TORSION UNITS IN INTEGRAL GROUP RINGS OF JANKO SIMPLE GROUPS

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Dedicated to the memory of Professor I.S. Luthar

ABSTRACT. Using the Luthar-Passi method, we investigate the classical Zassenhaus conjecture for the normalized unit group of integral group rings of Janko simple groups. As a consequence, for the Janko groups J_1, J_2 and J_3 we confirm Kimmerle's conjecture on prime graphs.

1. INTRODUCTION AND MAIN RESULTS

Let G be a finite group and $\mathbb{Z}G$ its integral group ring. By $V(\mathbb{Z}G)$ we denote the normalized unit group of $\mathbb{Z}G$. A long standing conjecture (attributed to H. Zassenhaus [24]) says that every torsion unit $u \in V(\mathbb{Z}G)$ is conjugate within the rational group algebra $\mathbb{Q}G$ to an element in G.

For finite simple groups, the main tool for the investigation of the Zassenhaus conjecture is the Luthar-Passi method, introduced in [21] to solve it for A_5. Later in [17] M. Hertweck extended it and applied for the investigation of the Zassenhaus conjecture for $PSL(2,p^n)$. This method proved to be useful for groups containing non-trivial normal subgroups as well. For some recent results we refer to [5, 6, 14, 17, 16, 18]. Also some related properties can be found in [1, 22] and [3, 20]. In the latter papers some weakened variations of the conjecture have been made.

In order to state one of these we introduce some notation. By $\#(G)$ we denote the set of all primes dividing the order of G. The Gruenberg-Kegel graph (or the prime graph) of G is the graph $\pi(G)$ with vertices labeled by the primes in $\#(G)$ and with an edge from p to q if there is an element of order pq in the group G. Kimmerle in [20] made the following

Conjecture (KC): if G is a finite group then $\pi(G) = \pi(V(\mathbb{Z}G))$.

Of course the Zassenhaus conjecture implies **(KC)**. In [20] it is shown, in particular, that **(KC)** holds for finite Frobenius and solvable groups. For solvable groups this result was improved recently by M. Hertweck in [15], where it was shown that orders of torsion units $V(\mathbb{Z}G)$ are exactly orders of elements of the group G. Note that with respect to the so-called p-version of the Zassenhaus conjecture the investigation of Frobenius groups was completed by M. Hertweck and the first author in [4]. In [6, 7, 9], **(KC)** is also confirmed for some Mathieu simple groups.

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In this paper we continue these investigations for the Janko simple groups. The main results give a lot of information on possible torsion units in $V(\mathbb{Z}J_i)$, $1 \le i \le 3$. An immediate consequence is a positive answer to **(KC)** for these groups. In the final section we give remarks for **(KC)** on J_4: the Luthar-Passi method usage is limited by the fact that not all Brauer character tables are known for J_4, so **(KC)** remains open for J_4.

In order to state the result we need to introduce some notation. Let G be a group. Put $\mathcal{C} = \{C_1, \dots, C_{nt}, \dots\}$, the collection of all conjugacy classes of G, where the first index denotes the order of the elements of this conjugacy class and $C_1 = \{1\}$ (throughout the paper we will use the ordering of conjugacy classes as used in the GAP Character Table Library). Suppose $u = \sum \alpha_g g \in V(\mathbb{Z}G)$ has finite order k. Denote by $\nu_{nt} = \nu_{nt}(u) = \varepsilon_{C_{nt}}(u) = \sum_{g \in C_{nt}} \alpha_g$, the partial augmentation of u with respect to C_{nt}. From S.D. Berman's Theorem [2] one knows that $\nu_1 = \alpha_1 = 0$ and, clearly,

$$\sum_{C_{nt} \in \mathcal{C}} \nu_{nt} = 1. \tag{1}$$

Hence, for any character χ of G, we get that $\chi(u) = \sum \nu_{nt}\chi(h_{nt})$, where h_{nt} is a representative of a conjugacy class C_{nt}.

The main result for J_1 is the following.

Theorem 1. *Let G denote the first Janko simple group J_1. Let u be a torsion unit of $V(\mathbb{Z}G)$ of order $|u|$. Denote by $\mathfrak{P}(u)$ the following tuple of partial augmentations of u in $V(\mathbb{Z}G)$:*

$$(\nu_{2a}, \nu_{3a}, \nu_{5a}, \nu_{5b}, \nu_{6a}, \nu_{7a}, \nu_{10a}, \nu_{10b}, \nu_{11a}, \nu_{15a}, \nu_{15b}, \nu_{19a}, \nu_{19b}, \nu_{19c}) \in \mathbb{Z}^{14}.$$

The following properties hold.

- (i) *There are no elements of orders 14, 21, 22, 33, 35, 38, 55, 57, 77, 95, 133 and 209 in $V(\mathbb{Z}G)$. Equivalently, if $|u| \ne 30$, then $|u|$ coincides with the order of some element $g \in G$.*
- (ii) *If $|u| \in \{2, 3, 7, 11, 19\}$, then u is rationally conjugate to some $g \in G$.*
- (iii) *If $|u| = 5$, then $\mathfrak{P}(u)$ satisfies the conditions*

$$(\nu_{5a}, \nu_{5b}) \in \{\ (-1, 2),\ (0, 1),\ (1, 0),\ (2, -1)\ \} \quad \textit{and} \quad \nu_{kx} = 0,\ kx \notin \{5a, 5b\}.$$

- (iv) *If $|u| = 6$, then $\mathfrak{P}(u)$ satisfies the conditions*

$$\begin{gathered}(\nu_{2a}, \nu_{3a}, \nu_{6a}) \in \{\ (-4, 3, 2),\ (-2, 0, 3),\ (-2, 3, 0),\ (0, 0, 1),\\ (0, 3, -2),\ (2, 0, -1)\ \} \quad \textit{and} \quad \nu_{kx} = 0,\ kx \notin \{2a, 3a, 6a\}.\end{gathered}$$

- (v) *If $|u| = 10$, then $\mathfrak{P}(u)$ satisfies the conditions*

$$\begin{gathered}(\nu_{5a}, \nu_{5b}, \nu_{10a}, \nu_{10b}) \in \{\ (2, -2, 0, 1),\ (0, 0, 2, -1),\ (0, 0, 0, 1),\ (-1, 1, 1, 0),\\ (1, -1, -1, 2),\ (1, -1, 1, 0),\ (-2, 2, 1, 0),\ (0, 0, -1, 2),\ (0, 0, 1, 0),\ (-1, 1, 2, -1),\\ (-1, 1, 0, 1),\ (1, -1, 0, 1)\ \} \quad \textit{and} \quad \nu_{kx} = 0, kx \notin \{5a, 5b, 10a, 10b\}.\end{gathered}$$

- (vi) *If $|u| = 15$, then $\mathfrak{P}(u)$ satisfies the conditions*

$$\begin{gathered}(\nu_{5a}, \nu_{5b}, \nu_{15a}, \nu_{15b}) \in \{\ (-1, 1, 0, 1),\ (0, 0, 0, 1), (0, 0, 1, 0),\ (1, -1, 1, 0)\ \}\\ \textit{and} \quad \nu_{kx} = 0, \quad kx \notin \{5a, 5b, 15a, 15b\}.\end{gathered}$$

(vii) *If* $|u| = 30$*, then the tuple of partial augmentations of* u *cannot belong to the set* $\mathbb{Z}^{14} \setminus M$*, where*

$$\begin{aligned} M = \{\ \mathfrak{P}(u)\ \mid (\nu_{5a}, \nu_{5b}, \nu_{10a}, \nu_{10b}, \nu_{15a}, \nu_{15b}) \in \{\ &(-1, 1, -1, -1, 1, 2), \\ (-1, 1, -2, 0, 1, 2),\ (0, 0, -2, 0, 1, 2),\ &(0, 0, 0, -2, 2, 1), \\ (1, -1, -1, -1, 2, 1),\ &(1, -1, 0, -2, 2, 1)\ \}, \\ \nu_{kx} = 0, \quad kx \notin &\{5a, 5b, 10a, 10b, 15a, 15b\}\ \}. \end{aligned}$$

Theorem 2. *Let* G *denote the second Janko simple group* J_2*. Let* u *be a torsion unit of* $V(\mathbb{Z}G)$ *of order* $|u|$*. Denote by* $\mathfrak{P}(u)$

$$\begin{aligned} (\nu_{2a}, \nu_{2b}, \nu_{3a}, \nu_{3b}, \nu_{4a}, \nu_{5a}, \nu_{5b}, \nu_{5c}, \nu_{5d}, \nu_{6a}, \nu_{6b}, \nu_{7a}, \nu_{8a}, \\ \nu_{10a}, \nu_{10b}, \nu_{10c}, \nu_{10d}, \nu_{12a}, \nu_{15a}, \nu_{15b}) \in \mathbb{Z}^{20} \end{aligned}$$

the tuple of partial augmentations of u *in* $V(\mathbb{Z}G)$*. The following properties hold.*

(i) *There are no elements of orders* 14*,* 21 *and* 35 *in* $V(\mathbb{Z}G)$*. Equivalently, if* $|u| \notin \{20, 24, 30, 40, 60, 120\}$*, then* $|u|$ *coincides with the order of some element* $g \in G$*.*

(ii) *If* $|u| \in \{7, 15\}$*, then* u *is rationally conjugate to some* $g \in G$*.*

(iii) *If* $|u| = 2$*, then* $\mathfrak{P}(u)$ *satisfies the conditions*

$$\begin{aligned} (\nu_{2a}, \nu_{2b}) \in \{\ (0, 1),\ (-2, 3),\ (2, -1),\ (1, 0), \\ (3, -2),\ (-1, 2)\ \} \quad \textit{and} \quad \nu_{kx} = 0,\ kx \notin \{2a, 2b\}. \end{aligned}$$

(iv) *If* $|u| = 3$*, then* $\mathfrak{P}(u)$ *satisfies the conditions*

$$(\nu_{3a}, \nu_{3b}) \in \{\ (0, 1),\ (1, 0),\ (-1, 2)\ \} \quad \textit{and} \quad \nu_{kx} = 0,\ kx \notin \{3a, 3b\}.$$

