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Some Erdős-Ko-Rado Theorems for Injections

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Abstract

This paper investigates t-intersecting families of injections, where two injections a, b from $[k]$ to $[n]$ t-intersect if there exists $X \subseteq [k]$ with $|X| \geq t$ such that $a(x) = b(x)$ for all $x \in X$. We prove that if $\mathcal{F}$ is a 1-intersecting injection family of maximal size then all elements of $\mathcal{F}$ have a fixed image point in common. We show that when n is large in terms of k and t, the set of injections which fix the first t points is the only t-intersecting injection family of maximal size, up to permutations of $[k]$ and $[n]$. This is not the case for small n. Indeed, we prove that if k is large in terms of $k - t$ and $n - k$, the largest t-intersecting injection families are obtained from a process of saturation rather than fixing.

Key words: Erdős-Ko-Rado, t-intersecting, injections, fixing, saturation, intersecting families
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1. Introduction

1.1. Definitions

Let $\mathcal{S}_n$ denote the symmetric group of permutations on n points, and let $\mathcal{I}_n^k$ be the set of injections from $[k]$ to $[n]$. Writing injections in image form, we may view $\mathcal{I}_n^k$ as the set of words of length k over $[n]$ with no repeated symbols.

Two injections in $\mathcal{I}_n^k$ *t-intersect* if they agree on the image of at least t domain points. For $a = a_1a_2 \dots a_k$, $b = b_1b_2 \dots b_k \in \mathcal{I}_n^k$, set

$$\operatorname{int}(a, b) = \{\, i \in [k] : a_i = b_i \,\}\,.$$

A subset $\mathcal{F}$ of $\mathcal{I}_n^k$ is *t-intersecting* if, for all $a, b \in \mathcal{F}$, we have $|\operatorname{int}(a, b)| \geq t$. When $t = 1$, we usually say *intersecting* rather than 1-intersecting. We call a t-intersecting family *maximal* if it is maximal under set inclusion, and *maximum* if there is no larger t-intersecting family.

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We say that $A \subseteq \mathcal{I}_n^k$ is *equivalent* to $B \subseteq \mathcal{I}_n^k$ if A can be obtained from B by permutations of the domain $[k]$ and the image $[n]$. Let $\mathcal{K}_0(t,k,n)$ be the t-intersecting subset of $\mathcal{I}_n^k$ obtained by including all injections which fix the first t points:

$$\mathcal{K}_0(t,k,n) = \left\{ a \in \mathcal{I}_n^k : a(i) = i,\ 1 \le i \le t \right\}.$$

We will refer to this and any family equivalent to $\mathcal{K}_0(t,k,n)$ as a *fix-family.* Note that when $t = 1$, the fix-family of permutations has size $|\mathcal{K}_0(1,n,n)| = (n-1)!$.

1.2. Background

Following the investigation of t-intersecting families of subsets of a set by Erdős, Ko, Rado, Katona and others in the 1960s, research into t-intersecting sets of injections began with the study of intersecting permutation families in the 1970s when Deza & Frankl showed in [6] that an intersecting subset of $\mathcal{S}_n$ has size at most $(n-1)!$. Much later, Cameron & Ku as well as Larose & Malvenuto independently obtained the classification of maximum intersecting permutation families quoted below.

Theorem 1.1. *(Cameron, Ku [5]; Larose, Malvenuto [17].)*
If $n \ge 2$ *and* $\mathcal{F}$ *is an intersecting subset of* $\mathcal{S}_n$ *with* $|\mathcal{F}| = (n-1)!$ *then* $\mathcal{F}$ *is equivalent to the fix-family.*

This result inspired numerous investigations of intersecting permutation families. It has since been shown that fixing is the unique optimal strategy for obtaining large intersecting subsets of the following global sets:

- the set of k-partial permutations of $[n]$ [16, 18],
- the alternating group $\mathcal{A}_n \subset \mathcal{S}_n$ [15],
- a direct product $\mathcal{S}_{n_1} \times \cdots \times \mathcal{S}_{n_q}$ of symmetric groups [15],
- Coxeter groups of types B and D [20].

We point out that $\mathcal{I}_n^k$ is strictly contained in the set of k-partial permutations on n points studied in [16, 18], since the domain of a k-partial permutation is not fixed to be $[k]$, but can be any k-subset of $[n]$.

The results described so far follow the spirit of Erdős-Ko-Rado: they show that, for certain values of the parameters or in general, a maximum intersecting family consists of all such elements which have a fixed set of image points in common. To generalise this idea, it helps to view this *fixing* concept as a special case of a *saturation* process. This is best illustrated by an example: consider the extension of the Erdős-Ko-Rado Theorem by Ahlswede and Khachatrian.

Erdős, Ko & Rado proved in 1938 that for $n \ge n_0(k,t)$, a t-intersecting family of k-subsets of an n-set has size at most $\binom{n-t}{k-t}$. This result is often referred to as the EKR-Theorem. For the 1-intersecting case, they conjectured that all optimal families are obtained by *fixing*, i.e. that the largest possible intersecting families of k-subsets of an n-set are precisely those families all of whose members contain some *fixed* element of the n-set. This was proved by

Katona [13] in 1964, three years after the publication of the EKR-Theorem in [7].

Theorem 1.2. *(Erdős, Ko, Rado [7]; Katona [13]).*
Let $k \leq n/2$ and let $\mathcal{F}$ be an intersecting family of k-subsets of $[n]$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ and equality implies that all members of $\mathcal{F}$ have a fixed element of $[n]$ in common.

Considering the t-intersecting case, Frankl [8] and Wilson [21] subsequently proved that $n_0(k,t) = (k-t+1)(t+1)$. But what happens when $n < n_0$? For $0 \leq i \leq (n-t)/2$, let S_i be some fixed $(t+2i)$-subset of $[n]$. The t-intersecting family F_i is obtained by *saturation*:

$$F_i = \{\, X \subseteq [n] : |X| = k,\ |X \cap S_i| \geq t+i \,\}\,.$$

Ahlswede & Khachatrian proved in [1] that given k, t and $n < n_0(k,t)$, a t-intersecting family of k-subsets of an n-set has size at most $|F_p|$, where p is given explicitly as a function of n, k and t. Moreover, they show that up to the choice of the saturation set S_i, F_p is the unique maximum t-intersecting family, unless the size of F_p equals that of F_{p+1} in which case both systems are optimal.

Saturation has since been shown to yield optimal t-intersecting families for other combinatorial structures as well, e.g. words [2]. The concept was first applied to permutations in [6] where Deza & Frankl showed that when k is large in terms of its differences with t and n, saturation yields a maximum t-intersecting family: if $k-t$ is even, set

$$\mathcal{G}(t,k,n) = \left\{\, w \in \mathcal{I}_n^k : w \text{ moves at most } (k-t)/2 \text{ points} \,\right\};$$

if $k-t$ is odd, set

$$\mathcal{G}(t,k,n) = \left\{\, w \in \mathcal{I}_n^k : w \text{ moves at most } (k-t-1)/2 \text{ elements of } [k-1] \,\right\}.$$

The saturation family $\mathcal{G}(t,k,n)$ is t-intersecting by the pigeonhole principle.

Theorem 1.3. *(Deza, Frankl [6])*
For each $T \in \mathbb{N}$ with $T \geq 3$, there exists $k_0(T) \in \mathbb{N}$ such that for $k \geq k_0(T)$, the saturation family $\mathcal{G}(k-T,k,k)$ is maximum $(k-T)$-intersecting in $\mathcal{S}_k$.

The proof of Theorem 1.3 depends on the bound in the EKR-Theorem. However, using Katona's classification result from [13] (cf. Theorem 1.2) in Deza & Frankl's proof of Theorem 1.3 demonstrates that for T and k as in Theorem 1.3, the saturation family $\mathcal{G}(k-T,k,k)$ is in fact the unique maximum $(k-T)$-intersecting subset of $\mathcal{S}_k$. This argument will be presented in detail in the concluding paragraphs of the proof of Theorem 3.5 which generalises Theorem 1.3 to injections.

1.3. Outline

We show that every maximum 1-intersecting subset of $\mathcal{I}_n^k$ is equivalent to the fix-family, a fact which was recently conjectured in [3]. In Section 2.1, we prove this for $k \leq (n+1)/2$, while in Section 2.2, the case $k \geq (n+1)/2$ is dealt with using an analogous approach to that of Cameron and Ku in [5]."

In Section 3, Corollary 3.3 shows that fixing is also the unique optimal strategy for $t > 1$ provided n is large in terms of k and t. By way of contrast, Theorem 3.5 fixes the differences between $k-t$ and $n-k$ and increases k: in this case, the maximum t-intersecting subsets of $\mathcal{I}_n^k$ are equivalent to the saturation family $\mathcal{G}$.

Computational evidence suggests that an Ahlswede-Khachatrian-type result holds for injections as well as sets [1] and words [2]: setting

$$\mathcal{K}_r(t,k,n) = \left\{ w \in \mathcal{I}_n^k : w \text{ fixes at least } t+r \text{ elements of } [t+2r] \right\},$$

we conjecture that there exists a function $r^*(t,k,n)$ such that all maximum t-intersecting families in $\mathcal{I}_n^k$ are equivalent to $\mathcal{K}_{r^*}$. We refer to the thesis of the first author for a proof that this is the case among injection families whose fixed point sets are t-intersecting and left-compressed. Whether there are any injection families which cannot be standardised in this way remains an open question.

2. Intersection Size 1

We begin by giving a bound on the size of an intersecting family in $\mathcal{I}_n^k$. Let $\pi = (1\,2\,\ldots\,n)$ denote the n-cycle in $\mathcal{S}_n$, and let permutations act on injections in $\mathcal{I}_n^k$ by acting on each position separately:

$$(w_1 w_2 \ldots w_k)\pi = (w_1\pi)(w_2\pi)\ldots(w_k\pi).$$

For $w \in \mathcal{I}_n^k$, denote by $O(w)$ the *orbit* of w in $\mathcal{I}_n^k$:

$$O(w) = \{w(1\,2\,\ldots n)^i \;:\; i \in \mathbb{N}\}.$$

These orbits provide a fairly standard way of obtaining bounds on intersecting families, see [4, 6].

Theorem 2.1. *If $\mathcal{F} \subseteq \mathcal{I}_n^k$ is intersecting then $|\mathcal{F}| \leq \frac{(n-1)!}{(n-k)!}$.*

Moreover, the set of orbits $\{O(w) \mid w \in \mathcal{I}_n^k\}$ forms a partition of $\mathcal{I}_n^k$ into disjoint sets of size n and if $|\mathcal{F}| = \frac{(n-1)!}{(n-k)!}$ then $\mathcal{F}$ is a transversal of the orbits.

