ_i and ρ_i.*

(ii) The elements $\sigma_3, \dots, \sigma_n$ stabilise both λ_1 and ρ_1; the elements $\sigma_1, \dots, \sigma_{n-2}$ stabilise both λ_n and ρ_n.

(iii) If $2 \leq i \leq n-1$ then the elements $\sigma_1, \dots, \sigma_{i-2}, \sigma_{i+1}, \dots, \sigma_n$ and $\sigma_{i-1,i+1}$ stabilise e_i.

(iv) The elements $\sigma_2, \dots, \sigma_{n-1}$ stabilise e_1; the elements $\sigma_1, \dots, \sigma_{n-2}$ stabilise e_n.

Proof. To prove the first claim, in view of (6), it suffices to show that $\sigma_{i-1,i+2}$ stabilises λ_i and ρ_i, $2 \leq i \leq n-2$. We do it only for λ_i:

$$\begin{aligned}
\sigma_{i-1,i+2}\lambda_i\sigma_{i-1,i+2} &= \sigma_{i-1}\sigma_i\sigma_{i+1}\sigma_i\sigma_{i-1}\lambda_i\sigma_{i-1}\sigma_i\sigma_{i+1}\sigma_i\sigma_{i-1} \\
&= \sigma_{i-1}\sigma_i\sigma_{i+1}\sigma_i\sigma_i\lambda_{i-1}\sigma_i\sigma_i\sigma_{i+1}\sigma_i\sigma_{i-1} && \text{(by (7))} \\
&= \sigma_{i-1}\sigma_i\sigma_{i+1}\lambda_{i-1}\sigma_{i+1}\sigma_i\sigma_{i-1} && \text{(by (1))} \\
&= \sigma_{i-1}\sigma_i\lambda_{i-1}\sigma_i\sigma_{i-1} && \text{(by (6) and (1))} \\
&= \sigma_{i-1}\sigma_{i-1}\lambda_i\sigma_{i-1}\sigma_{i-1} && \text{(by (7))} \\
&= \lambda_i. && \text{(by (1))}
\end{aligned}$$

To prove the third claim, due to (14), it suffices to prove that $\sigma_{i-1,i+1}$ stabilises e_i. We compute

$$\begin{aligned}
\sigma_{i-1,i+1}e_i\sigma_{i-1,i+1} &= \sigma_{i-1}\sigma_i\sigma_{i-1}e_i\sigma_{i-1}\sigma_i\sigma_{i-1} \\
&= \sigma_{i-1}\sigma_i e_{i-1}\sigma_i\sigma_{i-1} && \text{(by (15) and (1))} \\
&= \sigma_{i-1}e_{i-1}\sigma_i\sigma_i\sigma_{i-1} && \text{(by (14))} \\
&= \sigma_{i-1}e_{i-1}\sigma_{i-1} && \text{(by (1))} \\
&= e_i. && \text{(by (15) and (1))}
\end{aligned}$$

The remaining claims are proved similarly, we leave the details to the reader. ☐

To proceed, we need to introduce some more notation. Let $1 \le p, q \le n$ and $p \ne q$. For any $\pi \in \mathcal{S}_n$, such that $\pi(1) = p$ and $\pi(2) = q$, set

$$\lambda_{p,q} = \pi^{-1}\lambda_1\pi, \quad \rho_{p,q} = \pi^{-1}\rho_1\pi. \tag{23}$$ def-lambda

In view of Lemma 3.1 these are well-defined elements, i.e. independent of the choice of $\pi \in \mathcal{S}_n$ with $\pi(1) = p$ and $\pi(2) = q$. Moreover, it can be verified that $\lambda_{i,i+1} = \lambda_i$ and $\rho_{i,i+1} = \rho_i$ for all $1 \le i \le n-1$. Indeed, for $i = 1$ this is trivial. Let $i \ge 2$. Then we apply $(i-1)$ times the relations (7) and (1) and obtain

$$(\sigma_{i-1}\sigma_i)\cdots(\sigma_2\sigma_3)(\sigma_1\sigma_2)\lambda_1(\sigma_2\sigma_1)(\sigma_3\sigma_2)\cdots(\sigma_i\sigma_{i-1}) = \lambda_i.$$

In addition, the element $(\sigma_2\sigma_1)(\sigma_3\sigma_2)\cdots(\sigma_i\sigma_{i-1})$ maps 1 to i and 2 to $i+1$ respectively.

lem:lr **Lemma 3.2.** *Let $\pi \in \mathcal{S}_n$ be such that $\pi(p) = s$ and $\pi(q) = t$. Then $\pi^{-1}\lambda_{p,q}\pi = \lambda_{s,t}$ and $\pi^{-1}\rho_{p,q}\pi = \rho_{s,t}$.*

Proof. We prove only the first equality, the second one being proved similarly. Firstly, we note that every element $\alpha \in \mathcal{S}_n$ such that $\alpha(s) = s$ and $\alpha(t) = t$ stabilises $\lambda_{s,t}$. This follows from the definition of $\lambda_{s,t}$ and Lemma 3.1. Hence, to prove the lemma, it suffices to provide at least one $\pi \in \mathcal{S}_n$, such that $\pi(p) = s$ and $\pi(q) = t$, with $\pi^{-1}\lambda_{p,q}\pi = \lambda_{s,t}$. Now consider γ and δ from $\mathcal{S}_n$ such that $\gamma(1) = p$, $\gamma(2) = q$, $\delta(1) = s$ and $\delta(2) = t$. Then $\delta^{-1}\gamma\lambda_{p,q}\gamma^{-1}\delta = \delta^{-1}\lambda_1\delta = \lambda_{s,t}$, which yields the required statement. ☐

lem:e **Lemma 3.3.** *Let $\pi \in \mathcal{S}_n$ be such that $\pi(p) = s$. Then $\pi^{-1}e_p\pi = e_s$.*