(v) *If* $|u| = 4$*, then* $\mathfrak{P}(u)$ *satisfies the conditions*

$$\begin{aligned} (\nu_{2a}, \nu_{2b}, \nu_{4a}) \in \{\ (-2, -2, 5),\ (-1, -3, 5),\ (-1, -1, 3),\ (-1, 1, 1), \\ (0, -4, 5),\ (0, -2, 3),\ (0, 0, 1),\ (0, 2, -1),\ (0, 4, -3),\ (1, -3, 3),\ (1, -1, 1), \\ (1, 1, -1),\ (1, 3, -3),\ (2, 0, -1),\ (2, 2, -3)\ \} \quad \textit{and} \quad \nu_{kx} = 0,\ kx \notin \{2a, 2b, 4a\}. \end{aligned}$$

(vi) *If* $|u| = 5$*, then* $\mathfrak{P}(u)$ *satisfies the conditions*

$$\begin{aligned} (\nu_{5a}, \nu_{5b}, \nu_{5c}, \nu_{5d}) \in \{\ (0, 0, 2, -1),\ (1, 0, 0, 0),\ (0, 0, 0, 1), \\ (0, 0, -1, 2),\ (1, 0, -1, 1), (0, 0, 1, 0),\ (0, 1, 0, 0),\ (1, 1, 0, -1), \\ (1, 1, -1, 0),\ (0, 1, 1, -1)\ \} \quad \textit{and} \quad \nu_{kx} = 0,\ kx \notin \{5a, 5b, 5c, 5d\}. \end{aligned}$$

(vii) *If* $|u| = 8$*, then* $\mathfrak{P}(u)$ *satisfies the conditions*

$$\begin{aligned} (\nu_{2a}, \nu_{2b}, \nu_{4a}, \nu_{8a}) \in \{\ (-1, -1, 0, 3),\ (-1, -1, 2, 1),\ (-1, 1, -2, 3), \\ (-1, 1, 0, 1),\ (0, -2, 0, 3),\ (0, -2, 2, 1),\ (0, 0, -2, 3),\ (0, 0, 0, 1),\ (0, 0, 2, -1), \\ (0, 2, -2, 1),\ (0, 2, 0, -1), (0, 2, 2, -3),\ (1, -1, 0, 1),\ (1, -1, 2, -1),\ (1, 1, -2, 1), \\ (1, 1, 0, -1),\ (1, 1, 2, -3),\ (2, 0, 2, -3)\ \} \quad \textit{and} \quad \nu_{kx} = 0,\ kx \notin \{2a, 2b, 4a, 8a\}. \end{aligned}$$

Theorem 3. *Let* G *denote the third Janko simple group* J_3*. Let* u *be a torsion unit of* $V(\mathbb{Z}G)$ *of order* $|u|$*. Denote by* $\mathfrak{P}(u)$

$$\begin{aligned} (\nu_{2a}, \nu_{3a}, \nu_{3b}, \nu_{4a}, \nu_{5a}, \nu_{5b}, \nu_{6a}, \nu_{8a}, \nu_{9a}, \nu_{9b}, \nu_{9c}, \nu_{10a}, \nu_{10b}, \\ \nu_{12a}, \nu_{15a}, \nu_{15b}, \nu_{17a}, \nu_{17b}, \nu_{19a}, \nu_{19b}) \in \mathbb{Z}^{20} \end{aligned}$$

the tuple of partial augmentations of u *in* $V(\mathbb{Z}G)$*. The following properties hold.*

(i) *There are no elements of orders* 34, 38, 51, 57, 85, 95 *and* 323 *in* $V(\mathbb{Z}G)$. *Equivalently, if* $|u| \notin \{18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360\}$*, then* $|u|$ *coincides with the order of some element* $g \in G$.

(ii) *If* $|u| = 2$*,* u *is rationally conjugate to some* $g \in G$.

(iii) *If* $|u| = 3$*, then* $\mathfrak{P}(u)$ *satisfies the conditions*

$$(\nu_{3a}, \nu_{3b}) \in \{\ (5,-4),\ (0,1),\ (-2,3),\ (2,-1),\ (-3,4),\ (-4,5),\ (1,0),\ (3,-2),\ (-1,2),\ (4,-3)\ \} \quad \text{and} \quad \nu_{kx} = 0,\ kx \notin \{3a, 3b\}.$$

(iv) *If* $|u| = 4$*, then* $\mathfrak{P}(u)$ *satisfies the conditions*

$$(\nu_{2a}, \nu_{4a}) \in \{\ (0,1),\ (-2,3),\ (2,-1)\ \} \quad \text{and} \quad \nu_{kx} = 0,\ kx \notin \{2a, 4a\}.$$

(v) *If* $|u| = 5$*, then* $\mathfrak{P}(u)$ *satisfies the conditions*

$$(\nu_{5a}, \nu_{5b}) \in \{\ (0,1),\ (-2,3),\ (2,-1),\ (-3,4),\ (1,0),\ (3,-2),\ (-1,2),\ (4,-3)\ \} \quad \text{and} \quad \nu_{kx} = 0,\ kx \notin \{5a, 5b\}.$$

(vi) *If* $|u| = 8$*, then* $\mathfrak{P}(u)$ *satisfies the conditions*

$$(\nu_{2a}, \nu_{4a}, \nu_{8a}) \in \{\ (-2,-6,9),\ (-2,-4,7),\ (-2,-2,5),\ (-2,0,3),\ (-2,2,1),\ (0,-4,5),\ (0,-2,3),\ (0,0,1),\ (0,2,-1),\ (0,4,-3),\ (2,-2,1),\ (2,0,-1),\ (2,2,-3),\ (2,4,-5),\ (2,6,-7)\ \} \quad \text{and} \quad \nu_{kx} = 0,\ kx \notin \{2a, 4a, 8a\}.$$

(vii) *If* $|u| = 17$*, then* $\mathfrak{P}(u)$ *satisfies the conditions*

$$(\nu_{17a}, \nu_{17b}) \in \{\ (5,-4),\ (0,1),\ (-2,3),\ (2,-1),\ (-3,4),\ (-4,5),\ (1,0),\ (3,-2),\ (-1,2),\ (4,-3)\ \} \quad \text{and} \quad \nu_{kx} = 0,\ kx \notin \{17a, 17b\}.$$

(viii) *If* $|u| = 19$*, then* $\mathfrak{P}(u)$ *satisfies the conditions*

$$(\nu_{19a}, \nu_{19b}) \in \{\ (5,-4),\ (0,1),\ (-2,3),\ (2,-1),\ (-3,4),\ (-4,5),\ (1,0),\ (3,-2),\ (-1,2),\ (4,-3)\ \} \quad \text{and} \quad \nu_{kx} = 0,\ kx \notin \{19a, 19b\}.$$

As an immediate consequence of first parts of Theorems 1 - 3 one obtains Kimmerle's conjecture for these groups.

Corollary 1. *If* $G \in \{J_1, J_2, J_3\}$*, then* $\pi(G) = \pi(V(\mathbb{Z}G))$.

We refer to Section 6 for comments on $V(\mathbb{Z}J_4)$.

2. PRELIMINARIES

The following result relates the solution of the Zassenhaus conjecture to partial augmentations of torsion units.

Proposition 1. *(see* [21] *and Theorem 2.5 in* [23]*) Let* $u \in V(\mathbb{Z}G)$ *be a torsion unit of order* k*. Then* u *is conjugate in* $\mathbb{Q}G$ *to an element* $g \in G$ *if and only if for each* d *dividing* k *there is precisely one conjugacy class* C *with partial augmentation* $\varepsilon_C(u^d) \neq 0$.

The next result already yields that several partial augmentations are zero.

Proposition 2. *(see* [14]*, Proposition 3.1;* [17]*, Proposition 2.2) Let* G *be a finite group and let* u *be a torsion unit in* $V(\mathbb{Z}G)$*. If* x *is an element of* G *whose* p*-part, for some prime* p*, has order strictly greater than the order of the* p*-part of* u*, then* $\varepsilon_x(u) = 0$.

The key restriction on partial augmentations is given by the following result that is the cornerstone of the Luthar-Passi method.

Proposition 3. *(see [17, 21]) Let either $p = 0$ or p is a prime divisor of $|G|$. Suppose that $u \in V(\mathbb{Z}G)$ has finite order k and assume k and p are coprime in case $p \neq 0$. If z is a complex primitive k-th root of unity and χ is either a classical character or a p-Brauer character of G then, for every integer l, the number*

$$\mu_l(u, \chi, p) = \frac{1}{k} \sum_{d|k} Tr_{\mathbb{Q}(z^d)/\mathbb{Q}}\{\chi(u^d) z^{-dl}\}$$

is a non-negative integer.

Note that if $p = 0$, we will use the notation $\mu_l(u, \chi, *)$ for $\mu_l(u, \chi, 0)$.

Finally, we shall use the well-known bound for orders of torsion units.

Proposition 4. *(see [10]) The order of a torsion element $u \in V(\mathbb{Z}G)$ is a divisor of the exponent of G.*

In case of units of prime power order, the following Proposition may also be useful to eliminate some tuples of partial augmentations.

Proposition 5. *(see [10]) Let p be a prime, and let u be a torsion unit of $V(\mathbb{Z}G)$ of order p^n. Then for $m \neq n$ the sum of all partial augmentations of u with respect to conjugacy classes of elements of order p^m is divisible by p.*

3. Proof of Theorem 1

In this section we denote by G the first Janko simple group J_1. It is well known [12, 13] that $|G| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$ and $exp(G) = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$. The character table of G, as well as the Brauer character tables, denoted by $\mathfrak{BCT}(p)$, where $p \in \{2, 3, 5, 7, 11, 19\}$, can be found using the computational algebra system GAP [12], which derives its data from [11, 19]. Throughout the paper we will use the notation, inclusive the indexation, for the characters and conjugacy classes as used in the GAP Character Table Library.

Since the group G only possesses elements of orders 2, 3, 5, 6, 7, 10, 11, 15 and 19, we first shall investigate units of these orders. After this, by Proposition 4, the order of each torsion unit divides the exponent of G, and it will be enough to consider units of orders 14, 21, 22, 30, 33, 35, 38, 55, 57, 77, 95, 133 and 209, because if u will be a unit of another possible order, then there is $t \in \mathbb{N}$ such that u^t has an order from this list. We shall prove that units of all these orders except 30 do not appear in $V(\mathbb{Z}G)$.

Now we consider each case separately.