Proof. Since π is a permutation, two orbits are either equal or disjoint. Moreover, all orbits have size equal to the order of π which is n. Finally, if u, v are two distinct elements of $O(w)$ for some $w \in \mathcal{I}_n^k$ then u and v do not intersect. Hence $\mathcal{F}$ contains at most one word from each orbit. This yields

$$|\mathcal{F}| \leq |\{O(w) \,:\, w \in \mathcal{I}_n^k\}| = \frac{|\mathcal{I}_n^k|}{|O(w)|} = \frac{|\mathcal{I}_n^k|}{n} = \frac{(n-1)!}{(n-k)!}.$$

If equality holds, then $\mathcal{F}$ must contain precisely one word from each orbit. $\square$

Brockman & Kay considered intersecting subsets of $\mathcal{I}_n^k$ recently in [4]. They use a Katona-type argument [12] involving cyclic permutations to prove the bound of Theorem 2.1, but make no attempt at our structural result: that up to permutations of $[k]$ and $[n]$, the fix-family is the only maximum intersecting subset of $\mathcal{I}_n^k$. We prove this for small k in the next section by investigating some simple consequences of the orbit approach.

Note that if $k = n$ then $\mathcal{I}_n^k = \mathcal{S}_n$ and our structural result is equivalent to the main result of [5]. The case $n \leq 2$ is trivial. Thus we assume $1 \leq k < n$ and $n \geq 3$ in all remaining proofs in this section.

2.1. Classification for Small Domains

We say that two words a, b in $\mathcal{I}_n^k$ *strictly t-intersect* if they t-intersect, but do not $(t+1)$-intersect.

Lemma 2.2. *If $\mathcal{F}$ is a maximal intersecting subset of $\mathcal{I}_n^k$ for $1 \leq k \leq n$, then there exist two words $a, b \in \mathcal{F}$ which strictly 1-intersect.*

Proof. By Theorem 2.1, $\mathcal{F}$ contains precisely one word from $O(12\ldots k)$. Denote this word by c and let c_1 be the first letter in c. Since $\mathcal{F}$ is equivalent to

$$\mathcal{F}' = \mathcal{F}\pi^{n-c_1+1},$$

it suffices to prove the lemma about $\mathcal{F}'$, which contains $12\ldots k$.

Now $\mathcal{F}'$ contains precisely one word from $O(n(n-1)\ldots(n-k+1))$ by Theorem 2.1. Denote this word by u and set $v := 12\ldots k$. There are only two fundamentally different forms which u can take.

Suppose firstly that u is strictly decreasing, so $u = l(l-1)\ldots(l-k+1)$ for some $l \in [n]$. Then if u and v intersect in position p, we must have $u_p = v_p = p$ since $v_i = i$ for all i. In v, all entries in positions left of p are strictly less than p. In u on the other hand, all entries in positions left of p are strictly greater than p. Therefore, u and v cannot intersect in any position left of p. Similarly, u and v cannot intersect anywhere to the right of position p, so u and v strictly 1-intersect.

If u is not strictly decreasing then $u_j = 1$ for some $j \in [k-1]$ and

$$u = j(j-1)\ldots 1\, n(n-1)\ldots(n-k+j+1).$$

In this case, there is only one among the first j positions in which u and v can intersect: for $p \in [j]$ we have $u_p = j - p + 1$, so $u_p = v_p$ requires

$$j - p + 1 = p \implies p = (j+1)/2$$

since $v_i = i$ for all i. For the remaining positions q with $j < q \leq k$, we have $u_q = n - q + j + 1$, so u and v can intersect only in position $q = (n+j+1)/2$.

Suppose u and v intersect in both positions p and q. Since $n > k$, the word w obtained from v by replacing $(j+1)/2$ by $k+1$ is an element of $\mathcal{I}_n^k$. The element of $O(w) \cap \mathcal{F}'$ is unique by Theorem 2.1 and must intersect v. Thus either $w \in \mathcal{F}'$ or $z \in \mathcal{F}'$ where z is the unique element of $O(w)$ which has $(j+1)/2$ in position $(j+1)/2$, that is $z = w\pi^i$ where $i = n - (k+1) + (j+1)/2$.

Since u and v strictly 2-intersect and one of their intersecting positions is $(j+1)/2$, u and w strictly 1-intersect. Also, v and w only differ in one position, so v and $z = w\pi^i$ strictly 1-intersect. Thus in the case $w \in \mathcal{F}'$, the Lemma is satisfied with $(a, b) = (u, w)$ and if $z \in \mathcal{F}'$, then the result holds with $(a, b) = (v, z)$. □

Simply using the fact that these two strictly 1-intersecting words are in $\mathcal{F}$, it can be deduced that $\mathcal{F}$ contains a much larger set of mutually 1-intersecting elements. This is the key to the proof of the following structural result:

Theorem 2.3. *For $1 \leq k \leq (n+1)/2$, if $\mathcal{F}$ is a maximal intersecting subset of $\mathcal{I}_n^k$ then all words in $\mathcal{F}$ have a fixed position in common.*

Proof. By Lemma 2.2 there exist $\alpha, \beta \in \mathcal{F}$ such that, for some $p \in [k]$,

$$\alpha = a_1 a_2 \dots a_{p-1} c\, a_{p+1} \dots a_k, \quad \beta = b_1 b_2 \dots b_{p-1} c\, b_{p+1} \dots b_k$$

with $a_i \neq b_i$ for all i. Let $d \in [n] \setminus \mathrm{im}(\alpha)$ and set

$$\delta = a_1 a_2 \dots a_{p-1} d\, a_{p+1} \dots a_k,$$

then $\delta \in \mathcal{I}_n^k$. Since p is the only position in which α and δ differ, there are only two words in $O(\delta)$ which intersect $\alpha \in \mathcal{F}$. These two words are δ and $\delta\pi^{c-d}$ which has c in position p. Since $d \neq c$ and $a_i \neq b_i$ for all i, it is clear that δ does not intersect $\beta \in \mathcal{F}$. Therefore $\delta\pi^{c-d} \in \mathcal{F}$ which proves that

$$X = \{\alpha\} \cup \{(a_1 a_2 \dots a_{p-1} d a_{p+1} \dots a_k)\pi^{c-d} \ : \ d \in [n] \setminus \mathrm{im}(\alpha)\}$$

is a subset of $\mathcal{F}$. We have $|[n] \setminus \mathrm{im}(\alpha)| = n - k$, so $|X \setminus \{\alpha\}| = n - k$. Moreover, α is distinct from all elements of $X \setminus \{\alpha\}$, so $|X| = n - k + 1$.

Next we show that elements of X do not mutually 2-intersect. Let us label the elements of $[n] \setminus \mathrm{im}(\alpha)$ as $d_1, d_2, \dots, d_{n-k}$ in such a way that this labelling corresponds to their ordering as natural numbers, i.e. for $i, j \in [n-k]$, we have $d_i < d_j$ whenever $i < j$. Using this notation, X consists of the following words:

$$\begin{gathered}
a_1 a_2 \dots a_{p-1}\, c\, a_{p+1} \dots a_k = \alpha \\
(a_1 a_2 \dots a_{p-1} d_1 a_{p+1} \dots a_k)\pi^{c-d_1} \\
(a_1 a_2 \dots a_{p-1} d_2 a_{p+1} \dots a_k)\pi^{c-d_2} \\
\vdots \\
(a_1 a_2 \dots a_{p-1} d_{n-k} a_{p+1} \dots a_k)\pi^{c-d_{n-k}}
\end{gathered}$$

All of the above words have c in position p. Since $d_i \neq c$ and the d_i are distinct for all $i \in [n-k]$, it is apparent from the above list that X is a set of $n-k+1$ elements all of which mutually strictly 1-intersect.

Now suppose there exists $w \in \mathcal{F}$ such that $w(p) \neq c$. Since two distinct elements of X do not intersect in any position other than p, w can intersect at most one element of X in position i, for any $i \in [k]$. Since w does not intersect any element of X in position p, this implies that w intersects at most $k-1$ elements of X. Since $k < (n+2)/2$, this gives $k - 1 < |X|$. Thus w does not intersect all elements of X, contradicting the intersecting property of $\mathcal{F}$. We conclude that $w(p) = c$ for all $w \in \mathcal{F}$. $\square$

We complete this classification for $n/2 < k \leq n$ by extending the methods of [5] from permutations to general injections. The following section presents an abbreviated version of this work. In cases where technical details have been missed out, a fuller discussion can be found in the thesis of the first author.

2.2. Classification for Large Domains

For an injection $w \in \mathcal{I}_n^k$, its *fixed point set* is the set of points in $[k]$ which are fixed under w. That is,

$$\mathrm{Fix}(w) = \{x \in [k] \ : \ w(x) = x\}$$

and if S is a subset of $\mathcal{I}_n^k$ then $\mathrm{Fix}(S) = \{\mathrm{Fix}(w) \ : \ w \in S\}$.

Definition 2.4 introduces a fixing operation based on traditional shifting maps. Intuitively, for $x \in [n]$ and $w \in \mathcal{I}_n^k$, we obtain the injection $f(w, x)$ from w as follows: no changes are made if x is already fixed under w, or cannot be fixed because it is not an element of the domain. If no point maps to x under w, then we may fix x without having to make any further changes. Finally, if some point $y \in [k]$ maps to x then we swap the images of y and x. This map f on injections combines previous fixing maps for words and permutations: the naive 'insertion' of the second case is based on fixing maps for words in e.g. [14, 2], while the swapping map for the permutation case corresponds to that in Cameron & Ku's paper [5].

To formalise this fixing operation, we use the image notation for injections: in Definition 2.4, the image point is given underneath the corresponding domain point.

Definition 2.4. Let $x \in [n]$ and $w \in \mathcal{I}_n^k$.

- If either $x \leq k$ and $w(x) = x$, or if $x > k$, then $f(w, x) = w$.
- If $x \leq k$ and $x \notin \mathrm{im}(w)$, then

$$f(w,x) = \begin{pmatrix} x & \lambda \\ x & w(\lambda) \end{pmatrix}, \quad \lambda \in [k] \setminus \{x\}.$$

- If $x \leq k$ and $w(y) = x$ for some $y \in [k]$ with $y \neq x$, then

$$f(w,x) = \begin{pmatrix} x & y & \lambda \\ x & w(x) & w(\lambda) \end{pmatrix}, \quad \lambda \in [k] \setminus \{x, y\}.$$

Then $f(w, x)$ is an injection in $\mathcal{I}_n^k$ which fixes x.