Proof. The proof follows the same pattern as in Lemma 3.2. ☐

lm:prime **Lemma 3.4.** *There is a unique extension of the mapping $'$: $\mathfrak{M} \to \mathfrak{M}$ defined by $\sigma_i' = \sigma_i$, $\lambda_i' = \rho_i$, $\rho_i' = \lambda_i$ and $e_i' = e_i$, to an anti-isomorphism of $\mathfrak{M}$.*

Proof. Follows from the defining relations for $\mathfrak{M}$. ☐

In the following propositions we prove some relations satisfied by the products of the elements $\lambda_{p,q}$, $\rho_{p,q}$, $\sigma_{p,q}$ and e_i, which we will use in the following section.

prop:rel_lamb **Proposition 3.5.** *The following relations hold for all admissible and pairwise distinct p, q, k:*

$$e_p^2 = e_p \qquad e_pe_q = e_qe_p \tag{24}$$ aux1

$$e_k\sigma_{p,q} = \sigma_{p,q}e_k \qquad e_p\sigma_{p,q} = \sigma_{p,q}e_q \tag{25}$$ aux2

$$e_k\lambda_{p,q} = \lambda_{p,q}e_k \qquad e_q\lambda_{p,q} = \lambda_{p,q} \tag{26}$$ aux3

$$e_k\rho_{p,q} = \rho_{p,q}e_k \qquad \rho_{p,q}e_q = \rho_{p,q} \tag{27}$$ aux4

$$e_p\lambda_{p,q} = \lambda_{p,q}e_q = \lambda_{p,q}e_p = e_pe_q \quad (28) \quad \boxed{\texttt{aux5}}$$

$$\rho_{p,q}e_p = e_q\rho_{p,q} = e_p\rho_{p,q} = e_pe_q. \quad (29) \quad \boxed{\texttt{aux6}}$$

Proof. Involving Lemmas 3.2 and 3.3, it suffices to prove the statement only for the case $p = 1$, $q = 2$ and $k = 3$. It remains then to use the appropriate relations from (14)–(22). □

pr:crucial **Proposition 3.6.** *For all pairwise distinct p, q, k, l,*

$$\lambda_{k,l}\lambda_{p,q} = \lambda_{p,q}\lambda_{k,l} \qquad \rho_{k,l}\rho_{p,q} = \rho_{p,q}\rho_{k,l} \qquad \lambda_{k,l}\rho_{p,q} = \rho_{p,q}\lambda_{k,l} \quad (30) \quad \boxed{\texttt{u1}}$$

$$\lambda_{k,l}\rho_{l,k} = e_l\sigma_{k,l} \qquad \lambda_{k,l}\rho_{k,l} = e_l \qquad \rho_{k,l}\lambda_{l,k} = e_ke_l \quad (31) \quad \boxed{\texttt{u2}}$$

$$\lambda_{k,l}\lambda_{p,k} = \lambda_{p,k}e_l \qquad \rho_{p,k}\rho_{k,l} = e_l\rho_{p,k} \quad (32) \quad \boxed{\texttt{u3}}$$

$$\lambda_{k,l}\lambda_{p,l} = e_k\lambda_{p,l} \qquad \rho_{p,l}\rho_{k,l} = \rho_{p,l}e_k \quad (33) \quad \boxed{\texttt{u4}}$$

$$\rho_{k,l}\lambda_{p,k} = \lambda_{p,k}e_l \qquad \rho_{p,k}\lambda_{k,l} = e_l\rho_{p,k} \quad (34) \quad \boxed{\texttt{u5}}$$

$$\lambda_{k,l}\rho_{p,k} = \rho_{p,k}\lambda_{p,k}e_l\sigma_{k,l} \qquad \lambda_{p,k}\rho_{k,l} = \sigma_{k,l}e_l\rho_{p,k}\lambda_{p,k} \quad (35) \quad \boxed{\texttt{u6}}$$

$$\lambda_{k,l}\rho_{p,l} = \sigma_{p,l}e_p\rho_{k,l}\lambda_{k,l}\sigma_{p,l} \qquad \rho_{k,l}\lambda_{k,l} = \rho_{l,k}\lambda_{l,k} \quad (36) \quad \boxed{\texttt{u7}}$$

$$\lambda_{p,q}^2 = \rho_{p,q}^2 = \lambda_{p,q}\lambda_{q,p} = \rho_{p,q}\rho_{q,p} = e_pe_q \quad (37) \quad \boxed{\texttt{u8}}$$

$$\lambda_{k,q}\lambda_{k,l} = \lambda_{k,l}\lambda_{k,q} = \lambda_{k,q}\lambda_{q,l} = \lambda_{k,l}\lambda_{l,q} \quad (38) \quad \boxed{\texttt{u9}}$$

$$\rho_{k,l}\rho_{k,q} = \rho_{k,q}\rho_{k,l} = \rho_{q,l}\rho_{k,q} = \rho_{l,q}\rho_{k,l} \quad (39) \quad \boxed{\texttt{u10}}$$

$$\lambda_{k,l}\rho_{k,q} = \rho_{k,q}\lambda_{k,l} = \rho_{k,q}\lambda_{k,q}e_l\sigma_{q,l}. \quad (40) \quad \boxed{\texttt{u11}}$$

Proof. For the first we note that in view of Lemmas 3.2 and 3.3, it suffices to prove these relations only for some particularly chosen pairwise distinct p, q, k and l. We will use this fact without any reference.

The relations (30) follow from (4) and Lemma 3.2.

The first equality of (31) follows from $\lambda_2\rho_{3,2} = \lambda_2\rho_2\sigma_2 = e_3\sigma_2$. The second equality follows from the definition of e_i and the third one from (11) and (19).

To prove the first equality of (32) it is enough to show that $\lambda_2\lambda_1 = \lambda_1e_3$, which holds by (17). The second equality follows from the first one applying $'$.