- Let $|u| \in \{2, 3, 7, 11\}$. Since there is only one conjugacy class in G consisting of elements or order $|u|$, this case follows at once from Propositions 1 and 2.
- Let u be a unit of order 19. By (1) and Proposition 2 we have that

$$\nu_{19a} + \nu_{19b} + \nu_{19c} = 1.$$

Applying Proposition 3 to characters $\chi_2, \chi_8, \chi_{13}$ in $\mathfrak{BCT}(11)$ we get the following system of inequalities

$$\begin{aligned}
\mu_1(u,\chi_2,11) &= \tfrac{1}{19}(-7\nu_{19a}+12\nu_{19b}-7\nu_{19c}+7)\geq 0;\\
\mu_2(u,\chi_2,11) &= \tfrac{1}{19}(-7\nu_{19a}-7\nu_{19b}+12\nu_{19c}+7)\geq 0;\\
\mu_4(u,\chi_2,11) &= \tfrac{1}{19}(12\nu_{19a}-7\nu_{19b}-7\nu_{19c}+7)\geq 0;\\
\mu_1(u,\chi_8,11) &= \tfrac{1}{19}(7\nu_{19a}-12\nu_{19b}+7\nu_{19c}+69)\geq 0;\\
\mu_2(u,\chi_8,11) &= \tfrac{1}{19}(7\nu_{19a}+7\nu_{19b}-12\nu_{19c}+69)\geq 0;\\
\mu_4(u,\chi_8,11) &= \tfrac{1}{19}(-12\nu_{19a}+7\nu_{19b}+7\nu_{19c}+69)\geq 0;\\
\mu_1(u,\chi_{13},11) &= \tfrac{1}{19}(14\nu_{19a}-5\nu_{19b}-5\nu_{19c}+119)\geq 0;\\
\mu_2(u,\chi_{13},11) &= \tfrac{1}{19}(-5\nu_{19a}+14\nu_{19b}-5\nu_{19c}+119)\geq 0;\\
\mu_4(u,\chi_{13},11) &= \tfrac{1}{19}(-5\nu_{19a}-5\nu_{19b}+14\nu_{19c}+119)\geq 0.
\end{aligned}$$

From these restrictions and the requirement that all $\mu_i(u,\chi_j,p)$ must be non-negative integers it can be verified that $(\nu_{19a},\nu_{19b},\nu_{19c})\in\{(1,0,0),(0,1,0),(0,0,1)\}$.

Thus, for units of orders 2,3,7,11 and 19 we obtained that there is precisely one conjugacy class with non-zero partial augmentation. Proposition 1 then yields part (ii) of Theorem 1.

Note that using the LAGUNA package [8] we computed inequalities from Proposition 3 for every irreducible character from ordinary and Brauer character tables, and for every $0\leq l\leq |u|-1$ (it is enough to enumerate l in this range since $z^{|u|}=1$, so for bigger values of l we will not have new inequalities), but the only inequalities that really matter are those ones listed above. The same remark applies for all other orders of torsion units considered in the paper.

• Let u be a unit of order 5. By (1) and Proposition 2 we get $\nu_{5a}+\nu_{5b}=1$. Again applying Proposition 3 to characters in $\mathfrak{BCT}(11)$ we get the system of inequalities

$$\begin{aligned}
\mu_1(u,\chi_2,11) &= \tfrac{1}{5}(3\nu_{5a}-2\nu_{5b}+7)\geq 0; & \mu_2(u,\chi_2,11) &= \tfrac{1}{5}(-2\nu_{5a}+3\nu_{5b}+7)\geq 0;\\
\mu_1(u,\chi_3,11) &= \tfrac{1}{5}(-4\nu_{5a}+\nu_{5b}+14)\geq 0; & \mu_2(u,\chi_3,11) &= \tfrac{1}{5}(\nu_{5a}-4\nu_{5b}+14)\geq 0;\\
\mu_1(u,\chi_5,11) &= \tfrac{1}{5}(\nu_{5a}-4\nu_{5b}+49)\geq 0; & \mu_1(u,\chi_6,11) &= \tfrac{1}{5}(-6\nu_{5a}+4\nu_{5b}+56)\geq 0;\\
\mu_2(u,\chi_6,11) &= \tfrac{1}{5}(4\nu_{5a}-6\nu_{5b}+56)\geq 0; & \mu_2(u,\chi_7,11) &= \tfrac{1}{5}(-4\nu_{5a}+6\nu_{5b}+64)\geq 0;\\
\mu_2(u,\chi_8,11) &= \tfrac{1}{5}(\nu_{5a}-4\nu_{5b}+69)\geq 0; & \mu_1(u,\chi_{12},11) &= \tfrac{1}{5}(4\nu_{5a}-\nu_{5b}+106)\geq 0,
\end{aligned}$$

that has only four integer solutions $(\nu_{5a},\nu_{5b})\in\{(0,1),(2,-1),(1,0),(-1,2)\}$ such that all $\mu_i(u,\chi_j,11)$ are non-negative integers, so part (iii) of Theorem 1 is proved.

• Let u be a unit of order 6. By (1) and Proposition 2 we have that

$$\nu_{2a}+\nu_{3a}+\nu_{6a}=1.$$

Applying Proposition 3 to characters in $\mathfrak{BCT}(11)$ we get the system of inequalities

$$\begin{aligned}
&\mu_3(u,\chi_4,11) = \tfrac{1}{6}(-6\nu_{2a}+24) \geq 0; \qquad \mu_0(u,\chi_4,11) = \tfrac{1}{6}(6\nu_{2a}+30) \geq 0;\\
&\mu_0(u,\chi_6,11) = \tfrac{1}{6}(4\nu_{3a}+60) \geq 0; \qquad \mu_3(u,\chi_6,11) = \tfrac{1}{6}(-4\nu_{3a}+60) \geq 0;\\
&\mu_0(u,\chi_2,11) = \tfrac{1}{6}(-2\nu_{2a}+2\nu_{3a}-2\nu_{6a}+8) \geq 0;\\
&\mu_3(u,\chi_2,11) = \tfrac{1}{6}(2\nu_{2a}-2\nu_{3a}+2\nu_{6a}+10) \geq 0;\\
&\mu_0(u,\chi_3,11) = \tfrac{1}{6}(-4\nu_{2a}-2\nu_{3a}+2\nu_{6a}+10) \geq 0;\\
&\mu_1(u,\chi_3,11) = \tfrac{1}{6}(-2\nu_{2a}-\nu_{3a}+\nu_{6a}+17) \geq 0;\\
&\mu_3(u,\chi_3,11) = \tfrac{1}{6}(4\nu_{2a}+2\nu_{3a}-2\nu_{6a}+14) \geq 0,
\end{aligned}$$

that has only six integer solutions

$$(\nu_{2a},\nu_{3a},\nu_{6a}) \in \{\ (-4,3,2),\ (-2,0,3),\ (-2,3,0),\ (0,0,1),\ (0,3,-2),\ (2,0,-1)\ \}$$

such that all $\mu_i(u,\chi_j,11)$ are non-negative integers. So part (iv) of Theorem 1 is proved.

- Let u be a unit of order 10. By (1) and Proposition 2 we have that

$$\nu_{2a}+\nu_{5a}+\nu_{5b}+\nu_{10a}+\nu_{10b} = 1.$$

Since $|u^2| = 5$, for any character χ of G we need to consider four cases, defined by part (iii) of Theorem 1:

$$\begin{array}{lrcl@{\qquad}lrcl}
\text{Case 1.} & \chi(u^2) & = & \chi(5a). & \text{Case 3.} & \chi(u^2) & = & 2\chi(5a)-\chi(5b).\\
\text{Case 2.} & \chi(u^2) & = & \chi(5b). & \text{Case 4.} & \chi(u^2) & = & -\chi(5a)+2\chi(5b).
\end{array}$$

Here and below, $\chi(5a)$ denote the value of the character χ on the representative of the conjugacy class C_{5a}, etc.

Applying Proposition 3 to characters in $\mathfrak{BCT}(11)$ we get the system of inequalities

$$\begin{aligned}
&\mu_0(u,\chi_2,11) = \tfrac{1}{10}(-4\nu_{2a}-2\nu_{5a}-2\nu_{5b}+6\nu_{10a}+6\nu_{10b}+4) \geq 0;\\
&\mu_5(u,\chi_2,11) = \tfrac{1}{10}(4\nu_{2a}+2\nu_{5a}+2\nu_{5b}-6\nu_{10a}-6\nu_{10b}+6) \geq 0;\\
&\mu_2(u,\chi_3,11) = \tfrac{1}{10}(2\nu_{2a}-4\nu_{5a}+\nu_{5b}+2\nu_{10a}-3\nu_{10b}+\alpha_1) \geq 0;\\
&\mu_3(u,\chi_3,11) = \tfrac{1}{10}(-2\nu_{2a}+4\nu_{5a}-\nu_{5b}-2\nu_{10a}+3\nu_{10b}+\alpha_2) \geq 0;\\
&\mu_0(u,\chi_6,11) = \tfrac{1}{10}(4\nu_{5a}+4\nu_{5b}+60) \geq 0;\\
&\mu_1(u,\chi_6,11) = \tfrac{1}{10}(-4\nu_{5a}+6\nu_{5b}+\alpha_3) \geq 0;\\
&\mu_4(u,\chi_6,11) = \tfrac{1}{10}(4\nu_{5a}-6\nu_{5b}+\alpha_3) \geq 0;\\
&\mu_5(u,\chi_6,11) = \tfrac{1}{10}(-4\nu_{5a}-4\nu_{5b}+60) \geq 0,
\end{aligned}$$

where $(\alpha_1,\alpha_2,\alpha_3)$ is equal to $(13,17,50)$,$(8,12,60)$,$(18,22,40)$ and $(3,7,70)$ in cases 1-4 respectively.

Now denote $t_1 = 2\nu_{2a}+\nu_{5a}+\nu_{5b}-3\nu_{10a}-3\nu_{10b} \in \mathbb{Z}$, then from the first two inequalities (recall that all $\mu_i(u,\chi_j,p)$ must be non-negative integers) we obtain that $t_1 \in \{-3,2\}$. Put $t_2 = 2\nu_{2a}-4\nu_{5a}+\nu_{5b}+2\nu_{10a}-3\nu_{10b} \in \mathbb{Z}$. From the third and fourth inequalities t_2 belongs to the set $\{-13,-3,7,17\}$, $\{-8,2,12\}$, $\{-18,-8,2,12,22\}$ and $\{-3,7\}$ in cases 1-4 respectively. Put $t_3 = \nu_{5a}+\nu_{5b} \in \mathbb{Z}$. From fifth and eighth inequalities it follows that that $t_3 \in \{5k \mid -3 \leq k \leq 3\}$.