We may apply a sequence of fixing operations by using the inductive definition

$$f(w; x_1, \ldots, x_q) = f(f(w; x_1, \ldots, x_{q-1}), x_q).$$

If S is a subset of $\mathcal{I}_n^k$ such that $f(w, x) \in S$ for all $x \in [n]$ and $w \in S$, then we say that S is *closed under the fixing operation.*

Analogues of the following result are standard in the study of t-intersecting families of combinatorial structures other than sets, see for instance [9, 2, 5]. It shows how fixed point sets together with the fixing operation may enable us to build on the theory of t-intersecting set families.

Theorem 2.5. *If $\mathcal{F}$ is a t-intersecting subset of $\mathcal{I}_n^k$ which is closed under the fixing operation then* $\mathrm{Fix}(\mathcal{F})$ *is t-intersecting.*

Proof. Suppose $\mathrm{Fix}(\mathcal{F})$ is not t-intersecting. Then there exist $v, w \in \mathcal{F}$ with $|\,\mathrm{Fix}(v) \cap \mathrm{Fix}(w)| < t$. Note that

$$\mathrm{int}(v, w) = \{x_1, x_2, \ldots, x_s\}$$

has size $t > 0$ and that $u = f(v; x_1, \ldots, x_s) \in \mathcal{F}$ since $\mathcal{F}$ is closed under the fixing operation. We will show that u cannot t-intersect w.

First we consider positions $y \in [k] \setminus \mathrm{int}(v, w)$. It follows from Definition 2.4 that for an injection $a \in \mathcal{I}_n^k$ and points $x, z \in [k]$, if the images of z under a and $f(a; x)$ are different, then we must have either $z = x$ or $a(z) = x$. Thus unless v maps y to one of the points x_i which we are trying to fix, the image of y remains unchanged: if $v(y) \notin \mathrm{int}(v, w)$ then

$$u(y) = v(y) \neq w(y)$$

since $y \notin \mathrm{int}(v, w)$, as claimed.

If on the other hand $v(y) \in \mathrm{int}(v, w)$, say $v(y) = x_l$, then

$$\begin{aligned} f(v; x_1, \ldots, x_{l-1})(y) &= v(y) = x_l, \\ f(v; x_1, \ldots, x_l)(y) &= f(v; x_1, \ldots, x_{l-1})(x_l). \end{aligned}$$

Now whether or not the image of y is changed again under the fixing operation depends on whether or not $f(v; x_1, \ldots, x_l)(y)$ is one of the elements of $\mathrm{int}(v, w)$ which have not yet been fixed. In any case, we end up with $u(y) = v(z)$ for some $z \in \mathrm{int}(v, w)$. Therefore

$$\begin{aligned} u(y) &= v(z) \ \text{ for some } z \in \mathrm{int}(v, w) \\ &= w(z) \ \text{ by definition of } \mathrm{int}(v, w) \\ &\neq w(y) \end{aligned}$$

since $y \notin \mathrm{int}(v, w)$ implies $y \neq z$. We have shown that u and w do not intersect in positions $y \in [k] \setminus \mathrm{int}(v, w)$.

Finally, suppose $u(x_i) = w(x_i)$ for some $i \in [s]$. Then since u fixes all elements of $\operatorname{int}(v, w)$, we have

$$x_i = u(x_i) = w(x_i) = v(x_i)$$

because $x_i \in \operatorname{int}(v, w)$. Since $|\operatorname{Fix}(v) \cap \operatorname{Fix}(w)| < t$, this can occur for at most $t - 1$ values of i.

Hence u and w do not t-intersect, so the result follows from this contradiction to the t-intersection property of $\mathcal{F}$. $\square$

The notion of $\operatorname{Fix}(\mathcal{F})$ provides a map from injections to sets. Conversely, we introduce a map $\mathcal{V}$ which can be regarded as a map from sets back to injections. For a subset A of $[k]$, we denote by $\mathcal{V}(A)$ the set of injections in $\mathcal{I}_n^k$ which fix all elements of A:

$$\mathcal{V}(A) = \left\{ v \in \mathcal{I}_n^k : A \subseteq \operatorname{Fix}(v) \right\}.$$

Note that individual injections in $\mathcal{V}(A)$ may fix more points than just the elements of A. For a family $\mathcal{A}$ of subsets of $[k]$, we have $\mathcal{V}(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} \mathcal{V}(A)$.

If $\mathcal{F}$ is a t-intersecting subset of $\mathcal{I}_n^k$, we refer to the set of minimal elements of $\operatorname{Fix}(\mathcal{F})$ under set inclusion by

$$\mathcal{M}(\mathcal{F}) = \{ X \in \operatorname{Fix}(\mathcal{F}) : \text{no element of } \operatorname{Fix}(\mathcal{F}) \text{ is strictly contained in } X \}.$$

The following lemma clarifies why $\mathcal{M}(\mathcal{F})$ can be considered to 'generate' $\mathcal{F}$ in some sense. Again, similar results can be found in [2, 5].

Lemma 2.6. *If $\mathcal{F}$ is a maximal t-intersecting subset of $\mathcal{I}_n^k$ such that* $\operatorname{Fix}(\mathcal{F})$ *is t-intersecting then $\mathcal{F} = \mathcal{V}(\mathcal{M}(\mathcal{F}))$ and*

$$|\mathcal{F}| \le \sum_{X \in \mathcal{M}(\mathcal{F})} \frac{(n - |X|)!}{(n-k)!}.$$

Proof. The fixed point set of any element w of $\mathcal{F}$ contains some element X of $\mathcal{M}(\mathcal{F})$, so $w \in \mathcal{V}(X) \subseteq \mathcal{V}(\mathcal{M}(\mathcal{F}))$. For the reverse containment, let $w \in \mathcal{V}(\mathcal{M}(\mathcal{F}))$ and X be an element of $\mathcal{M}(\mathcal{F})$ such that $w \in \mathcal{V}(X)$. Since X is an element of the t-intersecting set $\operatorname{Fix}(\mathcal{F})$, we have $|X \cap Y| \ge t$ for all $Y \in \operatorname{Fix}(\mathcal{F})$, so w t-intersects all elements of $\mathcal{F}$. Since $\mathcal{F}$ is maximal, this implies $w \in \mathcal{F}$.

Hence $\mathcal{F} = \mathcal{V}(\mathcal{M}(\mathcal{F}))$, giving

$$|\mathcal{F}| \le \sum_{X \in \mathcal{M}(\mathcal{F})} |\mathcal{V}(X)| = \sum_{X \in \mathcal{M}(\mathcal{F})} \frac{(n - |X|)!}{(n-k)!},$$

as required. $\square$

We prove the main result of this section by building on the latin square approach of [5]. We define the ith k-row of a latin square L to be the word of length k obtained by taking the first k symbols of the ith row of L. It can then be shown (using the clique-coclique bound of [6]) that if $\mathcal{F}$ is a maximum intersecting subset of $\mathcal{I}_n^k$ then $\mathcal{F}$ contains exactly one k-row of each latin square of order n. Using this result and extending some specific elements of $\mathcal{I}_n^k$ to elements of $\mathcal{S}_n$ yields Theorem 2.7 (cf. the proof of the analogous result in [5]). The subsequent lemma follows from the so-called LYM inequality; see [5] for details.

Theorem 2.7. *If $n \geq 6$ and $\mathcal{F}$ is a maximum intersecting subset of $\mathcal{I}_n^k$ containing $12\dots k$ then $\mathcal{F}$ is closed under the fixing operation.*

Lemma 2.8. *If $\mathcal{Z}$ is an antichain of subsets of a k-set such that $|A| \geq j$ for all $A \in \mathcal{Z}$ then $\sum_{A\in\mathcal{Z}}(k-|A|)! \leq k!/j!$.*

We are now in a position to complete the classification of maximum intersecting injection families.

Theorem 2.9. *For $n/2 < k \leq n$, let $\mathcal{F}$ be a maximum intersecting subset of $\mathcal{I}_n^k$. Then all words in $\mathcal{F}$ have a fixed position in common.*

Proof. Note that applying permutations of $\mathcal{S}_n$ to a subset of $\mathcal{I}_n^k$ does not alter the cardinality or intersecting structure of that subset, so we can assume without loss of generality that $\mathcal{F}$ contains the identity $12\dots k$. Moreover, if $k = n$ then Theorem 2.9 is equivalent to the main result of [5], so we assume $k < n$. Lastly, small values of k and n can be checked in an elementary case analysis by hand or using a computational package such as GAP [10], so we will assume within the proof that $n \geq 6$ and $k \geq 4$.

By Theorems 2.7 and 2.5, $\mathrm{Fix}(\mathcal{F})$ is intersecting. Moreover, $12\dots k \in \mathcal{F}$ and so $\mathcal{M}(\mathcal{F})$ is a non-empty, intersecting antichain of subsets of $[k]$. We will establish bounds on the size of the elements of $\mathcal{M}(\mathcal{F})$. Since $\mathrm{Fix}(\mathcal{F})$ is intersecting, $\emptyset \notin \mathrm{Fix}(\mathcal{F})$. Moreover, if $\mathrm{Fix}(\mathcal{F})$ contains an element of size 1, then Theorem 2.9 follows by the intersection property of $\mathrm{Fix}(\mathcal{F})$. Thus we may assume that all elements of $\mathcal{M}(\mathcal{F})$ have size at least 2.

Pursuing a similar argument, if $Y = \bigcap_{X\in\mathcal{M}(\mathcal{F})} X$ is non-empty, then all elements of $\mathcal{F}$ fix all elements of Y, and Theorem 2.9 is immediate. We therefore assume $\bigcap_{X\in\mathcal{M}(\mathcal{F})} X = \emptyset$, implying $|\mathcal{M}(\mathcal{F})| \geq 2$. Since $\mathcal{M}(\mathcal{F})$ is an antichain of subsets of $[k]$, this gives $[k] \notin \mathcal{M}(\mathcal{F})$, and we have shown that all $X \in \mathcal{M}(\mathcal{F})$ satisfy $2 \leq |X| \leq k-1$.

For the remainder of this proof, the aim is to derive a contradiction to the assumption that $\mathcal{F}$ attains the bound given in Theorem 2.1, but there exists no $i \in [k]$ such that $w(i) = i$ for all $w \in \mathcal{F}$. As in [5], we consider two cases.