To prove (33) it is again enough to check that $\lambda_2\lambda_{1,3} = e_2\lambda_{1,3}$. Conjugating both sides by σ_2 we obtain the equivalent equality $\sigma_2\lambda_2\lambda_1 = e_3\lambda_1$, which holds by (13) and (17).

Again it suffices to prove only the first equality from (34) which is equivalent to $\rho_2\lambda_1 = \lambda_1e_3$, which is a direct consequence of (16).

The relation (35), up to applying $'$, is equivalent to $\lambda_2\rho_1 = \rho_1\lambda_1e_3\sigma_2$, which follows from (20).

The first equality of (36) is equivalent to $\lambda_1\rho_{3,2} = \sigma_2e_3\rho_1\lambda_1\sigma_2$ which is the same as $\lambda_1\rho_2 = \sigma_2e_3\rho_1\lambda_1$, which, in turn, follows from (20). The second equality of (36) follows from $\sigma_1\rho_1\sigma_1\sigma_1\lambda_1\sigma_1 = \rho_1\lambda_1$.

To prove (37), in view of (19) and applying $'$, it is enough to check that $\lambda_1\lambda_{2,1} = e_1e_2$: using (8) and (19), we have $\lambda_1\lambda_{2,1} = \lambda_1\sigma_1\lambda_1\sigma_1 = \lambda_1^2 = e_1e_2$.

To establish (38), it suffices to prove $\lambda_1\lambda_{1,3} = \lambda_{1,3}\lambda_1 = \lambda_1\lambda_2 = \lambda_{1,3}\lambda_{3,2}$. Firstly, using (8) and then (9), we obtain

$$\lambda_1\lambda_{1,3} = \lambda_1\sigma_1\lambda_2\sigma_1 = \lambda_1\lambda_2\sigma_1 = \lambda_1\lambda_2.$$

Then we use successively (9), (8), (9), (8), (7) and (1) to compute

$$\lambda_1\lambda_2 = \sigma_2\lambda_1\lambda_2 = \sigma_2\lambda_1\lambda_2\sigma_2 = \sigma_2\lambda_1\lambda_2\sigma_1\sigma_2 \\ = \sigma_2\lambda_1\sigma_1\lambda_2\sigma_1\sigma_2 = \sigma_2\lambda_1\sigma_2\lambda_1\sigma_2\sigma_2 = \lambda_{1,3}\lambda_1;$$

and finally applying (1), (9) and (8) we have

$$\lambda_{1,3}\lambda_{3,2} = \sigma_2\lambda_1\sigma_2\sigma_2\lambda_2\sigma_2 = \sigma_2\lambda_1\lambda_2\sigma_2 = \lambda_1\lambda_2\sigma_2 = \lambda_1\lambda_2.$$

The relation (39) follows from (38) using $'$.

Finally let us prove (40). It is again enough to verify that $\rho_{1,3}\lambda_1 = \lambda_1\rho_{1,3} = \rho_{1,3}\lambda_{1,3}e_2\sigma_2$. We compute

$$\begin{aligned} \lambda_1\rho_{1,3} = \lambda_1\sigma_1\rho_2\sigma_1 &= \lambda_1\rho_2\sigma_1 && \text{(by (8))} \\ &= \sigma_2\rho_1\lambda_1 e_3\sigma_1 && \text{(by (20))} \\ &= \sigma_2\rho_1\lambda_1 e_3 && \text{(by (8))} \\ &= \sigma_2\rho_1\lambda_1\sigma_2 e_2\sigma_2 && \text{(by Lemma 3.3)} \\ &= \sigma_2\rho_1\sigma_2\sigma_2\lambda_1\sigma_2 e_2\sigma_2 && \text{(by (1))} \\ &= \sigma_1\rho_2\sigma_1\sigma_1\lambda_2\sigma_1 e_2\sigma_2 && \text{(by (7))} \\ &= \rho_{1,3}\lambda_{1,3}e_2\sigma_2. \end{aligned}$$

and

$$\begin{aligned} \lambda_1\rho_{1,3} &= \lambda_1\rho_2\sigma_1 && \text{(by (8))} \\ &= \sigma_1\rho_2\sigma_1\lambda_1\sigma_1 && \text{(by (12))} \\ &= \rho_{1,3}\lambda_1\sigma_1 \\ &= \rho_{1,3}\lambda_1. && \text{(by (8))} \end{aligned}$$

The proof is complete. □

4 Rewriting Technique for Words from $\mathfrak{M}$

sec:rewrite

In this section we are going to develop some rewriting technique for the elements of $\mathfrak{M}$ in order to construct canonical forms for them. We start with the following observation.

lem:rewr1

Lemma 4.1. *Every element of $\mathfrak{M}$ can be written in the form $\alpha_1 \cdots \alpha_k\beta$, where $k \geq 0$; each α_i is equal to some $\lambda_{p,q}$ or $\rho_{p,q}$; and $\beta \in \mathcal{S}_n$.*

Proof. For the first we note that λ_i and ρ_i lie in the orbits of λ_1 and ρ_1 respectively (with respect to the action of $\mathcal{S}_n$ on $\mathfrak{M}$ by inner automorphisms). This implies that $\mathfrak{M} = \langle \mathcal{S}_n, \lambda_1, \rho_1\rangle$. Therefore, an arbitrary $\mu \in \mathfrak{M}$ can be written as

$$\mu = \pi_1\gamma_1\pi_2\gamma_2 \dots \pi_k\gamma_k\pi_{k+1}$$

with $k \geq 0$, $\pi_i \in \mathcal{S}_n$ and $\gamma_i \in \{\lambda_1, \rho_1\}$ for all i. In view of (23) we can rewrite the expression for μ as follows:

$$\mu = \pi_1\gamma_1\pi_1^{-1}(\pi_1\pi_2)\gamma_2(\pi_1\pi_2)^{-1} \cdots (\pi_1 \cdots \pi_k)\gamma_k(\pi_1 \cdots \pi_k)^{-1} \\ \cdot (\pi_1 \cdots \pi_k\pi_{k+1}) = \gamma_{p_1,q_1} \cdots \gamma_{p_k,q_k}\beta,$$

where $p_i = (\pi_1 \cdots \pi_i)^{-1}(1)$, $q_i = (\pi_1 \cdots \pi_i)^{-1}(2)$ and

$$\gamma_{p_i,q_i} = \begin{cases} \lambda_{p_i,q_i} & \text{if } \gamma_i = \lambda_1 \\ \rho_{p_i,q_i} & \text{if } \gamma_i = \rho_1, \end{cases}$$