Finally, put $t_4 = 2\nu_{5a} - 3\nu_{5b}$. Considering the sixth and seventh inequalities we get that t_4 belongs to the set $\{5k \mid -5 \le k \le 5\}$, $\{5k \mid -6 \le k \le 6\}$, $\{5k \mid -4 \le k \le 4\}$ and $\{5k \mid -7 \le k \le 7\}$ in cases 1-4 respectively.

We obtain the system of linear equations

$$\begin{aligned} \nu_{2a} + \nu_{5a} + \nu_{5b} + \nu_{10a} + \nu_{10b} &= 1; \\ 2\nu_{2a} + \nu_{5a} + \nu_{5b} - 3\nu_{10a} - 3\nu_{10b} &= t_1; \\ 2\nu_{2a} - 4\nu_{5a} + \nu_{5b} + 2\nu_{10a} - 3\nu_{10b} &= t_2; \\ \nu_{5a} + \nu_{5b} &= t_3; \\ 2\nu_{5a} - 3\nu_{5b} &= t_4. \end{aligned}$$

Since the matrix of the system is non-degenerate, such system has a unique solution for any values of parameters t_i. For each of the allowable values of t_1, t_2, t_3 and t_4 we thus can compute the unique integer solution of this system of equations.

Now Proposition 3 for $\mathfrak{BCT}(11)$ also gives the following additional inequalities

$$\begin{aligned} \mu_1(u,\chi_2,11) &= \tfrac{1}{10}(-\nu_{2a} + 2\nu_{5a} - 3\nu_{5b} + 4\nu_{10a} - \nu_{10b} + \beta_1) \ge 0; \\ \mu_2(u,\chi_2,11) &= \tfrac{1}{10}(\nu_{2a} + 3\nu_{5a} - 2\nu_{5b} + \nu_{10a} - 4\nu_{10b} + \beta_2) \ge 0; \\ \mu_3(u,\chi_2,11) &= \tfrac{1}{10}(-\nu_{2a} - 3\nu_{5a} + 2\nu_{5b} - \nu_{10a} + 4\nu_{10b} + \beta_3) \ge 0; \\ \mu_4(u,\chi_2,11) &= \tfrac{1}{10}(\nu_{2a} - 2\nu_{5a} + 3\nu_{5b} - 4\nu_{10a} + \nu_{10b} + \beta_4) \ge 0; \\ \mu_0(u,\chi_3,11) &= \tfrac{1}{10}(-8\nu_{2a} + 6\nu_{5a} + 6\nu_{5b} + 2\nu_{10a} + 2\nu_{10b} + 18) \ge 0; \\ \mu_1(u,\chi_3,11) &= \tfrac{1}{10}(-2\nu_{2a} - \nu_{5a} + 4\nu_{5b} + 3\nu_{10a} - 2\nu_{10b} + \beta_5) \ge 0; \\ \mu_4(u,\chi_3,11) &= \tfrac{1}{10}(2\nu_{2a} + \nu_{5a} - 4\nu_{5b} - 3\nu_{10a} + 2\nu_{10b} + \beta_6) \ge 0, \end{aligned}$$

where

$$\begin{aligned} (\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6) \in \{ \quad & (11,-4,6,9,12,8),\ (6,9,11,4,17,13), \\ & (16,-1,1,14,7,3),\ (1,14,16,-1,22,18) \quad \} \end{aligned}$$

in cases 1-4 respectively. It then follows that the only integer solutions with non-negative integers $\mu_i(u,\chi_j,11)$ are such that $\nu_{2a} = 0$ and

$$\begin{aligned} (\nu_{5a},\nu_{5b},\nu_{10a},\nu_{10b}) \in \{ \quad & (2,-2,0,1),\ (0,0,2,-1),\ (0,0,0,1),\ (-1,1,1,0), \\ & (1,-1,-1,2),\ (1,-1,1,0),\ (-2,2,1,0),\ (0,0,-1,2), \\ & (0,0,1,0),\ (-1,1,2,-1),\ (-1,1,0,1),\ (1,-1,0,1) \quad \}. \end{aligned}$$

So part (v) of Theorem 1 is proved.

- Let u be a unit of order 15. By (1) and Proposition 2 we have that

$$\nu_{3a} + \nu_{5a} + \nu_{5b} + \nu_{15a} + \nu_{15b} = 1.$$

Since $|u^3| = 5$, for any character χ of G we need to consider four cases, defined by part (iii) of Theorem 1:

$$\begin{array}{llcl} \text{Case 1.} & \chi(u^3) & = & \chi(5a). \\ \text{Case 2.} & \chi(u^3) & = & \chi(5b). \end{array} \qquad \begin{array}{llcl} \text{Case 3.} & \chi(u^3) & = & 2\chi(5a) - \chi(5b). \\ \text{Case 4.} & \chi(u^3) & = & -\chi(5a) + 2\chi(5b). \end{array}$$

Again applying Proposition 3 to characters in $\mathfrak{BCT}(11)$ we get the system

$$\begin{aligned}
\mu_0(u,\chi_2,11) &= \tfrac{1}{15}(8\nu_{3a}-4\nu_{5a}-4\nu_{5b}+8\nu_{15a}+8\nu_{15b}+7)\geq 0;\\
\mu_5(u,\chi_2,11) &= \tfrac{1}{15}(-4\nu_{3a}+2\nu_{5a}+2\nu_{5b}-4\nu_{15a}-4\nu_{15b}+4)\geq 0;\\
\mu_0(u,\chi_3,11) &= \tfrac{1}{15}(-8\nu_{3a}+12\nu_{5a}+12\nu_{5b}+12\nu_{15a}+12\nu_{15b}+18)\geq 0;\\
\mu_5(u,\chi_3,11) &= \tfrac{1}{15}(4\nu_{3a}-6\nu_{5a}-6\nu_{5b}-6\nu_{15a}-6\nu_{15b}+21)\geq 0;\\
\mu_1(u,\chi_4,11) &= \tfrac{1}{15}(-3\nu_{5a}+2\nu_{5b}+5\nu_{15b}+\alpha_1)\geq 0;\\
\mu_6(u,\chi_4,11) &= \tfrac{1}{15}(+6\nu_{5a}-4\nu_{5b}-10\nu_{15b}+\alpha_1)\geq 0;\\
\mu_1(u,\chi_6,11) &= \tfrac{1}{15}(2\nu_{3a}-4\nu_{5a}+6\nu_{5b}+2\nu_{15a}-3\nu_{15b}+\alpha_2)\geq 0;\\
\mu_1(u,\chi_7,11) &= \tfrac{1}{15}(-2\nu_{3a}+4\nu_{5a}-6\nu_{5b}-2\nu_{15a}+3\nu_{15b}+\alpha_3)\geq 0,
\end{aligned}$$

where $(\alpha_1,\alpha_2,\alpha_3)$ is equal to $(25,48,72)$,$(30,58,62)$,$(20,38,82)$ and $(35,68,52)$ in cases 1-4 respectively.

Now denote $t_1=2\nu_{3a}-\nu_{5a}-\nu_{5b}+2\nu_{15a}+2\nu_{15b}\in\mathbb{Z}$, then from the first two inequalities (again recall that all $\mu_i(u,\chi_j,p)$ must be non-negative integers) we obtain that $t_1=2$. Put $t_2=2\nu_{3a}-3\nu_{5a}-3\nu_{5b}-3\nu_{15a}-3\nu_{15b}\in\mathbb{Z}$. From the third and fourth inequalities $t_2=-3$. Put $t_3=3\nu_{5a}-2\nu_{5b}-5\nu_{15b}\in\mathbb{Z}$. From the fifth and sixth inequalities it follows that that t_3 belongs to the set $\{-5,10,25\}$, $\{-15,0,15,30\}$, $\{-10,5,20\}$ and $\{-10,5,20,35\}$ in cases 1-4 respectively.

Finally, put $t_4=2\nu_{3a}-4\nu_{5a}+6\nu_{5b}+2\nu_{15a}-3\nu_{15b}\in\mathbb{Z}$. Considering the seventh and eighth inequalities we get that t_4 belongs to the set $\{\ \gamma+15k\ \mid\ k=0,\dots,8\ \}$, where γ is equal to -48, -58, -38, -68 in cases 1-4 respectively. We obtain the system of linear equations

$$\begin{aligned}
\nu_{3a}+\nu_{5a}+\nu_{5b}+\nu_{15a}+\nu_{15b}&=1;\\
2\nu_{3a}-\nu_{5a}-\nu_{5b}+2\nu_{15a}+2\nu_{15b}&=2;\\
2\nu_{3a}-3\nu_{5a}-3\nu_{5b}-3\nu_{15a}-3\nu_{15b}&=-3;\\
3\nu_{5a}-2\nu_{5b}-5\nu_{15b}&=t_3;\\
2\nu_{3a}-4\nu_{5a}+6\nu_{5b}+2\nu_{15a}-3\nu_{15b}&=t_4.
\end{aligned}$$

Since the matrix of the system is non-degenerate, again such system has a unique solution for any values of the parameters t_i. For each of the allowable values of t_3 and t_4, we thus can compute the unique integer solution of this system of equations.