Case 1 *$\mathcal{M}(\mathcal{F})$ contains no element of size 2.*

By Lemma 2.6 we have

$$\begin{aligned}
|\mathcal{F}|\cdot(n-k)! &\le \sum_{X\in\mathcal{M}(\mathcal{F})}(n-|X|)! \\
&= \sum_{\substack{X\in\mathcal{M}(\mathcal{F})\\ 3\le|X|\le\lfloor k/2\rfloor}}(n-|X|)! + \sum_{\substack{X\in\mathcal{M}(\mathcal{F})\\ |X|>\lfloor k/2\rfloor}}(n-|X|)! \\
&\le \sum_{i=3}^{\lfloor k/2\rfloor}|\mathcal{M}^{(i)}(\mathcal{F})|(n-i)! + \frac{n!}{(\lfloor k/2\rfloor+1)!}
\end{aligned}$$

where $|\mathcal{M}^{(i)}(\mathcal{F})|$ is the number of elements in $\mathcal{M}(\mathcal{F})$ of size i. The inequality follows from Lemma 2.8 upon noting that $X\subseteq[k]\subset[n]$ for all $X\in\mathcal{M}(\mathcal{F})$. Using the Erdős-Ko-Rado Theorem 1.2, this inequality becomes

$$|\mathcal{F}|\cdot(n-k)! \le \sum_{i=3}^{\lfloor k/2\rfloor}\binom{k-1}{i-1}(n-i)! + \frac{n!}{(\lfloor k/2\rfloor+1)!}.$$

We are assuming that $|\mathcal{F}|=(n-1)!/(n-k)!$, so this gives

$$(n-1)! \le \sum_{i=3}^{\lfloor k/2\rfloor}\frac{(k-1)!(n-i)!}{(i-1)!(k-i)!} + \frac{n!}{(\lfloor k/2\rfloor+1)!}. \tag{1}$$

Let us denote the right hand side of (1) by $f(n,k)$.

To provide the required contradiction to (1), straightforward numerical calculation demonstrates that $f(n,k)<(n-1)!$ for $n<16$, unless

$$(n,k)\in\{(6,4),(6,5),(7,4),(7,5),(8,5),(9,5)\}.$$

These special cases have been checked by a more involved recursive algorithm using GAP [10], see the thesis of the first author for details. For the remainder of Case 1, we therefore assume $n\ge16$.

Since $k<n$, we have

$$\begin{aligned}
f(n,k) &= \sum_{i=3}^{\lfloor k/2\rfloor}\frac{(k-1)(k-2)\dots(k-i+1)(n-i)!}{(i-1)!} + \frac{n!}{(\lfloor k/2\rfloor+1)!} \\
&< (n-1)!\sum_{i=3}^{\lfloor k/2\rfloor}\frac{1}{(i-1)!} + \frac{n!}{(\lfloor k/2\rfloor+1)!} \\
&\le (n-1)!\sum_{i=3}^{\lfloor n/2\rfloor}\frac{1}{(i-1)!} + \frac{n!}{(\lfloor k/2\rfloor+1)!}.
\end{aligned}$$

Now if e is the natural exponent then $e=2+\sum_{i=3}^{\infty}\frac{1}{(i-1)!}$, so $\sum_{i=3}^{\lfloor n/2\rfloor}\frac{1}{(i-1)!}<$

$e-2<\frac{4}{5}$. Since $k>n/2$, this gives

$$\begin{aligned} f(n,k) &< (n-1)!\cdot\frac{4}{5}+\frac{n!}{(\lfloor k/2\rfloor+1)!} \\ &< (n-1)!\cdot\frac{4}{5}+\frac{n!}{(\lfloor n/4\rfloor+1)!}=(n-1)!\left(\frac{4}{5}+\frac{n}{(\lfloor n/4\rfloor+1)!}\right). \end{aligned}$$

It is easily verified that $\frac{n}{(\lfloor n/4\rfloor+1)!}<\frac{1}{5}$ for $n\geq 16$, so $f(n,k)<(n-1)!$, giving the required contradiction to (1).

Case 2 $\mathcal{R}_2=\{X\in\mathcal{M}(\mathcal{F})\ :\ |X|=2\}$ *is non-empty.*

If $\bigcap_{X\in\mathcal{R}_2}X=\emptyset$ then, by the intersection property of $\mathcal{M}(\mathcal{F})$, there exist distinct $a,b,c\in[k]$ such that $\{\{a,b\},\{a,c\},\{b,c\}\}\subseteq\mathcal{R}_2$. Suppose there exists $X\in\mathcal{M}(\mathcal{F})\setminus\{\{a,b\},\{a,c\},\{b,c\}\}$. Since $X\cap\{b,c\}\neq\emptyset$, we have either $b\in X$ or $c\in X$. This implies $a\notin X$ because otherwise either $\{a,b\}\subseteq X$ or $\{a,c\}\subseteq X$ which would contradict the fact that $\mathcal{M}(\mathcal{F})$ is an antichain. However, we must also have $X\cap\{a,b\}\neq\emptyset$ and $X\cap\{a,c\}\neq\emptyset$, so $a\notin X$ implies $\{b,c\}\subseteq X$ which again contradicts the antichain property of $\mathcal{M}(\mathcal{F})$. We conclude

$$\mathcal{M}(\mathcal{F})=\mathcal{R}_2=\{\{a,b\},\{a,c\},\{b,c\}\}$$

and applying Lemma 2.6 gives

$$|\mathcal{F}|\leq\sum_{X\in\mathcal{M}(\mathcal{F})}\frac{(n-|X|)!}{(n-k)!}=3\frac{(n-2)!}{(n-k)!}<\frac{(n-1)!}{(n-k)!}$$

for $n\geq 5$, giving the contradiction $|\mathcal{F}|<|\mathcal{F}|$.

Hence we must have $\bigcap_{X\in\mathcal{R}_2}X\neq\emptyset$, so we may assume without loss of generality that

$$\mathcal{R}_2=\{\{1,i\}:\ 2\leq i\leq c\}$$

for some $c\in\{2,3,\ldots,k\}$. Set

$$\mathcal{Y}=\{X\in\mathcal{M}(\mathcal{F})\setminus\mathcal{R}_2\ :\ 1\in X\},\quad \mathcal{N}=\{X\in\mathcal{M}(\mathcal{F})\setminus\mathcal{R}_2\ :\ 1\notin X\}.$$

Then it follows from the definition of $\mathcal{Y}\subset\mathcal{M}(\mathcal{F})$ that all $Y\in\mathcal{Y}$ contain 1 as well as two distinct elements x, y of $[k]\setminus[c]$ since $\mathcal{M}(\mathcal{F})$ is an antichain. If $Y\in\mathcal{Y}$ and $w\in\mathcal{I}_n^k$ is a word whose fixed point set $\mathrm{Fix}(w)$ contains Y, then $w\in\mathcal{V}(\{1,x,y\})$.

By the intersection property of $\mathcal{M}(\mathcal{F})\supseteq\mathcal{R}_2$, we have $\{2,3,\ldots,c\}\subseteq N$ for all $N\in\mathcal{N}$. Thus if $w\in\mathcal{I}_n^k$ is a word whose fixed point set $\mathrm{Fix}(w)$ contains $N\in\mathcal{N}$, then $w\in\mathcal{V}(\{2,3,\ldots,c\})$. By an argument analogous to the proof of Lemma 2.6, we therefore have

$$\begin{aligned} |\mathcal{F}| &\leq \sum_{X\in\mathcal{R}_2}\frac{(n-|X|)!}{(n-k)!}+\sum_{\substack{x\neq y\\ x,y\in\{c+1,\ldots,k\}}}|\mathcal{V}(\{1,x,y\})|\ +\ |\mathcal{V}(\{2,3,\ldots,c\})| \\ &= (c-1)\frac{(n-2)!}{(n-k)!}+\binom{k-c}{2}\frac{(n-3)!}{(n-k)!}+\frac{(n-c+1)!}{(n-k)!}. \end{aligned}$$

Since $|\mathcal{F}| = \frac{(n-1)!}{(n-k)!}$, this may be simplified to

$$(n-1)! \ \le \ (c-1)(n-2)! + \binom{k-c}{2}(n-3)! + (n-c+1)!. \tag{2}$$

We will now investigate the range of values which c can take. Firstly, suppose $3 \le c \le k-2$. Then $(n-c+1)! \le (n-2)!$ and so

$$(n-1)! \le c(n-2)! + \binom{k-c}{2}(n-3)! := f(c).$$

Now $c > 2$ implies $\frac{n-c}{2} < n-2$, giving $\frac{(n-c)(n-c-1)}{2} < (n-2)(n-c-1)$ since $n-c-1 > 0$. Using $k < n$, this implies

$$\binom{k-c}{2}(n-3)! < \binom{n-c}{2}(n-3)! < (n-2)!(n-c-1)$$

which yields $f(c) < (n-1)!$. We now have the contradiction $(n-1)! \le f(c) < (n-1)!$, so we conclude that we cannot have $3 \le c \le k-2$.

Next suppose $c \ge k-1$. Recall that all $Y \in \mathcal{Y}$ satisfy $\{1, x, y\} \subseteq Y$ for some distinct $x, y \in [k] \setminus [c]$. Thus $c \ge k-1$ implies $\mathcal{Y} = \emptyset$ and $\mathcal{M}(\mathcal{F}) = \mathcal{R}_2 \cup \mathcal{N}$. If $c = k-1$ then

$$\begin{aligned} |\mathcal{F}| \ &\le \ \sum_{X \in \mathcal{R}_2} \frac{(n-|X|)!}{(n-k)!} \ + \ |\mathcal{V}(\{2, 3, \ldots, k-1\})| \\ &= \ (k-2)\frac{(n-2)!}{(n-k)!} \ + \ (n-k+2)(n-k+1) \end{aligned}$$

and multiplying through by $(n-k)!$ gives

$$\begin{aligned} (n-1)! \ &\le \ (k-2)(n-2)! \ + \ (n-k+2)! \\ &< \ (n-2)(n-2)! \ + \ (n-2)! \ = \ (n-1)! \end{aligned}$$

since $4 \le k < n$. Similarly, if $c = k$ then

$$\begin{aligned} |\mathcal{F}| \ &\le \ |\mathcal{R}_2| \cdot \frac{(n-2)!}{(n-k)!} \ + \ |\mathcal{V}(\{2, 3, \ldots, k\})| \\ &= \ (k-1)\frac{(n-2)!}{(n-k)!} \ + \ (n-k+1)(n-k), \end{aligned}$$

so

$$\begin{aligned} (n-1)! \ &\le \ (k-1)(n-2)! \ + \ (n-k)(n-k+1)! \\ &< \ (k-1)(n-2)! \ + \ (n-k)(n-2)! \ = \ (n-1)!. \end{aligned}$$

It follows from these contradictions that $c = 2$.