$0 \leq i \leq k$, and $\beta = \pi_1 \cdots \pi_{k+1}$. □

Now we strengthen this result to

lem:rewr2 **Lemma 4.2.** *Every element of* $\mathfrak{M}$ *can be written as*

$$\alpha_1 \cdots \alpha_k \beta_1 \cdots \beta_l E\sigma, \tag{41}$$ eq:product

where $k, l \geq 0$*, each* α_i *equals some* $\rho_{p,q}$*, each* β_i *equals some* $\lambda_{p,q}$*,* $\sigma \in \mathcal{S}_n$ *and* $E = 1$ *or* E *is a product of several* e_p*-s.*

Proof. Take $\mu \in \mathfrak{M}$. It follows from Lemma 4.1 that we can express μ as a product $\gamma_1 \cdots \gamma_k \sigma$ for some $k \geq 0$, and such that each γ_i is either one of $\lambda_{p,q}$-s or one of $\rho_{p,q}$-s, and $\sigma \in \mathcal{S}_n$.

Suppose that $\gamma_i = \lambda_{p,q}$ and $\gamma_{i+1} = \rho_{k,l}$ for some i. If the sets $\{p, q\}$ and $\{k, l\}$ are disjoint, we have $\alpha_i \alpha_{i+1} = \alpha_{i+1} \alpha_i$ by (30). If the sets $\{p, q\}$ and $\{k, l\}$ are not disjoint, we apply the appropriate relation of (31), (35), (36) or (40). As a result we obtain an expression for μ containing less subwords of the form $\lambda_{i,j} \rho_{s,t}$.

However, after such a rewriting in the expression for μ there might appear some e_i-s and $\sigma_{s,t}$-s. If some e_i-s appear, using (25)–(29), we rewrite our expression such that it has the occurrence of e_i-s at the right position with respect to λ-s and ρ-s, and at the left with respect to σ, while the number of subwords of the form $\lambda_{i,j} \rho_{s,t}$ remains the same. If some $\sigma_{s,t}$-s appear, using the action of $\mathcal{S}_n$ on $\mathfrak{M}$ by inner automorphisms, we can, similarly to as this is done in the proof of Lemma 4.1, rewrite it such that the group element occurs to the right of all occurrences of $\lambda_{i,j}$-s and $\rho_{s,t}$-s. As the mentioned rewriting does not affect the number of the subwords of the form $\lambda_{i,j} \rho_{s,t}$, the statement of the lemma follows by induction on the number of subwords of the form $\lambda_{i,j} \rho_{s,t}$ in the initial expression for μ. □

Our next step is to improve the decomposition (41) for a typical element of $\mathfrak{M}$. For this we require some more notation: for a set $A = \{a_1, \ldots, a_s\} \subseteq \{1, \ldots, n\}$ and $p \in \{1, \ldots, n\}$, $p \notin A$, define

$$R_{p,A} = \rho_{p,a_1} \cdots \rho_{p,a_s} \text{ and } L_{p,A} = \lambda_{p,a_1} \cdots \lambda_{p,a_s}.$$

Note that due to (38) and (39) these are well-defined. Analogously, for $M = \{m_1, \ldots, m_r\} \subseteq \{1, \ldots, n\}$ we put $E_M = e_{m_1} \cdots e_{m_r}$ and it is well-defined by (24).

lem:rewr3 **Lemma 4.3.** *Every element of* $\mathfrak{M}$ *can be written as*

$$R_{p_1,A_1} \cdots R_{p_k,A_k} L_{q_1,B_1} \cdots L_{q_l,B_l} E_M \sigma, \tag{42}$$ eq:expression

where $k, l \geq 0$*,* $p_1, \ldots, p_k$ *are pairwise distinct,* $q_1, \ldots, q_l$ *are pairwise distinct,* $A_1, \ldots, A_k$ *are pairwise disjoint,* $B_1, \ldots, B_l$ *are pairwise disjoint,* $M \subseteq \{1, \ldots, n\}$*,* $\sigma \in \mathcal{S}_n$ *and* $p_i \notin \cup_{i \leq k} A_i$*,* $q_j \notin \cup_{j \leq l} B_j$*.*

Proof. Take an arbitrary $\mu \in \mathfrak{M}$. Write it as $\mu = \alpha_1 \cdots \alpha_k \beta_1 \cdots \beta_l E_M \sigma$ with all the conditions from Lemma 4.2.