Now Proposition 3 for $\mathfrak{BCT}(11)$ also gives the following additional inequalities:

$$\begin{aligned}
\mu_1(u,\chi_2,11) &= \tfrac{1}{15}(\nu_{3a}+2\nu_{5a}-3\nu_{5b}-4\nu_{15a}+6\nu_{15b}+\beta_1)\geq 0;\\
\mu_2(u,\chi_2,11) &= \tfrac{1}{15}(\nu_{3a}-3\nu_{5a}+2\nu_{5b}+6\nu_{15a}-4\nu_{15b}+\beta_2)\geq 0;\\
\mu_3(u,\chi_2,11) &= \tfrac{1}{15}(-2\nu_{3a}+6\nu_{5a}-4\nu_{5b}-12\nu_{15a}+8\nu_{15b}+\beta_3)\geq 0;\\
\mu_6(u,\chi_2,11) &= \tfrac{1}{15}(-2\nu_{3a}-4\nu_{5a}+6\nu_{5b}+8\nu_{15a}-12\nu_{15b}+\beta_4)\geq 0;\\
\mu_1(u,\chi_3,11) &= \tfrac{1}{15}(-\nu_{3a}-\nu_{5a}+4\nu_{5b}-\nu_{15a}+4\nu_{15b}+\beta_5)\geq 0;\\
\mu_3(u,\chi_3,11) &= \tfrac{1}{15}(2\nu_{3a}-8\nu_{5a}+2\nu_{5b}-8\nu_{15a}+2\nu_{15b}+\beta_6)\geq 0;\\
\mu_6(u,\chi_3,11) &= \tfrac{1}{15}(2\nu_{3a}+2\nu_{5a}-8\nu_{5b}+2\nu_{15a}-8\nu_{15b}+\beta_7)\geq 0,
\end{aligned}$$

where $(\beta_1,\beta_2,\beta_3,\beta_4,\beta_5,\beta_6,\beta_7)$ is equal to $(9,4,7,12,11,13,8)$, $(4,9,12,7,16,8,13)$, $(14,-1,2,17,6,18,3)$ and $(-1,14,17,2,21,3,18)$ in cases 1-4 respectively. It then follows that the only integer solutions with non-negative integers $\mu_i(u,\chi_j,11)$ are such that $\nu_{3a}=0$ and

$$(\nu_{5a},\nu_{5b},\nu_{15a},\nu_{15b})\in\{\ (0,0,0,1),\ (0,0,1,0),\ (1,-1,1,0),\ (-1,1,0,1)\ \}.$$

So part (vi) of Theorem 1 is proved.

Now we will prove part (i). To do this we will show that $V(\mathbb{Z}G)$ has no elements of orders 14, 21, 22, 33, 35, 38, 55, 57, 77, 95, 133 and 209.

- Let u be a unit of order 14. By (1) and Proposition 2 we have $\nu_{2a}+\nu_{7a}=1$. By Proposition 3 we obtain the system of three inequalities

$$\begin{aligned}\mu_0(u,\chi_2,*)&=\tfrac{1}{14}(-6\nu_{2a}+6)\geq 0;\\ \mu_7(u,\chi_2,*)&=\tfrac{1}{14}(6\nu_{2a}+8)\geq 0;\\ \mu_6(u,\chi_2,*)&=\tfrac{1}{14}(\nu_{2a}+6)\geq 0,\end{aligned}$$

that has no integral solution.

- Let u be a unit of order 21. By (1) and Proposition 2 we have $\nu_{3a}+\nu_{7a}=1$. By Proposition 3 we obtain the system of three inequalities

$$\begin{aligned}\mu_7(u,\chi_2,*)&=\tfrac{1}{21}(-6\nu_{3a}+6)\geq 0;\\ \mu_0(u,\chi_2,*)&=\tfrac{1}{21}(12\nu_{3a}+9)\geq 0;\\ \mu_6(u,\chi_2,*)&=\tfrac{1}{21}(-2\nu_{3a}+9)\geq 0,\end{aligned}$$

that has no integral solution.

- Let u be a unit of order 22. By (1) and Proposition 2 we have $\nu_{2a}+\nu_{11a}=1$. Using Proposition 3 to characters in $\mathfrak{BCT}(19)$ we obtain that

$$\begin{aligned}\mu_0(u,\chi_2,19)&=\tfrac{1}{22}(-20\nu_{2a}+20)\geq 0;\\ \mu_{11}(u,\chi_2,19)&=\tfrac{1}{22}(20\nu_{2a}+24)\geq 0;\\ \mu_2(u,\chi_5,19)&=\tfrac{1}{22}(\nu_{2a}+54)\geq 0,\end{aligned}$$

and this system of inequalities has no integral solutions such that $\mu_0(u,\chi_2,19)$, $\mu_{11}(u,\chi_2,19)$ and $\mu_2(u,\chi_5,19)$ are non-negative integers.

- Let u be a unit of order 33. Obviously, $\nu_{3a}+\nu_{11a}=1$ by (1) and Proposition 2. Applying Proposition 3 to characters in $\mathfrak{BCT}(5)$ we get

$$\mu_0(u,\chi_5,5)=\tfrac{1}{33}(-20\nu_{2a}+75)\geq 0,\qquad \mu_{11}(u,\chi_5,5)=\tfrac{1}{33}(10\nu_{2a}+78)\geq 0.$$

This system of inequalities has no integral solutions such that $\mu_0(u,\chi_5,5)$ and $\mu_{11}(u,\chi_5,5)$ are non-negative integers.

- Let u be a unit of order 35. By (1) and Proposition 2 we have

$$\nu_{5a}+\nu_{5b}+\nu_{7a}=1.$$

Since u^7 has order 5 and by (iii) of Theorem 1 we have 4 different partial augmentations, we need to consider four cases. Again applying Proposition 3 to characters in $\mathfrak{BCT}(11)$ we obtain that in all of these four cases

$$\begin{aligned}\mu_0(u,\chi_2,11)&=\tfrac{1}{35}(-12(\nu_{5a}+\nu_{5b})+5)\geq 0;\\ \mu_5(u,\chi_2,11)&=\tfrac{1}{35}(2(\nu_{5a}+\nu_{5b})+5)\geq 0.\end{aligned}$$

Clearly, this system has no integral solutions such that $\mu_0(u,\chi_2,11),\mu_5(u,\chi_2,11)$ are non-negative integers.

- Let u be a unit of order 38. From (1) and Proposition 2 we obtain that

$$\nu_{2a}+\nu_{19a}+\nu_{19b}+\nu_{19c}=1.$$

Now u^2 has order 19, but we already proved that torsion units of this order are rationally conjugate to group elements. Because there are three conjugacy classes of elements of order 19, we need to consider three cases determined by a class representative that is rationally conjugate to u^2. By Proposition 3, using $\mathfrak{BCT}(11)$, in all three cases we have the system of inequalities

$$\begin{aligned}\mu_0(u,\chi_{14},11) =&\tfrac{1}{38}(18\nu_{2a}+210)\geq 0;\\ \mu_1(u,\chi_{14},11) =&\tfrac{1}{38}(\nu_{2a}+208)\geq 0;\\ \mu_{19}(u,\chi_{14},11) =&\tfrac{1}{38}(-18\nu_{2a}+208)\geq 0,\end{aligned}$$

that has no integral solutions such that $\mu_0(u,\chi_{14},11), \mu_1(u,\chi_{14},11)$ are non-negative integers.

- Let u be a unit of order 55. By (1) and Proposition 2 we have

$$\nu_{5a}+\nu_{5b}+\nu_{11a}=1.$$

By Proposition 3 we have the system of inequalities

$$\begin{aligned}\mu_0(u,\chi_9,*) =& \tfrac{1}{55}(-40\nu_{11a}+110)\geq 0;\\ \mu_5(u,\chi_9,*) =& \tfrac{1}{55}(4\nu_{11a}+121)\geq 0;\\ \mu_{11}(u,\chi_9,*) =& \tfrac{1}{55}(10\nu_{11a}+110)\geq 0,\end{aligned}$$

that has no integral solutions such that all $\mu_i(u,\chi_j,*)$ are non-negative integers.

- Let u be a unit of order 57. By (1) and Proposition 2 it yields that

$$\nu_{3a}+\nu_{19a}+\nu_{19b}+\nu_{19c}=1.$$

Since u^3 has order 19 and we have three conjugacy classes of elements of order 19, we need to consider three cases. By Proposition 3 using $\mathfrak{BCT}(11)$ we obtain that in all these three cases

$$\begin{aligned}\mu_0(u,\chi_{14},11) =& \tfrac{1}{57}(-36\nu_{3a}+207)\geq 0;\\ \mu_1(u,\chi_{14},11) =& \tfrac{1}{57}(-\nu_{3a}+210)\geq 0;\\ \mu_{19}(u,\chi_{14},11) =& \tfrac{1}{57}(18\nu_{2a}+210)\geq 0,\end{aligned}$$

that has no integral solutions such that $\mu_0(u,\chi_{14},11), \mu_1(u,\chi_{14},11)$ are non-negative integers.

- Let u be a unit of order 77. By (1) and Proposition 2 we have $\nu_{7a}+\nu_{11a}=1$. Using Proposition 3, we get the system of inequalities

$$\mu_0(u,\chi_2,*) = \tfrac{1}{77}(60\nu_{11a}+66)\geq 0, \quad \mu_{11}(u,\chi_2,*) = \tfrac{1}{77}(-10\nu_{11a}+66)\geq 0,$$

that has no integral solutions such that $\mu_0(u,\chi_2,*), \mu_{11}(u,\chi_2,*)$ are integers.

- Let u be a unit of order 95. By (1) and Proposition 2 we have

$$\nu_{5a}+\nu_{5b}+\nu_{19a}+\nu_{19b}+\nu_{19c}=1.$$

Clearly, u^{19} has order 5 and by part (iii) of Theorem 1 we have 4 different partial augmentations for elements of order 5. Moreover, u^5 has order 19 and we have three conjugacy classes of elements of order 19. Thus, we need to consider 12 cases. Using Proposition 3 for $\mathfrak{BCT}(11)$ we obtain that in all 12 cases

$$\begin{aligned}\mu_0(u,\chi_{14},11) =&\tfrac{1}{95}(-72(\nu_{5a}+\nu_{5b})+205)\geq 0;\\ \mu_{19}(u,\chi_{14},11) =&\tfrac{1}{95}(18(\nu_{5a}+\nu_{5b})+210)\geq 0,\end{aligned}$$

that has no integral solutions such that $\mu_0(u,\chi_{14},11), \mu_1(u,\chi_{14},11)$ are integers.

• Let u be a unit of order 133. By (1) and Proposition 2 it yields that

$$\nu_{7a}+\nu_{19a}+\nu_{19b}+\nu_{19c}=1.$$

Since u^7 has order 19, again we need to consider three cases. Again using Proposition 3 for $\mathfrak{BCT}(11)$ we obtain that in all three cases

$$\mu_0(u,\chi_{14},11)=\tfrac{1}{133}(-108\nu_{7a}+203)\geq 0;\quad \mu_{19}(u,\chi_{14},11)=\tfrac{1}{133}(18\nu_{7a}+210)\geq 0,$$

that has no integral solution such that $\mu_0(u,\chi_{14},11),\mu_1(u,\chi_{14},11)$ are integers.