Hence we have $\mathcal{R}_2 = \{\{1, 2\}\}$ which implies $\mathcal{M}(\mathcal{F}) = \mathcal{R}_2 \cup \mathcal{B}_1 \cup \mathcal{B}_2$ where

$$\mathcal{B}_1 = \{X \in \mathcal{M}(\mathcal{F}) \setminus \mathcal{R}_2 \ : \ 1 \in X\}, \quad \mathcal{B}_2 = \{X \in \mathcal{M}(\mathcal{F}) \setminus \mathcal{R}_2 \ : \ 2 \in X\}.$$

Since $\mathcal{M}(\mathcal{F})$ is an antichain, $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$. Moreover, for $i \in \{1, 2\}$, each $X \in \mathcal{B}_i$ satisfies $\{i, a, b\} \subseteq X$ for some $a, b \in [k] \setminus \{1, 2\}$. Therefore we deduce

$$\begin{aligned} |\mathcal{F}| &\leq \sum_{X \in \mathcal{R}_2} \frac{(n-|X|)!}{(n-k)!} + \sum_{\substack{a \neq b \\ a,b \in [k]\setminus\{1,2\}}} |\mathcal{V}(\{1, a, b\})| + \sum_{\substack{a \neq b \\ a,b \in [k]\setminus\{1,2\}}} |\mathcal{V}(\{2, a, b\})| \\ &= \frac{(n-2)!}{(n-k)!} + 2\binom{k-2}{2}\frac{(n-3)!}{(n-k)!}, \end{aligned}$$

and simplifying yields the usual contradiction:

$$\begin{aligned} (n-1)! &\leq (n-2)! + 2\binom{k-2}{2}(n-3)! \\ &< (n-2)! + 2\binom{n-2}{2}(n-3)! \;<\; (n-1)!. \end{aligned}$$

We started the proof by assuming that not all elements of $\mathcal{F}$ have a fixed position in common. We have shown that this assumption leads to a contradiction in all possible cases, so the result now follows. □

Corollary 2.10. *If $\mathcal{F}$ is a maximum intersecting subset of $\mathcal{I}_n^k$ then all words in $\mathcal{F}$ have a fixed position in common.*

Proof. Follows from Theorems 2.3 and 2.9. □

Having completed the classification of maximum intersecting injection families, we now turn our attention to t-intersecting injection families. In the following section we prove that fixing is eventually optimal for all t and k, provided n is large.

3. Arbitrary Intersection Size: Classifications in the Limit

3.1. Injections with Large Images

We use a version of the so-called *kernel method* as presented in [19], where Meagher & Moura attribute the origins of this method to Hajnal & Rothschild [11].

Lemma 3.1. *Let $\mathcal{F}$ be a t-intersecting subset of $\mathcal{I}_n^k$. If there do not exist $x \in [k]$, $y \in [n]$ such that all elements of $\mathcal{F}$ map x to y, then*

$$|\mathcal{F}| \leq \frac{k!(n-t-1)!}{t!(k-t-1)!(n-k)!}.$$

Proof. Let $\alpha \in \mathcal{F}$. By assumption, there exists $x \in [k]$ and $\beta \in \mathcal{F}$ such that $\alpha(x) \neq \beta(x)$. Setting

$$\mathcal{F}_{\alpha(x)} = \{\, \gamma \in \mathcal{F} : \gamma(x) = \alpha(x) \,\},$$

it is then clear that $\operatorname{int}(\gamma, \beta) \subseteq [k] \setminus \{x\}$ for all $\gamma \in \mathcal{F}_{\alpha(x)}$. On the other hand, $\operatorname{int}(\gamma, \beta)$ has size at least t and so

$$|\mathcal{F}_{\alpha(x)}| \leq \binom{k-1}{t} \frac{(n-(t+1))!}{(n-k)!}.$$

By the intersecting property of $\mathcal{F}$, we have $\mathcal{F} = \bigcup_{x=1}^{k} \mathcal{F}_{\alpha(x)}$, giving

$$|\mathcal{F}| \leq \sum_{x=1}^{k} |\mathcal{F}_{\alpha(x)}| = k \cdot |\mathcal{F}_{\alpha(x)}| = \frac{k!(n-t-1)!}{t!(k-t-1)!(n-k)!}$$

as required. □

Theorem 3.2. *Let $1 \leq t \leq k \leq n$ and suppose that*

$$(k-c)! < (n-t)(t-c)!(k-t-1)!$$

for all $0 \leq c < t$. Then any maximum t-intersecting subset of $\mathcal{I}_n^k$ is equivalent to the fix-family $\mathcal{K}_0(t, k, n)$.

Proof. Let $\mathcal{F}$ be a t-intersecting subset of $\mathcal{I}_n^k$ which is not equivalent to the fix-family $\mathcal{K}_0(t, k, n)$. It suffices to show that $|\mathcal{F}| < |\mathcal{K}_0(t, k, n)| = (n-t)!/(n-k)!$.

Let $\mathcal{C}$ be the intersection of all elements of $\mathcal{F}$, so

$$\mathcal{C} = \{\, (x, y) \in [k] \times [n] : \alpha(x) = y \text{ for all } \alpha \in \mathcal{F} \,\},$$

and set $c = |\mathcal{C}|$, $X = \{\, x : (x, y) \in \mathcal{C} \,\}$ and $Y = \{\, y : (x, y) \in \mathcal{C} \,\}$. Then $0 \leq c < t$ since $\mathcal{F}$ is not a fix-family.

Let $\mathcal{F}'$ be the family obtained from $\mathcal{F}$ by first deleting all elements of $\mathcal{C}$ from each element of $\mathcal{F}$, and then relabelling $[k] \setminus X$ and $[n] \setminus Y$ to eliminate the resulting gaps. Then $\mathcal{F}'$ is a $(t-c)$-intersecting subset of $\mathcal{I}_{n-c}^{k-c}$ with $|\mathcal{F}'| = |\mathcal{F}|$. Thus we may employ Lemma 3.1 to obtain

$$\begin{aligned} |\mathcal{F}| = |\mathcal{F}'| \quad &\leq \quad \frac{(k-c)!(n-t-1)!}{(t-c)!(k-t-1)!(n-k)!} \\ &= \frac{(k-c)!}{(n-t)(t-c)!(k-t-1)!} \cdot \frac{(n-t)!}{(n-k)!} \end{aligned}$$

as required. □

Corollary 3.3. *There exists a function $n_0(k, t) : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $n > n_0(k, t)$, every maximum t-intersecting subset of $\mathcal{I}_n^k$ is equivalent to the fix-family $\mathcal{K}_0(t, k, n)$.*

Proof. Given $0 \le c < t \le k \le n$, we have $(k-c)! \le k!$ and $(t-c)! \ge 1$, and these bounds cannot be simultaneously achieved since c is fixed. Thus if

$$k! \le (n-t)(k-t-1)! \tag{3}$$

then

$$(k-c)! \le k! \le (n-t)(k-t-1)! \le (n-t)(k-t-1)!(t-c)!$$

and one of these inequalities is strict. By Theorem 3.2, inequality (3) therefore implies that no t-intersecting subset of $\mathcal{I}_n^k$ is larger than $\mathcal{K}_0(t,k,n)$. For fixed k and t, inequality (3) can clearly be achieved by taking

$$n > n_0(k,t) = t + \frac{k!}{(k-t-1)!},$$

which completes the proof. □

We have shown that t-intersecting injection families eventually behave like 1-intersecting families in the sense that for large n, fixing is the unique optimal strategy. Note however, that Corollary 3.3 is a result strictly about injections, not including the case of permutations, since n is required to be large in terms of t as well as k.

The remainder of this section is devoted to generalising Theorem 1.3 to injection families: Theorem 3.5 classifies the optimal t-intersecting injection families for large k, given that both $k-t$ and $n-k$ are fixed. For these parameter values, Theorem 3.5 shows that fixing is not optimal, since the saturation family $\mathcal{G}$ is not equivalent to the fix-family $\mathcal{K}_0$ (see definition of $\mathcal{G}$ on page 5).

3.2. Injections with Large Domains

Denoting the number of injections from $[k]$ to $[n]$ with no fixed points by $d(k,n)$, we have

$$d(k,n) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{(n-i)!}{(n-k)!} \tag{4}$$

by the inclusion-exclusion principle. (This requires the convention that there is one injection with no fixed points from the empty set into any other set.) Moreover, it is not difficult to show that

$$a \ge 1 \implies d(a+1, b+1) > d(a,b) \tag{5}$$

for $0 \le a \le b$.

Lemma 3.4 establishes a lower bound for the size of $\mathcal{G}(t,k,n)$. The *moved point set* of an injection $w \in \mathcal{I}_n^k$ is defined as

$$E(w) = \{\, x \in [k] : w(x) \ne x \,\}\,.$$

If S is a subset of $\mathcal{I}_n^k$ then $E(S) = \{E(w) \;:\; w \in S\}$ is a family of subsets of $[k]$.

Lemma 3.4. *For fixed natural numbers T, N and $c_{N,T}$ with $T \geq 2$, there exists $k_0(T, N) \in \mathbb{N}$ such that*

$$|\mathcal{G}(k-T, k, k+N)| > c_{N,T} \sum_{j=0}^{\lfloor T/2 \rfloor - 1} \binom{k}{j}$$

for all $k \geq k_0(T, N)$.

Proof. Setting $t = k - T$ and $n = k + N$, we abbreviate $\mathcal{G}(t, k, n)$ by $\mathcal{G}$.

Suppose $T = 2h + 1$ is odd. Let

$$\mathcal{B} = \{ X \subseteq [k] : |X| \leq h \} \cup \{ X \subseteq [k] : |X| = h+1, \ k \in X \},$$

then $\mathcal{G}$ is a disjoint union

$$\mathcal{G} = \bigcup_{X \in \mathcal{B}} \{ w \in \mathcal{I}_n^k : E(w) = X \}. \tag{6}$$

For given $X \subseteq [k]$, an injection w in $\mathcal{I}_n^k$ with $E(w) = X$ must fix all elements of $[k] \setminus X$, so the image points of X under w are all in $[n] \setminus ([k] \setminus X)$. Hence

$$\begin{aligned} |\mathcal{G}| &= \sum_{X \in \mathcal{B}} |\{ w \in \mathcal{I}_n^k : E(w) = X \}| = \sum_{X \in \mathcal{B}} d(|X|, n - (k - |X|)) \\ &= \sum_{j=0}^{(T-1)/2} \binom{k}{j} d(j, N+j) + \binom{k-1}{h} d(h+1, N+h+1) \qquad (7) \\ &> \sum_{j=0}^{(T-1)/2} \binom{k}{j} d(j, N+j). \end{aligned}$$

Since $T \geq 2$, and both $d(j, N+j)$ and $c_{N,T}$ depend only on the constants N and T, we may choose k sufficiently large to ensure

$$|\mathcal{G}| > c_{N,T} \sum_{j=0}^{(T-1)/2-1} \binom{k}{j} = c_{N,T} \sum_{j=0}^{\lfloor T/2 \rfloor - 1} \binom{k}{j}.$$

The case when T is even is similar. $\square$

The symmetric difference $A \Delta B$ of two sets A and B is the set of points contained in one but not both of A and B, i.e. $A \Delta B = (A \cup B) \setminus (A \cap B)$. The following result generalises Theorem 1.3 from [6].