Now we implement the following procedure. If $k = 0$ then do nothing, otherwise take $\alpha_1 = \rho_{p,x_1} = R_{p,\{x_1\}}$. Then, if $k = 1$ then do nothing otherwise take $\alpha_2 = \rho_{y,z}$. If $\{p, x_1\} \cap \{y, z\} = \varnothing$ then put $p_1 = p$, $A_1 = \{x_1\}$ and start over the procedure for the word $\alpha_2 \cdots \alpha_k \beta_1 \cdots \beta_l E_M \sigma$. Alternatively, i.e. if $\{p, x_1\} \cap \{y, z\} \neq \varnothing$, we have six possible cases:

a1 (a) $p = y$ and $x_1 \neq z$ then do nothing;

a2 (b) $p = z$ and $x_1 \neq y$ then, using (39), swap $\alpha_1\alpha_2 = \rho_{p,x_1}\rho_{y,p}$ with $\rho_{y,x_1}\rho_{y,p}$;

a3 (c) $p = y$ and $x_1 = z$ then, using (37), swap $\alpha_1\alpha_2 = \rho_{p,x_1}\rho_{p,x_1}$ with $e_p e_{x_1}$;

a4 (d) $p = z$ and $x_1 = y$ then, using (37), swap $\alpha_1\alpha_2 = \rho_{p,x_1}\rho_{x_1,p}$ with $e_p e_{x_1}$;

a5 (e) $p \neq y$ and $x_1 = z$ then, using (33), swap $\alpha_1\alpha_2 = \rho_{p,x_1}\rho_{y,x_1}$ with $\rho_{p,x_1}e_y$;

a6 (f) $p \neq z$ and $x_1 = y$ then, using (32), swap $\alpha_1\alpha_2 = \rho_{p,x_1}\rho_{x_1,z}$ with $e_z\rho_{p,x_1}$.

In the cases (c)–(f), similarly to as it has been done in the proof of Lemma 4.2, write a new expression (41) for μ, with less entries of α_i-s, and start over the procedure for it. In the cases (a) and (b) we have (slightly abusing notation) that $\alpha_1\alpha_2 = R_{p,\{x_1,x_2\}} = \rho_{p,x_1}\rho_{p,x_2}$. Now we will involve inductive arguments: suppose at some stage of our procedure we obtain that $\alpha_1\cdots\alpha_r = R_{p,\{x_1,\ldots,x_r\}}$. If $k = r$ then do nothing. Otherwise take $\alpha_{r+1} = \rho_{y,z}$. If $\{y,z\}\cap\{p,x_1,\ldots,x_r\} = \varnothing$ then put $p_1 = p$, $A_1 = \{x_1,\ldots,x_r\}$ and start over the procedure for $\alpha_{r+1}\cdots\alpha_k\beta_1\cdots\beta_l E_M\sigma$. Alternatively, if $\{y,z\}\cap\{p,x_1,\ldots,x_r\} \neq \varnothing$, then for the similar reasonings as above, and up to swapping the factors in $\rho_{p,x_1}\cdots\rho_{p,x_r}$ (which is possible due to (39)), either we can rewrite μ as a product (41) with less entries of α_i-s, or have that:

Case 1. $p = y$ and $z \notin \{x_1,\ldots,x_r\}$ then do nothing.

Case 2. $p = z$ and $y \notin \{x_1,\ldots,x_r\}$ and then, involving (39), we have

$$\begin{aligned}
\alpha_1\cdots\alpha_{r+1} &= \rho_{p,x_1}\cdots\rho_{p,x_{r-1}}\rho_{p,x_r}\rho_{y,p}\\
&= \rho_{p,x_1}\cdots\rho_{p,x_{r-1}}\rho_{y,x_r}\rho_{y,p}\\
&= \rho_{p,x_1}\cdots\rho_{p,x_{r-2}}\rho_{y,x_{r-1}}\rho_{y,x_r}\rho_{y,p}\\
&\cdots\\
&= \rho_{y,x_1}\cdots\rho_{y,x_r}\rho_{y,p}\\
&= R_{y,\{x_1,\ldots,x_r,p\}}.
\end{aligned} \tag{43}$$

eq:needed

Eventually, going along this procedure, we shall arrive at the case when, for some r, $\alpha_1\cdots\alpha_r = R_{p,\{x_1,\ldots,x_r\}}$ and either $k = r$, or $\alpha_{r+1} = \rho_{y,z}$ with $\{y,z\}\cap\{p,x_1,\ldots,x_r\} = \varnothing$. Then we put $p_1 = p$, $A_1 = \{x_1,\ldots,x_r\}$ and start the procedure over for $\alpha_{r+1}\cdots\alpha_k\beta_1\cdots\beta_l E_M\sigma$.

Thus, implementing the above procedure we can rewrite μ in the form

$$\mu = R_{p_1,A_1}\cdots R_{p_k,A_k}\beta_1\cdots\beta_l E_M\sigma$$

with $p_i \notin A_i$ and $p_{i+1} \notin \{p_i\}\cup A_i$.

Further, we may also assume that $A_i \cap A_{i+1} \neq \varnothing$. Indeed, assume the contrary and let $a \in A_i \cap A_{i+1}$. Then permute the factors in the products so that we can spot a subword $\rho_{p_i,a}\rho_{p_{i+1},a}$ and then, using (33), replace this subword by $\rho_{p_i,a}e_{p_{i+1}}$, thereby decreasing the number of α_i-s in the product and the cardinality of the intersection $A_i \cap A_{i+1}$.

Furthermore, we may assume that $p_i \notin A_{i+1}$. Suppose the contrary. Then we have $R_{p_i,A_i}R_{p_{i+1},A_{i+1}} = R_{p_i,A_i}\rho_{p_{i+1},p_i}R_{p_{i+1},A_{i+1}-\{p_i\}}$ and using the same arguments as in (43), we obtain that

$$R_{p_i,A_i}R_{p_{i+1},A_{i+1}} = R_{p_{i+1},A_i\cup\{p_i\}}R_{p_{i+1},A_{i+1}-\{p_i\}} = R_{p_{i+1},A_i\cup A_{i+1}}$$

and we decrease the number k.