• Let u be a unit of order 209. By (1) and Proposition 2 we have

$$\nu_{11a}+\nu_{19a}+\nu_{19b}+\nu_{19c}=1.$$

Since u^{11} has order 19 we need to consider three cases. Again applying Proposition 3 to $\mathfrak{BCT}(7)$ we obtain that in all three cases

$$\mu_0(u,\chi_{12},7)=\tfrac{1}{209}(180\nu_{11a}+143)\geq 0;\quad \mu_{19}(u,\chi_{12},7)=\tfrac{1}{209}(-18\nu_{7a}+132)\geq 0,$$

that has no integral solution such that $\mu_0(u,\chi_{12},7),\mu_{19}(u,\chi_{12},7)$ are integers.

It remains to prove part (vii) of the Theorem.

• Let u be a unit of order 30. Clearly, u^6, u^5, u^3 and u^2 have orders 5, 6, 10 and 15 respectively, and there are 4, 6, 12 and 4 tuples of partial augmentations for elements of these orders, given by parts (iii)-(vi) of Theorem 1. Thus we need to consider $4\cdot 6\cdot 12\cdot 4=1152$ cases.

Using the LAGUNA package [8] we constructed and solved all these cases, and it turns out that only the six cases in the table below yield a non-trivial solution.

$\chi(u^6)$	$\chi(u^5)$	$\chi(u^3)$	$\chi(u^2)$
$\chi(5b)$	$3\chi(3a)-2\chi(2a)$	$2\chi(5b)-2\chi(5a)+\chi(10a)$	$\chi(15a)$
$\chi(5b)$	$3\chi(3a)-2\chi(2a)$	$2\chi(10b)-\chi(10a)$	$\chi(15a)$
$\chi(5a)$	$3\chi(3a)-2\chi(2a)$	$2\chi(10a)-\chi(10b)$	$\chi(15b)$
$\chi(5a)$	$3\chi(3a)-2\chi(2a)$	$2\chi(5a)-2\chi(5b)+\chi(10b)$	$\chi(15b)$
$2\chi(5a)-\chi(5b)$	$3\chi(3a)-2\chi(2a)$	$\chi(10a)$	$\chi(5a)-\chi(5b)+\chi(15a)$
$2\chi(5b)-\chi(5a)$	$3\chi(3a)-2\chi(2a)$	$\chi(10b)$	$\chi(5b)-\chi(5a)+\chi(15b)$

Each of corresponding systems contains more than thirty inequalities, and has exactly one solution. We omit the technical computations and immediately list all these six solutions in part (vii) of Theorem 1.

4. PROOF OF THEOREM 2

Let G be the second Janko simple group J_2. It is well known [12, 13] that $|G|=2^7\cdot 3^3\cdot 5^2\cdot 7$ and $exp(G)=2^3\cdot 3\cdot 5\cdot 7$.

Since the group G only possesses elements of orders 2, 3, 4, 5, 6, 7, 8, 10, 12 and 15, we first shall investigate units of these orders. Because of Proposition 4, the order of each torsion unit divides the exponent of G, so it remains to consider units of orders 14, 20, 21, 24, 30 and 35. We shall prove that units of all these orders except 20, 24 and 30 do not appear in $V(\mathbb{Z}G)$.

We consider each case separately.

• Let u be a unit of order 2. By (1) and Proposition 2 we get $\nu_{2a}+\nu_{2b}=1$. By Proposition 3 we get the system of inequalities

$$\mu_0(u,\chi_2,*)=\tfrac{1}{2}(-2\nu_{2a}+2\nu_{2b}+14)\geq 0;\quad \mu_1(u,\chi_2,*)=\tfrac{1}{2}(2\nu_{2a}-2\nu_{2b}+14)\geq 0;$$

$$\mu_0(u,\chi_4,*)=\tfrac{1}{2}(5\nu_{2a}-3\nu_{2b}+21)\geq 0;\quad \mu_1(u,\chi_4,*)=\tfrac{1}{2}(-5\nu_{2a}+3\nu_{2b}+21)\geq 0,$$

that has only six integer solutions (ν_{2a},ν_{2b}) as listed in part (iii) of Theorem 2.

- Let u be a unit of order 3. By (1) and Proposition 2 we get $\nu_{2a} + \nu_{2b} = 1$. Again using Proposition 3 we get the system of inequalities

$$\begin{aligned}
\mu_0(u, \chi_2, *) &= \tfrac{1}{3}(10\nu_{3a} - 2\nu_{3b} + 14) \geq 0;\\
\mu_1(u, \chi_2, *) &= \tfrac{1}{3}(-5\nu_{3a} + \nu_{3b} + 14) \geq 0;\\
\mu_0(u, \chi_2, 2) &= \tfrac{1}{3}(-6\nu_{3a} + 6) \geq 0,
\end{aligned}$$

that has only three integer solutions (ν_{3a}, ν_{3b}) as listed in part (iv) of Theorem 2.

- Let u be a unit of order 4. By (1) and Proposition 2 we get $\nu_{2a} + \nu_{2b} + \nu_{4a} = 1$. We need to consider six cases defined by part (iii) of Theorem 2. In each of these cases, we apply Proposition 3 to get the following systems of inequalities:

$$\begin{aligned}
\mu_0(u, \chi_2, *) &= \tfrac{1}{4}(-4\nu_{2a} + 4\nu_{2b} + 4\nu_{4a} + \alpha) \geq 0;\\
\mu_2(u, \chi_2, *) &= \tfrac{1}{4}(4\nu_{2a} - 4\nu_{2b} - 4\nu_{4a} + \alpha) \geq 0;\\
\mu_0(u, \chi_4, *) &= \tfrac{1}{4}(10\nu_{2a} - 6\nu_{2b} + 2\nu_{4a} + \beta) \geq 0;\\
\mu_2(u, \chi_4, *) &= \tfrac{1}{4}(-10\nu_{2a} + 6\nu_{2b} - 2\nu_{4a} + \beta) \geq 0,
\end{aligned}$$

$$\text{where} \quad (\alpha, \beta) = \begin{cases}
(12, 26) \quad \text{when} \quad \chi(u^2) = \chi(2a);\\
(16, 18) \quad \text{when} \quad \chi(u^2) = \chi(2b);\\
(24, 2) \quad \text{when} \quad \chi(u^2) = -2\chi(2a) + 3\chi(2b);\\
(8, 34) \quad \text{when} \quad \chi(u^2) = 2\chi(2a) - \chi(2b);\\
(4, 42) \quad \text{when} \quad \chi(u^2) = 3\chi(2a) - 2\chi(2b);\\
(20, 10) \quad \text{when} \quad \chi(u^2) = -\chi(2a) + 2\chi(2b).
\end{cases}$$

Additionally we need to consider the following case-dependent inequalities:

$$\begin{aligned}
\mu_0(u, \chi_2, 3) &= \tfrac{1}{4}(-6\nu_{2a} + 2\nu_{2b} + 2\nu_{4a} + 10) \geq 0 \quad \text{for } \chi(u^2) = \chi(2a);\\
\mu_0(u, \chi_8, *) &= \tfrac{1}{4}(-20\nu_{2a} - 4\nu_{2b} + 4\nu_{4a} + 68) \geq 0 \quad \text{for } \chi(u^2) = \chi(2b);\\
\mu_2(u, \chi_7, *) &= \tfrac{1}{4}(-30\nu_{2a} + 2\nu_{2b} - 6\nu_{4a} + 30) \geq 0 \quad \text{for } \chi(u^2) = -2\chi(2a) + 3\chi(2b);\\
\mu_0(u, \chi_4, 5) &= \tfrac{1}{4}(18\nu_{2a} + 2\nu_{2b} + 2\nu_{4a} + 26) \geq 0 \quad \text{for } \chi(u^2) = -2\chi(2a) + 3\chi(2b);\\
\mu_0(u, \chi_8, *) &= \tfrac{1}{4}(-20\nu_{2a} - 4\nu_{2b} + 4\nu_{4a} + 52) \geq 0 \quad \text{for } \chi(u^2) = 2\chi(2a) - \chi(2b);\\
\mu_0(u, \chi_2, 3) &= \tfrac{1}{4}(-6\nu_{2a} + 2\nu_{2b} + 2\nu_{4a} + 2) \geq 0 \quad \text{for } \chi(u^2) = 3\chi(2a) - 2\chi(2b);\\
\mu_0(u, \chi_8, *) &= \tfrac{1}{4}(-20\nu_{2a} - 4\nu_{2b} + 4\nu_{4a} + 76) \geq 0 \quad \text{for } \chi(u^2) = -\chi(2a) + 2\chi(2b);\\
\mu_0(u, \chi_{10}, *) &= \tfrac{1}{4}(20\nu_{2a} + 12\nu_{2b} - 4\nu_{4a} + 92) \geq 0 \quad \text{for } \chi(u^2) = -\chi(2a) + 2\chi(2b).
\end{aligned}$$

Solving these systems and applying Proposition 5 to the obtained solutions, we get only fifteen integer solutions $(\nu_{2a}, \nu_{2b}, \nu_{4a})$ as listed in part (v) of Theorem 2.