Theorem 3.5. *For positive integers T and N with $T \geq 2$, there exists $k_0(T, N) \in \mathbb{N}$ such that for $k \geq k_0(T, N)$, every maximum $(k-T)$-intersecting subset of $\mathcal{I}_{k+N}^k$ is equivalent to the saturation family $\mathcal{G}(k-T, k, k+N)$.*

Proof. Set $t = k - T$, $n = k + N$, abbreviate $\mathcal{G}(t, k, n)$ by $\mathcal{G}$ as before and let $\mathcal{F}$ be a maximum t-intersecting subset of $\mathcal{I}_n^k$. Following the proof outline of Theorem 1.3 in [6], we begin by establishing some useful technicalities (9 – 11) before picking a set $\mathcal{W}$ of $3T+1$ elements of $\mathcal{F}$ according to condition (12). We then prove that in the case where T is even, all elements of $\mathcal{W}$ act in the same way on the set of points moved by all of them, and applying the inverse of this action to the whole of $\mathcal{F}$ maps $\mathcal{F}$ into $\mathcal{G}$. In the case where T is odd we proceed similarly, though mapping $\mathcal{F}$ into $\mathcal{G}$ in this situation requires an application of the Erdős-Ko-Rado Theorem.

Without loss of generality, we may assume that the identity $12\ldots k$ is an element of $\mathcal{F}$. Then each $w \in \mathcal{F}$ must t-intersect the identity, so

$$|E(w)| \leq k - t = T, \quad \forall w \in \mathcal{F}. \tag{8}$$

Indeed, the t-intersecting property of $\mathcal{F}$ implies that for all $v, w \in \mathcal{F}$,

$$\begin{aligned} |E(v)\Delta E(w)| &\leq T, & (9)\\ |(E(v) \cap E(w)) \setminus \operatorname{int}(v, w)| &\leq T - |E(v)\Delta E(w)|. & (10) \end{aligned}$$

Pick $w_0 \in \mathcal{F}$ with $|E(w_0)|$ maximal. We wish to show that all remaining $w \in \mathcal{F}$ move at most $\lfloor T/2 \rfloor$ of the points which are fixed by w_0. So suppose the opposite holds for some $w \in \mathcal{F}$, then the maximality of $|E(w_0)|$ forces

$$|E(w_0) \setminus E(w)| \geq |E(w) \setminus E(w_0)| > \left\lfloor \frac{T}{2} \right\rfloor .$$

But this implies that the symmetric difference of $E(w)$ and $E(w_0)$ is larger than T, contradicting (9). Thus we have shown that

$$|E(w) \setminus E(w_0)| \leq \left\lfloor \frac{T}{2} \right\rfloor \tag{11}$$

for all $w \in \mathcal{F}$.

3.2.1. Picking the Elements of $\mathcal{W}$

We wish to pick $w_1 \in \mathcal{F}$ which achieves equality in (11), and subsequently continue to pick $w_{i+1} \in \mathcal{F}$ such that w_{i+1} moves exactly $\lfloor T/2 \rfloor$ points which are not moved by any of the injections $w_0, \ldots, w_i$ chosen so far:

$$\left| E(w_{i+1}) \setminus \bigcup_{j=0}^{i} E(w_j) \right| = \left\lfloor \frac{T}{2} \right\rfloor . \tag{12}$$

We will use the maximality of $\mathcal{F}$ as a t-intersecting subset of $\mathcal{I}_n^k$ to show, by contradiction, that we can pick elements of $\mathcal{F}$ in this way. Suppose that for some

$$i < 3T, \tag{13}$$

we cannot find such a w_{i+1} in $\mathcal{F}$. Then we must have

$$\left| E(w) \setminus \bigcup_{j=0}^{i} E(w_j) \right| \neq \left\lfloor \frac{T}{2} \right\rfloor$$

for all $w \in \mathcal{F}$. Also,

$$\left| E(w) \setminus \bigcup_{j=0}^{i} E(w_j) \right| \leq |E(w) \setminus E(w_0)| \leq \left\lfloor \frac{T}{2} \right\rfloor$$

by (11), and combining the previous two equations gives

$$\left| E(w) \setminus \bigcup_{j=0}^{i} E(w_j) \right| < \left\lfloor \frac{T}{2} \right\rfloor \tag{14}$$

for all $w \in \mathcal{F}$.

Note that due to condition (12) according to which the elements $w_0, \ldots, w_i$ were picked, we have

$$\left| \bigcup_{j=0}^{i} E(w_j) \right| = |E(w_0)| + i \left\lfloor \frac{T}{2} \right\rfloor < T + 3T \left\lfloor \frac{T}{2} \right\rfloor < 3T^2 \tag{15}$$

using (8, 13) and $T \geq 2$. We use these arguments to establish an upper bound on the size of

$$E(\mathcal{F}) = \{\, E(w) : w \in \mathcal{F} \,\}$$

as follows: denote $\bigcup_{j=0}^{i} E(w_j)$ by U. Then (14) tells us that each element of $\mathcal{F}$ moves less than $\lfloor T/2 \rfloor$ of the points which are not in U. Since U has less than 2^{3T^2} subsets by (15), this yields

$$|E(\mathcal{F})| < 2^{3T^2} \sum_{j=0}^{\lfloor T/2 \rfloor - 1} \binom{k}{j}. \tag{16}$$

Clearly we have

$$\mathcal{F} \subseteq \bigcup_{X \in E(\mathcal{F})} \{\, w \in \mathcal{I}_n^k : E(w) = X \,\}$$

and, since this union is disjoint,

$$\begin{aligned} |\mathcal{F}| &\leq \sum_{X \in E(\mathcal{F})} |\{\, w \in \mathcal{I}_n^k : E(w) = X \,\}| \\ &\leq \sum_{X \in E(\mathcal{F})} d(|X|, N + |X|). \end{aligned} \tag{17}$$

Recall from (5) that $|X| \geq 1$ implies $d(|X|, N + |X|) < d(|X| + 1, N + |X| + 1)$ and since $N \geq 1$ we also have

$$d(0, N) = 1 \leq d(1, N + 1) = 1.$$

Therefore we may use (8) together with the above bound on $|\mathcal{F}|$ to conclude

$$\begin{aligned} |\mathcal{F}| &\leq \sum_{X \in E(\mathcal{F})} d(T, N + T) \\ &= d(T, N + T) \cdot |E(\mathcal{F})| \\ &< d(T, N + T) 2^{3T^2} \sum_{j=0}^{\lfloor T/2 \rfloor - 1} \binom{k}{j} \end{aligned}$$

by (16). Since $c_{N,T} = d(T, N + T)2^{3T^2}$ depends only on the fixed constants T and N, Lemma 3.4 now implies that $|\mathcal{F}| < |\mathcal{G}|$. This contradicts the fact that $\mathcal{F}$ is maximum $(k - T)$-intersecting in $\mathcal{I}^k_{N+k}$, so we conclude that we can indeed pick $w_0, \dots, w_{3T}$ as described above.

Note that if $|E(w_0)| < \lfloor T/2 \rfloor$ then the maximality of $|E(w_0)|$ would force all elements of $E(\mathcal{F})$ to have size less than $\lfloor T/2 \rfloor$, making it impossible to pick the w_{i+1} according to (12). Since we have just shown that we can pick such w_{i+1} for $i < 3T$, we conclude that

$$|E(w_0)| \geq \left\lfloor \frac{T}{2} \right\rfloor \tag{18}$$

and set

$$\mathcal{W} = \{w_0, \dots, w_{3T}\}. \tag{19}$$

As in the proof of Lemma 3.4, we need to consider the possible parities of T separately.

Case 1 *$T = 2h$ is even.*

We will show that w_0 t-intersects all other elements w_i of $\mathcal{W}$ in the same t positions. In the process, we establish the sizes of the moved point sets $E(w_i)$ as well as their respective intersections and symmetric differences with $E(w_0)$.

3.2.2. The Intersection of w_0 with Other Elements of $\mathcal{W}$

By (18, 8) the number of points moved by w_0 is between h and $2h$. Thus setting

$$s = |E(w_0)| - h,$$

we have $0 \leq s \leq h$ and the maximality of $|E(w_0)|$ implies $|E(w)| \leq h + s$ for all $w \in \mathcal{F}$. Indeed, our next claim is that all $w \in \mathcal{W}$ satisfy $|E(w)| = h + s$.

For $w_i \in \mathcal{W} \subseteq \mathcal{F}$, it follows from the way the w_i were picked (12) that

$$|E(w_i) \setminus E(w_0)| \geq \left| E(w_i) \setminus \bigcup_{j=0}^{i-1} E(w_j) \right| = h.$$

Therefore setting $|E(w_i)| = h + s - j$ for some $j \geq 0$, we have

$$\begin{aligned}
|E(w_0) \setminus E(w_i)| &= |E(w_0)| - |E(w_0) \cap E(w_i)| \\
&= h + s - (|E(w_i)| - |E(w_i) \setminus E(w_0)|) \\
&= h + s - (h + s - j) + |E(w_i) \setminus E(w_0)| \\
&\geq j + h.
\end{aligned}$$

Thus the size of the symmetric difference $E(w_i)\Delta E(w_0)$ is given by

$$|E(w_i) \setminus E(w_0)| + |E(w_0) \setminus E(w_i)| \geq 2h + j = T + j.$$

Using (9, 10) this implies $j = 0$ and

$$\operatorname{int}(w_i, w_0) = [k] \setminus (E(w_i)\Delta E(w_0)), \tag{20}$$

i.e. w_0 and w_i intersect in all points which they both move. Observe that by proving $j = 0$ we have shown

$$\begin{aligned}
|E(w_i)| &= h + s, && (21) \\
|E(w_0)\Delta E(w_i)| &= T = 2h, && (22) \\
|E(w_0) \setminus E(w_i)| &= h
\end{aligned}$$

for all $w_i \in \mathcal{W}$. Therefore

$$\begin{aligned}
|E(w_i) \cap E(w_0)| &= |E(w_0)| - |E(w_0) \setminus E(w_i)| \\
&= h + s - h = s, \qquad \forall w_i \in \mathcal{W}. && (23)
\end{aligned}$$

Together with the arguments preceding (9), equation (22) implies that w_0 does not $(t + 1)$-intersect any element of $\mathcal{W} \setminus \{w_0\}$. It remains to be shown that all of these intersections coincide.