Thus now, by (30), we have $R_{p_i,A_i}R_{p_{i+1},A_{i+1}} = R_{p_{i+1},A_{i+1}}R_{p_i,A_i}$ for all i. So, involving the above reasonings, we may even assume that for $i \neq j$:

$$p_i \neq p_j,\ A_i \cap A_j = \varnothing \text{ and } p_i \notin A_j. \tag{44}$$ eq:conditions

Note also that $p_i \notin A_i$. Now involve the same reasonings with respect to β_j-s and finally we will obtain that

$$\mu = R_{p_1,A_1}\cdots R_{p_k,A_k}L_{q_1,B_1}\cdots L_{q_l,B_l}E_M\sigma$$

with p_i-s, q_j-s, A_i-s and B_j-s satisfying all the required conditions. □

Now we are ready to prove the crucial ingredient of our rewriting technique:

prop:rewr4 **Proposition 4.4.** *Every element of $\mathfrak{M}$ can be written as a product* (42) *with all conditions from Lemma 4.3, such that, in addition:*

i *(i)* $p_i \notin B_1 \cup \cdots \cup B_l$, $1 \leq i \leq k$;

ii *(ii)* $q_j \notin A_1 \cup \cdots \cup A_k$, $1 \leq j \leq l$;

iii *(iii)* M *is disjoint with* $\left(\cup_{i=1}^{k}(\{p_i\} \cup A_i)\right) \cup \left(\cup_{j=1}^{l}(\{q_j\} \cup B_j)\right)$.

Proof. Take an arbitrary $\mu \in \mathfrak{M}$. Consider the set of all expressions (42) for μ, with all conditions from Lemma 4.3 satisfied. Choose among them those with minimal possible k, and then, within the latter, choose arbitrarily one with minimal possible l.

Prove then that (i) holds. Seeking a contradiction, suppose that $p_i \in B_j$, for some $i \leq k$ and $j \leq l$. Up to swapping R_{p_i,A_i}-s, and L_{q_j,B_j}-s, we may assume that $i = k$ and $j = 1$. Then one spots in μ the factor $R_{p_k,A_k}\lambda_{q_1,p_k}$. Now we have two cases to consider:

Case 1. $q_1 \notin A_k$. Then using (34), that is $\rho_{p_k,a}\lambda_{q_1,p_k} = \lambda_{q_1,p_k}e_a$ for every $a \in A_k$, we obtain that $R_{p_k,A_k}\lambda_{q_1,p_k} = \lambda_{q_1,p_k}\prod_{a\in A_k} e_a$. Then, as in Lemma 4.2, we can rewrite the expression (42) for μ with a number of R_{p_i,A_i}-s less than k, a contradiction.

Case 2. $q_1 \in A_k$. Notice that $R_{p_k,A_k}\lambda_{q_1,p_k} = R_{p_k,A_k-\{q_1\}}\rho_{p_k,q_1}\lambda_{q_1,p_k}$. Now, involving (31) and then (28), we have $\rho_{p_k,q_1}\lambda_{q_1,p_k} = e_{p_k}e_{q_1} = \lambda_{q_1,p_k}e_{p_k}e_{q_1}$. Hence, by Case 1, we obtain

$$R_{p_k,A_k}\lambda_{q_1,p_k} = \lambda_{q_1,p_k}\prod_{a\in A_k-\{q_1\}} e_a \cdot e_{p_k}e_{q_1} = \prod_{a\in A_k\cup\{p_k\}} e_a.$$

and by the similar arguments as in Case 1, we get a contradiction.

Applying analogous reasonings it can be shown that (ii) holds. It remains to establish that (iii) holds.

Suppose there is $m \in M$ such that $m \in \cup_{j=1}^{l}\left(\{q_j\} \cup B_j\right)$. Then we may assume that $m \in \{q_l\} \cup B_l$. Then applying (28), we have $L_{q_l,B_l}e_m = e_{q_l}\prod_{b\in B_l} e_b$, a contradiction with the choice of l.

In particular, now we have that each L_{q_j,B_j} commutes with E_M. Now, assuming that (iii) does not hold, we may suppose that there exists $m \in M$ with $m \in \{p_k\} \cup A_k$. Then, as shown above, we have $R_{p_k,A_k}e_m = e_{p_k}\prod_{a\in A_k} e_a$ and so we can rewrite an expression (42) for μ (using the method from Lemma 4.2) with number of R_{p_i,A_i}-s less than k. This completes the proof. □

In light of the previous proposition, we now introduce some more notation: take k, l, p_i-s, q_j-s, A_i-s and B_j-s from Lemma 4.3, satisfying the conditions of Lemma 4.3 and Proposition 4.4. Let also $T = \{t_1, \ldots, t_s\}$ be the set comprising all p_i-s and q_j-s. Let r be a natural number $\leq s$.

- If $t_r = p_i$ for some i, set $C_r = A_i \cup \{p_i\}$ and $R^{t_r}_{C_r} = R_{p_i,A_i}$. Otherwise put $C_r = \{t_r\}$ and $R^{t_r}_{C_r} = 1$.
- If $t_r = q_j$ for some j, set $D_r = B_j \cup \{q_j\}$ and $L^{t_r}_{D_r} = L_{q_j,B_j}$. Otherwise put $D_r = \{t_r\}$ and $L^{t_r}_{D_r} = 1$.

The sets $C_1, \ldots, C_s$ are pairwise disjoint and their union coincides with $T \cup \left(\cup_{i=1}^{k} A_i\right)$. Analogously, the sets $D_1, \ldots, D_s$ are pairwise disjoint and their union coincides with $T \cup \left(\cup_{j=1}^{l} B_j\right)$. In addition, $t_r \in C_r \cap D_r$ for every r.

Using this notation, we can rewrite Proposition 4.4 to

cor:can_form **Corollary 4.5.** *Every element of $\mathfrak{M}$ can be presented in the form*

$$R^{t_1}_{C_1} \cdots R^{t_s}_{C_s} L^{t_1}_{D_1} \cdots L^{t_s}_{D_s} E_M \sigma, \tag{45}$$

eq:canonical

where $C_1, \ldots, C_s$ are pairwise disjoint, $D_1, \ldots, D_s$ are pairwise disjoint, $t_i \in C_i \cap D_i$, $1 \leq i \leq s$, M is disjoint with $\cup_{i=1}^{s}(C_i \cup D_i)$ and $\sigma \in \mathcal{S}_n$.

Call an expression of the form (45), such that the conditions of Corollary 4.5 are satisfied, a *canonical form*. Now we will develop a notion of equivalence of two canonical forms. For this we require some more notation: for $B \subseteq \{1, \ldots, n\}$ denote by $\mathcal{S}_B$ the subgroup of $\mathcal{S}_n$ generated by all $\sigma_{i,j}$ with $i, j \in B$. Put $F = M \cup (\cup_{i=1}^{s}(C_i \setminus D_i))$ and $G = \mathcal{S}_{D_1} \oplus \cdots \oplus \mathcal{S}_{D_s} \oplus \mathcal{S}_F$. Call two canonical forms

$$R^{t_1}_{C_1} \cdots R^{t_s}_{C_s} L^{t_1}_{D_1} \cdots L^{t_s}_{D_s} E_{M_1} \sigma_1 \text{ and } R^{t'_1}_{C'_1} \cdots R^{t'_s}_{C'_s} L^{t'_1}_{D'_1} \cdots L^{t'_s}_{D'_s} E_{M_2} \sigma_2$$

equivalent provided that there is a permutation $\tau \in \mathcal{S}_s$ such that $C_i = C'_{\tau(i)}$, $D_i = D'_{\tau(i)}$, $1 \leq i \leq s$, $M_1 = M_2$ and $\sigma_1 \sigma_2^{-1} \in G$.

prop:canonical **Proposition 4.6.** *If two canonical forms are equivalent, they are equal in $\mathfrak{M}$.*

We shall derive the proof from a series of the following lemmas.

lem:aux **Lemma 4.7.** *For pairwise distinct i, j and q:*

$$\lambda_{q,i}\lambda_{q,j}\sigma_{i,j} = \lambda_{q,i}\lambda_{q,j} \quad \text{and} \quad \lambda_{q,i}\lambda_{q,j}\sigma_{q,j} = \lambda_{q,i}\lambda_{q,j}. \tag{46}$$

eq:eat_lambda

Proof. Take an arbitrary $\pi \in \mathcal{S}_n$ with $\pi(1) = i$ and $\pi(2) = j$. Then $\pi\sigma_{i,j} = \sigma_1\pi$ and so

$$\begin{aligned}
\lambda_{q,i}\lambda_{q,j}\sigma_{i,j} &= \lambda_{q,i}\lambda_{i,j}\sigma_{i,j} && \text{(by (38))} \\
&= \lambda_{q,i}\pi^{-1}\lambda_1\pi\sigma_{i,j} && \text{(by (23))} \\
&= \lambda_{q,i}\pi^{-1}\lambda_1\sigma_1\pi && \\
&= \lambda_{q,i}\pi^{-1}\lambda_1\pi && \text{(by (8))} \\
&= \lambda_{q,i}\lambda_{i,j} && \text{(by (23))} \\
&= \lambda_{q,i}\lambda_{q,j}. && \text{(by (38))}
\end{aligned}$$

The second equality follows by the same arguments. ∎

lm:aux2 **Lemma 4.8.** *Let $i \neq j$. Then $e_i e_j \sigma_{i,j} = e_i e_j$.*

Proof. Notice first that according to Lemma 3.2 and (8), $\rho_{i,j}\sigma_{i,j} = \rho_{j,i}$. Then

$$\begin{aligned} e_i e_j \sigma_{i,j} &= e_i \lambda_{i,j} \rho_{i,j} \sigma_{i,j} && \text{(by (31))} \\ &= e_i \lambda_{i,j} \rho_{j,i} \\ &= \lambda_{j,i} \rho_{j,i} \lambda_{i,j} \rho_{j,i} && \text{(by (31))} \\ &= \lambda_{j,i} e_i e_j \rho_{j,i} && \text{(by (31))} \\ &= e_i e_j. && \text{(by (28) and (29))} \end{aligned}$$

□

lem:stabil_S **Lemma 4.9.** *Let $\mu = R_{C_1}^{t_1} \cdots R_{C_s}^{t_s} L_{D_1}^{t_1} \cdots L_{D_s}^{t_s} E_M$ be a canonical form. Then $\mu\sigma = \mu$ for all $\sigma \in G$.*

Proof. Obviously it suffices to prove the statement only for $\sigma \in \mathcal{S}_{D_r}$ and $\sigma \in \mathcal{S}_F$.

So, take $\sigma = \sigma_{i,j}$ with $i, j \in D_r$. Since E_M commutes with $L_{D_r}^{t_r}$, we are only to establish that $L_{D_r}^{t_r} \sigma_{i,j} = L_{D_r}^{t_r}$. But this equality follows from (46).

Suppose now that $\sigma = \sigma_{i,j}$ with $i, j \in F$. To prove the statement it is enough to show that $\mu = \mu e_i e_j$. Then we will have $\mu \sigma_{i,j} = \mu e_i e_j \sigma_{i,j} = \mu$ by Lemma 4.8, which will complete the proof.

Firstly we note that if $i \in M$ then $\sigma e_i = \sigma$ by the definition of E_M and (24). Let now $i \in C_r \setminus D_r$ for some r. Then since e_i commutes with every $L_{D_j}^{t_j}$ and $R_{C_r}^{t_r} e_i = R_{C_r}^{t_r}$ (by (27)), we have $\mu = \mu e_i$ and the claim follows. □

Proof of Proposition 4.6. Take a canonical form $R_{C_1}^{t_1} \cdots R_{C_s}^{t_s} L_{D_1}^{t_1} \cdots L_{D_s}^{t_s} E_M$.