3.2.3. A Common Intersection

We concluded in (20) that w_i and w_0 agree on each point in $E(w_i) \cap E(w_0)$. Indeed, suppose that for some $w_i \in \mathcal{W}$, $i \geq 2$, we had

$$E(w_i) \cap E(w_0) \neq E(w_1) \cap E(w_0).$$

Then since both intersections have the same size by (23), we must have

$$|(E(w_i) \cap E(w_0)) \setminus (E(w_1) \cap E(w_0))| = |(E(w_i) \cap E(w_0)) \setminus E(w_1)| > 0.$$

But then

$$\begin{aligned}
|E(w_i) \cap E(w_1)| &= |E(w_i)| - |E(w_i) \setminus (E(w_0) \cup E(w_1))| \\
&\qquad - |(E(w_i) \cap E(w_0)) \setminus E(w_1)| \\
&< |E(w_i)| - |E(w_i) \setminus (E(w_0) \cup E(w_1))| \\
&\leq |E(w_i)| - \left| E(w_i) \setminus \bigcup_{j=0}^{i-1} E(w_j) \right| \\
&= h + s - h = s
\end{aligned}$$

by (21, 12). But both $E(w_i)$ and $E(w_1)$ are sets of size $h+s$ by (21), so if their intersection has size less than s, then their symmetric difference must have size greater than $2h = T$, contradicting (9).

In conclusion, there must exists a set $X \subseteq [k]$ such that

$$E(w_i) \cap E(w_0) = X, \qquad \forall i \in [3T],$$

which has size s by (23). Clearly this implies

$$X \subseteq E(w_i) \cap E(w_j), \qquad \forall i, j \in [3T] \cup \{0\},\ i \neq j. \tag{24}$$

Indeed, it does not require much further effort to show that we have equality there: we already know this when $i = 0$, so suppose that for some $1 \leq i < j \leq 3T$, the sets $E(w_i)$ and $E(w_j)$ intersect in some point outside X. Then combining this with (24), we see that at least $|X| + 1 = s + 1$ of the points moved by w_j are also moved by w_i. But w_j only moves $h + s$ points in total by (21), so w_j moves at most

$$h + s - (s + 1) = h - 1$$

of the points which are not moved by w_i, contradicting the way w_j was picked (12) since $i < j$. Hence

$$E(w_i) \cap E(w_j) = X, \qquad \forall i < j \in [3T] \cup \{0\}. \tag{25}$$

Moreover, combining this with (20) gives

$$X = E(w_i) \cap E(w_0) \subseteq [k] \setminus (E(w_i) \Delta E(w_0)) = \operatorname{int}(w_i, w_0),$$

telling us that all elements of $\mathcal{W}$ act on X in the same way as w_0, i.e. X is invariant under $\mathcal{W}$.

3.2.4. Mapping $\mathcal{F}$ into $\mathcal{G}$

Let $\sigma \in \mathcal{S}_n$ be the permutation which coincides with the elements of $\mathcal{W}$ on X and with the identity elsewhere:

$$\sigma(x) = \begin{cases} w_0(x) & x \in X \\ x & x \in [n] \setminus X \end{cases},$$

and let σ^{-1} be the inverse of σ in $\mathcal{S}_n$. We let permutations act on injections as in Section 2, so a permutation acts on each image point of an injection separately, and set

$$\mathcal{F}_\sigma = \left\{ v\sigma^{-1} : v \in \mathcal{F} \right\}.$$

Since all elements w_i of $\mathcal{W}$ as well as σ agree on X, the effect of postmultiplying w_i by σ^{-1} is to fix the elements of X:

$$|E(w_i\sigma^{-1})| = |E(w_i)| - |X| = h + s - s = h, \tag{26}$$

as each w_i moves $h+s$ points by (21) and X has size s by (23). Applying the same argument to (25) gives

$$E(w_i\sigma^{-1}) \cap E(w_j\sigma^{-1}) = \emptyset, \qquad 0 \le i < j \le 3T. \tag{27}$$

By definition σ, and therefore also σ^{-1}, move $|X| = s$ points and any $v \in \mathcal{F}$ moves at most $T = 2h$ points by (8). Moreover, $v\sigma^{-1}$ certainly cannot move more points than the sum of those moved by v and σ^{-1}, i.e.

$$|E(v\sigma^{-1})| \le |E(v)| + |E(\sigma^{-1})| \le 2h + s \le 3h, \qquad \forall v \in \mathcal{F}. \tag{28}$$

It follows from the definition of $\mathcal{G}$ that $|E(v\sigma^{-1})| \le h$ for all $v\sigma^{-1} \in \mathcal{F}_\sigma$ would imply $\mathcal{F}_\sigma \subseteq \mathcal{G}$. Showing this is our final objective in Case 1, so suppose that $|E(v\sigma^{-1})| > h$ for some $v\sigma^{-1} \in \mathcal{F}_\sigma$. For any $w_i \in \mathcal{W}$ the symmetric difference of $E(v\sigma^{-1})$ and $E(w_i\sigma^{-1})$ has size at most $2h$ by (9). But if two sets, one of size larger than h by assumption, the other of size h by (26), have symmetric difference of size at most $2h$, then their intersection must be non-empty. In other words, $E(v\sigma^{-1})$ intersects each of the $3T+1$ sets $E(w_i\sigma^{-1})$, which are mutually disjoint by (27). This gives

$$|E(v\sigma^{-1})| \ge 3T + 1 = 6h + 1,$$

clearly contradicting (28). We have completed Case 1.

Case 2 $\quad$ *$T = 2h+1$ is odd.*

By (18, 8) the number of points moved by w_0 is between h and $2h+1$, so setting

$$s = |E(w_0)| - h$$

as in Case 1, we have $0 \le s \le h+1$ here.

Once again, the maximality of $|E(w_0)|$ implies $|E(w)| \le h+s$ for all $w \in \mathcal{F}$. We wish to show that the moved point set of each $w_i \in \mathcal{W}$ has size either $h+s$ or $h+s-1$. So suppose that for some $1 \le i \le 3T$, the injection $w_i \in \mathcal{W}$ moves at least two points less than w_0. Then

$$|E(w_0) \setminus E(w_i)| \ge |E(w_i) \setminus E(w_0)| + 2 \ge h + 2$$

since Condition 12, according to which w_i was picked, ensures that w_i moves at least h of the points not moved by w_0. The symmetric difference of the two moved point sets then has size

$$\begin{aligned} |E(w_i)\Delta E(w_0)| &= |E(w_0) \setminus E(w_i)| + |E(w_i) \setminus E(w_0)| \\ &\ge 2h + 2 > T, \end{aligned}$$

contradicting (9). Thus we may partition $\mathcal{W}$ according to the cardinalities of the moved point sets: setting

$$\begin{aligned} W_0 &= \{\, w_i \in \mathcal{W} : |E(w_i)| = h+s \,\}, \\ W_1 &= \{\, w_i \in \mathcal{W} : |E(w_i)| = h+s-1 \,\}, \end{aligned}$$

we have $\mathcal{W} = W_0 \cup W_1$. Now we reconsider the arguments employed in Case 1 with the new scenario in mind.

3.2.5. The Intersection of w_0 with Elements of W_0 and W_1

It follows from the way the w_i were picked (12) that any $w_i \in \mathcal{W}$ moves at least h of the points not moved by w_0. We therefore obtain

$$\begin{aligned} |E(w_0) \setminus E(w_i)| &= |E(w_0)| - |E(w_i)| + |E(w_i) \setminus E(w_0)| \\ &\geq h + s - |E(w_i)| + h \\ &= \begin{cases} h & w_i \in W_0 \\ h+1 & w_i \in W_1 \end{cases}. \end{aligned} \tag{29}$$

For $p \in \{0,1\}$ and $w_i \in W_p$ this implies $|E(w_i)\Delta E(w_0)| \geq 2h + p$, but two elements of $\mathcal{F}$ cannot have symmetric difference larger than $T = 2h+1$ by (9). Thus we conclude as in Case 1 that for $w_i \in W_1$,

$$|E(w_0) \setminus E(w_i)| = h+1, \tag{30}$$

$$|E(w_i) \setminus E(w_0)| = h,$$

$$|E(w_i) \cap E(w_0)| = s-1. \tag{31}$$

For elements of W_0 the situation is slightly different. Reconsidering how we obtained (29), it soon becomes clear that for $w_i \in W_0$,

$$|E(w_0) \setminus E(w_i)| = h+1 \iff |E(w_i) \setminus E(w_0)| = h+1.$$

Hence

$$|E(w_i)\Delta E(w_0)| = |E(w_0) \setminus E(w_i)| + |E(w_i) \setminus E(w_0)|$$

cannot be equal to $2h+1$, so we apply (9) to conclude that for all $w_i \in W_0$,

$$|E(w_i)\Delta E(w_0)| = 2h,$$

$$|E(w_0) \setminus E(w_i)| = |E(w_i) \setminus E(w_0)| = h, \tag{32}$$

$$|E(w_0) \cap E(w_i)| = s. \tag{33}$$

Next we investigate to what extent the intersections of elements of $E(\mathcal{W})$ overlap.

3.2.6. A Common Intersection

Let $p \in \{0,1\}$ and let a_p be the smallest positive integer such that $w_{a_p} \in W_p$. Suppose there exists $w_i \in W_p$ with

$$E(w_i) \cap E(w_0) \neq E(w_{a_p}) \cap E(w_0).$$

Neither of these intersections can be contained in the other since they have the same size by (31, 33). Also, $E(w_i)$ has size $h+s-p$ and so

$$\begin{aligned} |E(w_i) \cap E(w_{a_p})| &= |E(w_i)| - |E(w_i) \setminus (E(w_0) \cup E(w_{a_p}))| \\ &\qquad - |(E(w_i) \cap E(w_0)) \setminus E(w_{a_p})| \\ &\leq |E(w_i)| - |E(w_i) \setminus (E(w_0) \cup E(w_{a_p}))| - 1 \\ &\leq h + s - p - \left| E(w_i) \setminus \bigcup_{\lambda=0}^{i-1} E(w_\lambda) \right| - 1 \\ &= s - p - 1. \end{aligned}$$

This yields

$$\begin{aligned} |E(w_i)\Delta E(w_{a_p})| &= |E(w_i)| + |E(w_{a_p})| - 2|E(w_i)\cap E(w_{a_p})| \\ &> 2(h+s-p) - 2(s-p-1) \\ &= 2h+2 > T, \end{aligned}$$

our familiar contradiction to (9). Hence we have

$$E(w_i)\cap E(w_0) = E(w_{a_p})\cap E(w_0), \qquad \forall w_i \in W_p,$$

implying that the intersection of any two elements of $E(W_p)$ contains

$$X_p = E(w_{a_p})\cap E(w_0).$$

If some $w_i, w_j \in W_p$ with $i < j$ both move a point outside X_p, then $E(w_i)\cap E(w_j)$ has size at least

$$|X_p| + 1 = s - p + 1$$

by (31, 33). Therefore the maximum number of points moved by w_j and not moved by w_i is

$$|E(w_j)| - (|X_p| + 1) = (h+s-p) - (s-p+1) = h-1.$$

This contradicts the way w_j was picked (12) and so we conclude that any two elements of $E(W_p)\cup\{E(w_0)\}$ have intersection precisely X_p.