Take also $x_i \in C_i \cap D_i$, $1 \leq i \leq s$. In view of Lemma 4.9, it is enough to show that if we replace t_i-s by x_i-s, we do not change the value of the canonical form in $\mathfrak{M}$. Now, taking to account

$$\begin{aligned} R_{C_i}^{t_i} L_{D_i}^{t_i} &= R_{t_i, C_i - \{x_i, t_i\}} \cdot \rho_{t_i, x_i} \lambda_{t_i, x_i} \cdot L_{t_i, D_i - \{x_i, t_i\}} \\ &= R_{t_i, C_i - \{x_i, t_i\}} \rho_{x_i, t_i} \cdot \lambda_{x_i, t_i} L_{t_i, D_i - \{x_i, t_i\}} && \text{(by (36))} \\ &= \rho_{x_i, t_i} R_{x_i, C_i - \{x_i, t_i\}} \cdot L_{x_i, D_i - \{x_i, t_i\}} \lambda_{x_i, t_i} && \text{(by (38) and (39))} \\ &= R_{C_i}^{x_i} L_{D_i}^{x_i}, \end{aligned}$$

and that $R_{C_i}^{t_i}$-s commute, and that $L_{D_i}^{t_i}$-s commute, the proof is complete. □

5 Proof of Main Theorem

sec:final

In this section we provide the last ingredients for the proof of Main Theorem. Introduce the notation for certain elements of the monoid $\mathcal{PI}_n^*$: for distinct x and y of $\mathbf{n}$ we set

$$\begin{aligned} s_{x,y} &= \{\{x, y'\}, \{x', y\}, \{t, t'\}_{t \in \mathbf{n} \setminus \{x,y\}}\} \\ r_{x,y} &= \{\{x, y, x'\}, \{y'\}, \{t, t'\}_{t \in \mathbf{n} \setminus \{x,y\}}\} \\ l_{x,y} &= \{\{x, x', y'\}, \{y\}, \{t, t'\}_{t \in \mathbf{n} \setminus \{x,y\}}\} \\ \varepsilon_x &= \{\{x\}, \{x'\}, \{t, t'\}_{t \in \mathbf{n} \setminus \{x\}}\}. \end{aligned}$$

Note that $l_{x,y} r_{x,y} = \varepsilon_y$. Furthermore, we set $s_i = s_{i,i+1}$, $r_i = r_{i,i+1}$ and $l_i = l_{i,i+1}$ for $1 \leq i \leq n-1$. The elements $s_1, \ldots, s_{n-1}$ generate the group of units of $\mathcal{PI}_n^*$ which is isomorphic to the symmetric group $\mathcal{S}_n$ and will be identified with it. Now we are ready to prove our main result:

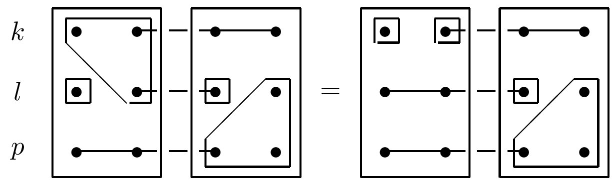

Figure 3: An illustration of the equality $l_{k,l}l_{p,l} = \varepsilon_k l_{p,l}$. fig:r5

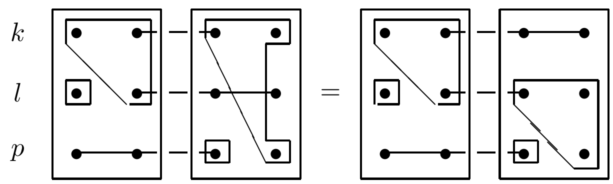

Figure 4: An illustration of the equality $l_{k,l}l_{k,p} = l_{k,l}l_{l,p}$. fig:r3

Proof of Main Theorem. Notice that s_i-s, r_i-s and l_i-s altogether generate $\mathcal{PI}_n^*$, see [7]. It is a routine matter to check that $\mathcal{PI}_n^*$ satisfies all the relations of $\mathfrak{M}$, with respect to these generators. We present some of them on Fig. 3–5.

Hence, it remains only to ensure that the natural surjective homomorphism $\phi : \mathfrak{M} \to \mathcal{PI}_n^*$ such that $\sigma_i \mapsto s_i$, $\lambda_i \mapsto l_i$ and $\rho_i \mapsto r_i$, is injective.

Applying Proposition 4.6 it is enough to show that if ϕ-images of values of two canonical forms are equal in $\mathcal{PI}_n^*$, then these canonical forms are equivalent. For this, we compute the value of the image of a typical canonical form in $\mathfrak{M}$. For the word (45) this is the element

$$\big\{\big(C_i \cup \sigma(D_i')\big)_{1\leq i\leq s}, \{x\}_{x\in K_1}, \{\sigma(x')\}_{x\in K_2}, \{x, \sigma(x')\}_{x\in K_3}\big\},$$

where

$$\begin{aligned}
K_1 &= M \cup \bigcup_{i=1}^{s}(D_i \setminus C_i)\\
K_2 &= F = M \cup \bigcup_{i=1}^{s}(C_i \setminus D_i)\\
K_3 &= X \setminus \big(M \cup \bigcup_{i=1}^{s}(C_i \cup D_i)\big).
\end{aligned}$$

The statement now follows from the definition of equivalent canonical forms. □

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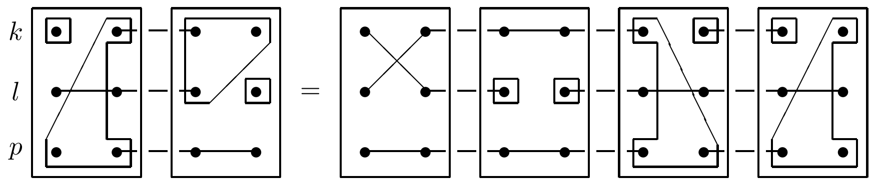

Figure 5: An illustration of the equality $l_{p,k}r_{k,l} = s_{k,l}\varepsilon_l r_{p,k} l_{p,k}$. fig:r8

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