This section may now be summarised as follows: let $p \in \{0,1\}$. For distinct $w_i, w_j \in W_p \cup \{w_0\}$,

$$(E(w_i)\cap E(w_j)) \subseteq X_p \subset E(w_0) \tag{34}$$

and for distinct $w_i, w_j \in W_p$, we have $E(w_i)\cap E(w_j) = X_p$ where $|X_p| = s-p$.

3.2.7. Mapping $\mathcal{F}$ into $\mathcal{G}$

We define $\sigma_p \in \mathcal{S}_n$ and $\mathcal{F}_p$ analogously to σ and $\mathcal{F}_\sigma$ in Case 1: let

$$\sigma_p(x) = \begin{cases} w_0(x) & x \in X_p \\ x & x \in [n]\setminus X_p \end{cases},$$

let σ_p^{-1} be the inverse of σ_p in $\mathcal{S}_n$ and set

$$\mathcal{F}_p = \left\{ v\sigma_p^{-1} : v \in \mathcal{F} \right\}.$$

Let $w_i \in W_p$ with $i > 0$. Clearly w_i intersects w_0 in at most $|X_p| = s-p$ elements of X_p, implying that postmultiplying w_i by ${\sigma_p}^{-1}$ can fix at most $s-p$ of the points moved by w_i. That is,

$$|E(w_i{\sigma_p}^{-1})| \geq |E(w_i)| - (s-p) = (h+s-p) - (s-p) = h \tag{35}$$

for all $w_i \in W_p$. Moreover, since elements of $E(W_p)$ do not intersect in points outside X_p by (34), we have

$$E(w_i {\sigma_p}^{-1}) \cap E(w_j {\sigma_p}^{-1}) = \emptyset, \qquad\qquad w_i, w_j \in W_p,\ i \neq j. \tag{36}$$

And for $v \in \mathcal{F}$ we have

$$\begin{aligned} |E(v{\sigma_p}^{-1})| &\leq |E(v)| + |E({\sigma_p}^{-1})| \\ &\leq h + s + |X_p| = h + 2s - p \\ &\leq 3h + 2 - p \end{aligned} \tag{37}$$

since $|X_p| = s - p$ by the summary of the previous section and $s \leq h + 1$ by definition.

Since T is odd, in order to prove $\mathcal{F}_p \subseteq \mathcal{G}$ we must demonstrate that for some $p \in \{0, 1\}$, all $v \in \mathcal{F}_p$ satisfy

$$|E(v) \cap [k-1]| \leq h.$$

We begin by proving that for at least one value of $p \in \{0, 1\}$, all $v \in \mathcal{F}_p$ satisfy $|E(v)| \leq h + 1$.

So suppose, for a contradiction, that for both $p = 0$ and $p = 1$ there exists $v_p \in \mathcal{F}_p$ with $|E(v_p)| > h + 1$. By (9) we have

$$|E(v_p) \Delta E(w_i {\sigma_p}^{-1})| \leq T = 2h + 1, \qquad\qquad \forall w_i \in W_p,$$

since the size of the symmetric difference is constant under the action of a permutation. Using (35), all $w_i \in W_p$ therefore satisfy

$$\begin{aligned} |E(v_p) \cap E(w_i {\sigma_p}^{-1})| &= \frac{1}{2}\left(|E(v_p)| + |E(w_i {\sigma_p}^{-1})| - |E(v_p) \Delta E(w_i {\sigma_p}^{-1})|\right) \\ &> \frac{1}{2}\left(h + 1 + h - (2h + 1)\right) = 0. \end{aligned}$$

Combining this with (36) we see that $E(v_p)$ intersects each of the mutually disjoint sets $E(w_i {\sigma_p}^{-1})$ for $w_i \in W_p$, implying

$$|E(v_p)| \geq |W_p|. \tag{38}$$

If $|W_1| > 3h + 1$ then considering the case $p = 1$ in (38) gives

$$|E(v_1)| > 3h + 1 = 3h + 2 - p,$$

contradicting (37). Therefore we must have $|W_1| \leq 3h + 1$ which, together with (19, 38), yields

$$|E(v_0)| \geq |\mathcal{W}| - |W_1| \geq 3T + 1 - (3h + 1) = 3h + 3 > 3h + 2,$$

this time contradicting (37) for $p = 0$. Hence we conclude that there exists $p^* \in \{0, 1\}$ such that all $v \in \mathcal{F}_{p^*}$ satisfy $|E(v)| \leq h + 1$.

The family $\mathcal{F}_{p^*}$ is t-intersecting, so if two elements $u, v \in \mathcal{F}_{p^*}$ do not intersect in any points they move, they must jointly fix at least t positions. Suppose, for a contradiction, that two elements $u, v \in \mathcal{F}_{p^*}$ have moved point sets of size $h+1$ which do not intersect. Then the number of points fixed by both u and v is

$$\begin{aligned} k - |E(u)| - |E(v)| &= k - 2h - 2 \\ &= k - (k - t - 1) - 2 = t - 1, \end{aligned}$$

a contradiction. We conclude that for $u, v \in \mathcal{F}_{p^*}$,

$$|E(u)| = |E(v)| = h + 1 \implies E(u) \cap E(v) \neq \emptyset,$$

so

$$\mathcal{A} = \{ A \in E(\mathcal{F}_{p^*}) : |A| = h + 1 \}$$

is intersecting. Furthermore,

$$h + 1 = \frac{k - t - 1}{2} + 1 \leq \frac{k - 2}{2} + 1 = \frac{k}{2},$$

so we may apply the Erdős-Ko-Rado Theorem 1.2 to deduce

$$|\mathcal{A}| \leq \binom{k-1}{h}. \tag{39}$$

If this inequality is strict, we combine (17) with the fact that all elements of $E(\mathcal{F}_{p^*})$ have size at most $h + 1$ to obtain

$$\begin{aligned} |\mathcal{F}_{p^*}| &\leq \sum_{X \in E(\mathcal{F}_{p^*})} d(|X|, N + |X|) \\ &< \sum_{j=0}^{h} \binom{k}{j} d(j, N + j) + \binom{k-1}{h} d(h + 1, N + h + 1) \quad = \quad |\mathcal{G}| \end{aligned}$$

by (7), contradicting the fact that $\mathcal{F}$, and therefore also $\mathcal{F}_{p^*}$, is maximum.

Hence we must have equality in (39), so Theorem 1.2 implies that all elements of $\mathcal{A}$ have a fixed point z in common: we have

$$E(\mathcal{F}_{p^*}) \quad \subseteq \quad \{ A \subseteq [k] : |A| \leq h \} \cup \{ A \subseteq [k] : |A| = h + 1, \; z \in A \}$$

and comparing this with (6), we conclude that $(z\ k)\mathcal{F}_{p^*} \subseteq \mathcal{G}$, where $(z\ k) \in \mathcal{S}_k$ is the transposition swapping z and k. We have demonstrated that $\mathcal{F}$ is equivalent to $\mathcal{G}$. Finally, this completes the proof of Theorem 3.5. □

4. Conclusion

In Section 2 we showed that when $t = 1$, the fix-family is the unique maximum intersecting injection family (Corollary 2.10). In Section 3, we considered the t-intersecting case in the limit, and found that whether saturation yields larger families than fixing or vice versa depends on the way we take the limit: when n is large in terms of k and t, all optimal families are fix-families (Corollary 3.3), whereas if we fix $k-t$ and $n-k$ and increase k, all maximum t-intersecting subsets of $\mathcal{I}_n^k$ are equivalent to $\mathcal{G}(t,k,n)$ (Theorem 3.5).

For $0 \le r \le (k-t)/2$, we define a more general saturation family by

$$\mathcal{K}_r(t,k,n) = \left\{ w \in \mathcal{I}_n^k : w \text{ fixes at least } t+r \text{ elements of } [t+2r] \right\}.$$

Then we have $\mathcal{G}(t,k,n) = \mathcal{K}_{\lfloor (k-t)/2 \rfloor}(t,k,n)$ and $\mathcal{K}_r(t,k,n)$, which we abbreviate by $\mathcal{K}_r$, is t-intersecting. Note that $\mathcal{K}_r$ is equivalent to the fix-family if, and only if, $r=0$.

Considering small parameter values for arbitrary t, we find that $n=6$ is the smallest value of n for which fixing is not the unique optimal strategy: it is easily seen that both $K_0(3,6,6)$ and $K_1(3,6,6)$ are 3-intersecting families in $\mathcal{S}_6$ of size 6. Moreover, there are many instances of the parameters where the fix-family is not maximum. For instance, a straightforward calculation shows that if $n/2 \le t \le (2n-4)/3$ then $|\mathcal{K}_1| > |\mathcal{K}_0|$ for all k. Concerning the general situation, we make the following conjecture.

Conjecture 4.1. *Let $t \le k \le n$ be natural numbers with $n \ge 8$.*

If $\mathcal{F}$ is a maximum t-intersecting subset of $\mathcal{I}_n^k$ then $\mathcal{F}$ is equivalent to $\mathcal{K}_r(t,k,n)$ where r is the largest integer in $\{0,1,\dots,(k-t)/2\}$ satisfying

$$(2r+t-1) \cdot \sum_{i=0}^{r} (-1)^i \binom{r}{i} (n-r-t-i)! \ge \\ r \cdot \sum_{i=0}^{r} (-1)^i \binom{r}{i} (n-r-t+1-i)!.$$

In her thesis, the first author proved Conjecture 4.1 for families $\mathcal{F}$ whose fixed point sets $\mathrm{Fix}(\mathcal{F})$ are t-intersecting and left-compressed. This was achieved by adapting methods of Ahlswede & Khachatrian from words [2] to injections. However, standardarisation maps for injections, which would allow our results to be extended to a proof of Conjecture 4.1 in the general case, have so far proved elusive.

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