- Let u be a unit of order 5. By (1) and Proposition 2 we get

$$\nu_{5a} + \nu_{5b} + \nu_{5c} + \nu_{5d} = 1.$$

By Proposition 3 we get the system of inequalities

$$\begin{aligned}
\mu_1(u, \chi_8, *) &= \tfrac{1}{5}(-15\nu_{5a} + 10\nu_{5b} + 70) \geq 0;\\
\mu_2(u, \chi_8, *) &= \tfrac{1}{5}(10\nu_{5a} - 15\nu_{5b} + 70) \geq 0;\\
\mu_0(u, \chi_{10}, *) &= \tfrac{1}{5}(20\nu_{5a} + 20\nu_{5b} + 90) \geq 0;\\
\mu_0(u, \chi_{12}, *) &= \tfrac{1}{5}(-20\nu_{5a} - 20\nu_{5b} + 160) \geq 0,
\end{aligned}$$

from which we can derive 71 possible pairs (ν_{5a}, ν_{5b}). From the inequalities

$$\begin{aligned}
\mu_0(u, \chi_7, 2) &= \tfrac{1}{5}(16\nu_{5a} + 16\nu_{5b} - 4\nu_{5c} - 4\nu_{5d} + 64) \geq 0;\\
\mu_0(u, \chi_6, 2) &= \tfrac{1}{5}(-16\nu_{5a} - 16\nu_{5b} + 4\nu_{5c} + 4\nu_{5d} + 36) \geq 0,
\end{aligned}$$

if follows that $t = 4\nu_{5a} + 4\nu_{5b} - \nu_{5c} - \nu_{5d} \in \{-16, -11, -6, -1, 4, 9\}$. Taking into account that $\nu_{5a} + \nu_{5b} + \nu_{5c} + \nu_{5d} = 1$ and considering the additional inequality

$$\mu_0(u, \chi_2, 2) = \tfrac{1}{5}(4\nu_{5a} + 4\nu_{5b} - 6\nu_{5c} - 6\nu_{5d} + 6) \geq 0;$$

it is easy to check that only the following 16 possibilities for $(\nu_{5a}, \nu_{5b}, \nu_{5c} + \nu_{5d})$ remain:

$$\begin{aligned}\{&(-2,2,1),(-2,3,0),(-1,1,1),(-1,2,0),(-1,3,-1),(0,0,1),(0,1,0),(0,2,-1),\\ &(1,-1,1),(1,0,0),(1,1,-1),(2,-2,1),(2,-1,0),(2,0,-1),(3,-2,0),(3,-1,-1)\}.\end{aligned}$$

Finally, using the inequalities

$$\begin{aligned}
\mu_1(u, \chi_2, 2) &= \tfrac{1}{5}(-6\nu_{5a} + 4\nu_{5b} - \nu_{5c} + 4\nu_{5d} + 6) \geq 0;\\
\mu_2(u, \chi_2, 2) &= \tfrac{1}{5}(4\nu_{5a} - 6\nu_{5b} + 4\nu_{5c} - \nu_{5d} + 6) \geq 0;\\
\mu_1(u, \chi_4, 2) &= \tfrac{1}{5}(-9\nu_{5a} + 6\nu_{5b} + \nu_{5c} - 4\nu_{5d} + 14) \geq 0;\\
\mu_2(u, \chi_4, 2) &= \tfrac{1}{5}(6\nu_{5a} - 9\nu_{5b} - 4\nu_{5c} + \nu_{5d} + 14) \geq 0;\\
\mu_1(u, \chi_7, 2) &= \tfrac{1}{5}(6\nu_{5a} - 14\nu_{5b} + 6\nu_{5c} - 4\nu_{5d} + 64) \geq 0;\\
\mu_2(u, \chi_7, 2) &= \tfrac{1}{5}(-14\nu_{5a} + 6\nu_{5b} - 4\nu_{5c} + 6\nu_{5d} + 64) \geq 0,
\end{aligned}$$

we obtain only ten integer solutions for $(\nu_{5a}, \nu_{5b}, \nu_{5c}, \nu_{5d})$. These are listed in part (vi) of Theorem 2.

• Let u be a unit of order 7. Since there is only one conjugacy class in G consisting of elements or order 7, this case follows immediately from Proposition 2.

• Let u be a unit of order 8. By (1) and Proposition 2 we get

$$\nu_{2a} + \nu_{2b} + \nu_{4a} + \nu_{8a} = 1.$$

Because $|u^2| = 4$ and $|u^4| = 2$, we need to consider 90 cases defined by parts (iii) and (v) of Theorem 2. First, in 45 of these cases, given in the following table, we have no units of order 8 because $\mu_1(u, \chi_2, *)$ is not an integer:

$\chi(u^2)$	$\chi(u^4) = \chi(2b)$	$\chi(u^4) = -2\chi(2a) + 3\chi(2b)$	$\chi(u^4) = 2\chi(2a) - \chi(2b)$
$\chi(4a)$ $-2\chi(2a) - 2\chi(2b) + 5\chi(4a)$ $-\chi(2a) - 3\chi(2b) + 5\chi(4a)$ $-\chi(2a) - \chi(2b) + 3\chi(4a)$ $-\chi(2a) + \chi(2b) + \chi(4a)$ $-4\chi(2b) + 5\chi(4a)$ $-2\chi(2b) + 3\chi(4a)$ $2\chi(2b) - \chi(4a)$ $4\chi(2b) - 3\chi(4a)$ $\chi(2a) - 3\chi(2b) + 3\chi(4a)$ $\chi(2a) - \chi(2b) + \chi(4a)$ $\chi(2a) + \chi(2b) - \chi(4a)$ $\chi(2a) + 3\chi(2b) - 3\chi(4a)$ $2\chi(2a) - \chi(4a)$ $2\chi(2a) + 2\chi(2b) - 3\chi(4a)$	$\mu_1(u, \chi_2, *) = \frac{3}{2}$	$\mu_1(u, \chi_2, *) = \frac{1}{2}$	$\mu_1(u, \chi_2, *) = \frac{5}{2}$

Then, when $\chi(u^4) = 3\chi(2a) - 2\chi(2b)$ and $\chi(u^2)$ is equal to $2\chi(2a) - \chi(4a)$ or $2\chi(2a) + 2\chi(2b) - 3\chi(4a)$, we obtain the system of inequalities

$$\begin{aligned}
\mu_0(u, \chi_2, *) &= \tfrac{1}{8}(-8\nu_{2a} + 8\nu_{2b} + 8\nu_{4a} - 8) \geq 0;\\
\mu_4(u, \chi_2, *) &= \tfrac{1}{8}(8\nu_{2a} - 8\nu_{2b} - 8\nu_{4a} - 8) \geq 0,
\end{aligned}$$

which have no integer solutions. Also, there is no solution for the system

$$\begin{aligned}
\mu_0(u, \chi_4, *) &= \tfrac{1}{8}(20\nu_{2a} - 12\nu_{2b} + 4\nu_{4a} - 4\nu_{8a} + \alpha) \geq 0;\\
\mu_4(u, \chi_4, *) &= \tfrac{1}{8}(-20\nu_{2a} + 12\nu_{2b} - 4\nu_{4a} + 4\nu_{8a} + \alpha) \geq 0,
\end{aligned}$$

where $\alpha = -4$ for $(\chi(u^4), \chi(u^2))$ in the following set

$$\{\ (\chi(2a),\ 4\chi(2b) - 3\chi(4a)),\ (-\chi(2a) + 2\chi(2b),\ -\chi(2a) + \chi(2b) + \chi(4a)),$$
$$(-\chi(2a) + 2\chi(2b),\ 2\chi(2b) - \chi(4a)),\ (-\chi(2a) + 2\chi(2b),\ \chi(2a) + 3\chi(2b) - 3\chi(4a))\ \}$$

and $\alpha = -20$ for $(\chi(u^4), \chi(u^2)) = (-\chi(2a) + 2\chi(2b), 4\chi(2b) - 3\chi(4a))$.

In the remaining 38 cases first we consider the following system of inequalities:

$$\begin{aligned}
\mu_0(u, \chi_2, *) &= \tfrac{1}{8}(-8\nu_{2a} + 8\nu_{2b} + 8\nu_{4a} + \alpha_1) \geq 0;\\
\mu_4(u, \chi_2, *) &= \tfrac{1}{8}(8\nu_{2a} - 8\nu_{2b} - 8\nu_{4a} + \alpha_1) \geq 0;\\
\mu_0(u, \chi_4, *) &= \tfrac{1}{8}(20\nu_{2a} - 12\nu_{2b} + 4\nu_{4a} - 4\nu_{8a} + \alpha_2) \geq 0;\\
\mu_4(u, \chi_4, *) &= \tfrac{1}{8}(-20\nu_{2a} + 12\nu_{2b} - 4\nu_{4a} + 4\nu_{8a} + \alpha_2) \geq 0;\\
\mu_0(u, \chi_7, *) &= \tfrac{1}{8}(60\nu_{2a} - 4\nu_{2b} + 12\nu_{4a} + 4\nu_{8a} + \alpha_3) \geq 0;\\
\mu_4(u, \chi_7, *) &= \tfrac{1}{8}(-60\nu_{2a} + 4\nu_{2b} - 12\nu_{4a} - 4\nu_{8a} + \alpha_3) \geq 0,
\end{aligned}$$

where the tuples $(\alpha_1, \alpha_2, \alpha_3)$ are given in the following table:

	$\chi(u^4)$	$\chi(u^2)$	$(\alpha_1, \alpha_2, \alpha_3)$
1	$\chi(2a)$	$\chi(4a)$	$(16, 28, 84)$
2		$-2\chi(2a) - 2\chi(2b) + 5\chi(4a)$	$(32, 28, 52)$
3		$-\chi(2a) - 3\chi(2b) + 5\chi(4a)$	$(24, 44, 84)$
4		$-\chi(2a) - \chi(2b) + 3\chi(4a)$	$(24, 28, 68)$
5		$-\chi(2a) + \chi(2b) + \chi(4a)$	$(24, 12, 52)$
6		$-4\chi(2b) + 5\chi(4a)$	$(16, 60, 116)$
7		$-2\chi(2b) + 3\chi(4a)$	$(16, 44, 100)$
8		$2\chi(2b) - \chi(4a)$	$(16, 12, 68)$
9		$\chi(2a) - 3\chi(2b) + 3\chi(4a)$	$(8, 60, 132)$
10		$\chi(2a) - \chi(2b) + \chi(4a)$	$(8, 44, 116)$
11		$\chi(2a) + \chi(2b) - \chi(4a)$	$(8, 28, 100)$
12		$\chi(2a) + 3\chi(2b) - 3\chi(4a)$	$(8, 12, 84)$
13		$2\chi(2a) - \chi(4a)$	$(0, 44, 132)$
14		$2\chi(2a) + 2\chi(2b) - 3\chi(4a)$	$(0, 28, 116)$
15	$3\chi(2a) - 2\chi(2b)$	$\chi(4a)$	$(8, 44, 116)$
16		$-2\chi(2a) - 2\chi(2b) + 5\chi(4a)$	$(24, 44, 84)$
17		$-\chi(2a) - 3\chi(2b) + 5\chi(4a)$	$(16, 60, 116)$
18		$-\chi(2a) - \chi(2b) + 3\chi(4a)$	$(16, 44, 100)$
19		$-\chi(2a) + \chi(2b) + \chi(4a)$	$(16, 28, 84)$
20		$-4\chi(2b) + 5\chi(4a)$	$(8, 76, 148)$
21		$-2\chi(2b) + 3\chi(4a)$	$(8, 60, 132)$
22		$2\chi(2b) - \chi(4a)$	$(8, 28, 100)$
23		$4\chi(2b) - 3\chi(4a)$	$(8, 12, 84)$
24		$\chi(2a) - 3\chi(2b) + 3\chi(4a)$	$(0, 76, 164)$
25		$\chi(2a) - \chi(2b) + \chi(4a)$	$(0, 60, 148)$
26		$\chi(2a) + \chi(2b) - \chi(4a)$	$(0, 44, 132)$
27		$\chi(2a) + 3\chi(2b) - 3\chi(4a)$	$(0, 28, 116)$
28	$-\chi(2a) + 2\chi(2b)$	$\chi(4a)$	$(24, 12, 52)$
29		$-2\chi(2a) - 2\chi(2b) + 5\chi(4a)$	$(40, 12, 20)$
30		$-\chi(2a) - 3\chi(2b) + 5\chi(4a)$	$(32, 28, 52)$
31		$-\chi(2a) - \chi(2b) + 3\chi(4a)$	$(32, 12, 36)$
32		$-4\chi(2b) + 5\chi(4a)$	$(24, 44, 84)$
33		$-2\chi(2b) + 3\chi(4a)$	$(24, 28, 68)$
34		$\chi(2a) - 3\chi(2b) + 3\chi(4a)$	$(16, 44, 100)$
35		$\chi(2a) - \chi(2b) + \chi(4a)$	$(16, 28, 84)$
36		$\chi(2a) + \chi(2b) - \chi(4a)$	$(16, 12, 68)$
37		$2\chi(2a) - \chi(4a)$	$(8, 28, 100)$
38		$2\chi(2a) + 2\chi(2b) - 3\chi(4a)$	$(8, 12, 84)$

In all 38 cases these inequalities allow us to compute admissible solutions, using the technique explained in detail in the proof of Theorem 1 in case of units of order 10. Having done this, we need to consider additional inequalities and apply Proposition 5 to reduce the number of solutions or, possibly, eliminate all of them.

In cases 1, 19 and 35 we use the system of inequalities

$$\begin{aligned}
\mu_0(u, \chi_2, 3) &= \tfrac{1}{8}(-12\nu_{2a} + 4\nu_{2b} + 4\nu_{4a} - 4\nu_{8a} + 12) \geq 0;\\
\mu_0(u, \chi_7, 3) &= \tfrac{1}{8}(-28\nu_{2a} - 12\nu_{2b} + 4\nu_{4a} + 4\nu_{8a} + 52) \geq 0;\\
\mu_4(u, \chi_7, 3) &= \tfrac{1}{8}(28\nu_{2a} + 12\nu_{2b} - 4\nu_{4a} - 4\nu_{8a} + 52) \geq 0,
\end{aligned}$$

to obtain in all three cases the same set of solutions $\{\ (0,2,0,-1),\ (0,-2,2,1),\ (0,0,2,-1),\ (0,0,0,1),\ (-1,-1,2,1),\ (-1,-1,0,3),\ (1,1,2,-3),\ (1,1,0,-1)\ \}$.

In cases 2, 3, 4, 5, 7, 16, 18, 28, 30, 31, 32, 33 and 34 we use the system

$$\begin{aligned}
\mu_4(u,\chi_{10},*) &= \tfrac{1}{8}(-40\nu_{2a} - 24\nu_{2b} + 8\nu_{4a} + \beta_1) \geq 0;\\
\mu_0(u,\chi_2,3) &= \tfrac{1}{8}(-12\nu_{2a} + 4\nu_{2b} + 4\nu_{4a} - 4\nu_{8a} + \beta_2) \geq 0;\\
\mu_0(u,\chi_{10},7) &= \tfrac{1}{8}(36\nu_{2a} + 20\nu_{2b} - 12\nu_{4a} - 4\nu_{8a} + \beta_3) \geq 0,
\end{aligned}$$

and the following table describes tuples $(\beta_1,\beta_2,\beta_3)$ and solutions for each case:

Cases	$(\beta_1,\beta_2,\beta_3)$	$(\nu_{2a},\ \nu_{2b},\ \nu_{4a},\ \nu_{8a})$
$2,30$	$(16,28,12)$	$(0,0,-2,3),\ (0,0,0,1)$
$3,16,32$	$(24,20,20)$	$(0,0,2,-1),\ (0,0,-2,3),\ (0,0,0,1),$ $(1,-1,2,-1),\ (1,-1,0,1),\ (-1,1,0,1)$
$4,33$	$(56,20,52)$	$(0,2,0,-1),\ (0,-2,0,3),\ (0,0,2,-1),\ (0,0,-2,3),$ $(0,0,0,1),\ (1,-1,0,1),\ (1,1,2,-3)$
$5,28$	$(88,20,84)$	$(0,-2,0,3),\ (0,0,2,-1),\ (0,0,0,1),$ $(-1,-1,2,1),\ (1,1,-2,1),\ (1,1,0,-1)$
$7,18,34$	$(64,12,60)$	$(0,2,-2,1),\ (0,2,0,-1),\ (0,0,2,-1),\ (0,0,0,1),$ $(1,-1,2,-1),\ (1,1,2,-3),\ (1,1,0,-1),\ (-1,1,0,1)$
31	$(48,28,44)$	$(0,2,2,-3),\ (0,0,2,-1),\ (0,0,0,1)$

In cases 6 and 17 we use the system of inequalities

$$\begin{aligned}
\mu_4(u,\chi_{10},*) &= \tfrac{1}{8}(-40\nu_{2a} - 24\nu_{2b} + 8\nu_{4a} + 32) \geq 0;\\
\mu_0(u,\chi_2,3) &= \tfrac{1}{8}(-12\nu_{2a} + 4\nu_{2b} + 4\nu_{4a} - 4\nu_{8a} + 12) \geq 0;\\
\mu_0(u,\chi_{10},7) &= \tfrac{1}{8}(36\nu_{2a} + 20\nu_{2b} - 12\nu_{4a} - 4\nu_{8a} + 28) \geq 0;\\
\mu_0(u,\chi_{12},7) &= \tfrac{1}{8}(-16\nu_{2a} + 16\nu_{2b} - 16\nu_{4a} + 48) \geq 0;\\
\mu_4(u,\chi_{12},7) &= \tfrac{1}{8}(16\nu_{2a} - 16\nu_{2b} + 16\nu_{4a} + 48) \geq 0,
\end{aligned}$$

to obtain in both cases the same set of solutions $\{\ (0,0,2,-1), (0,0,0,1), (-1,1,0,1)\}$.

In cases 8 and 36 we use the system of inequalities

$$\begin{aligned}
\mu_4(u,\chi_8,*) &= \tfrac{1}{8}(40\nu_{2a} + 8\nu_{2b} - 8\nu_{4a} + 48) \geq 0;\\
\mu_0(u,\chi_2,3) &= \tfrac{1}{8}(-12\nu_{2a} + 4\nu_{2b} + 4\nu_{4a} - 4\nu_{8a} + 12) \geq 0;\\
\mu_0(u,\chi_7,3) &= \tfrac{1}{8}(-28\nu_{2a} - 12\nu_{2b} + 4\nu_{4a} + 4\nu_{8a} + 36) \geq 0,
\end{aligned}$$

to obtain in both cases the same set of solutions $\{(0,0,2,-1), (0,0,0,1)\}$.

In cases 9, 20 and 21 we use the system

$$\begin{aligned}
\mu_4(u,\chi_{10},*) &= \tfrac{1}{8}(-40\nu_{2a} - 24\nu_{2b} + 8\nu_{4a} + \beta_1) \geq 0;\\
\mu_0(u,\chi_2,3) &= \tfrac{1}{8}(-12\nu_{2a} + 4\nu_{2b} + 4\nu_{4a} - 4\nu_{8a} + 4) \geq 0;\\
\mu_0(u,\chi_{10},7) &= \tfrac{1}{8}(36\nu_{2a} + 20\nu_{2b} - 12\nu_{4a} - 4\nu_{8a} + \beta_2) \geq 0;\\
\mu_0(u,\chi_{12},7) &= \tfrac{1}{8}(-16\nu_{2a} + 16\nu_{2b} - 16\nu_{4a} + \beta_3) \geq 0;\\
\mu_4(u,\chi_{12},7) &= \tfrac{1}{8}(16\nu_{2a} - 16\nu_{2b} + 16\nu_{4a} + \beta_3) \geq 0,
\end{aligned}$$

where $(\beta_1,\beta_2,\beta_3)$ is equal to $(72,68,64)$ in cases 9 and 21, and to $(40,36,32)$ in case 20. This leads to the solutions $(0,2,-2,1)$, $(0,-2,2,1)$, $(2,0,2,-3)$, $(0,0,0,1)$, $(1,-1,2,-1)$, $(-1,-1,0,3)$, $(-1,1,-2,3)$, $(1,1,0,-1)$ in cases 9 and 21 and to the unique solution $(0,0,0,1)$ in case 20.

In cases 10, 11, 12, 15, 22, 23, 37 and 38 we use the system

$$\begin{aligned}
\mu_0(u,\chi_8,*) &= \tfrac{1}{8}(-40\nu_{2a} - 8\nu_{2b} + 8\nu_{4a} + \beta_1) \geq 0;\\
\mu_0(u,\chi_2,3) &= \tfrac{1}{8}(-12\nu_{2a} + 4\nu_{2b} + 4\nu_{4a} - 4\nu_{8a} + 4) \geq 0;\\
\mu_4(u,\chi_7,3) &= \tfrac{1}{8}(28\nu_{2a} + 12\nu_{2b} - 4\nu_{4a} - 4\nu_{8a} + \beta_2) \geq 0,
\end{aligned}$$

and the following table describes tuples (β_1, β_2) and solutions for each case:

Cases	(β_1,β_2)	$(\nu_{2a},\ \nu_{2b},\ \nu_{4a},\ \nu_{8a})$
$10, 15$	$(48, 44)$	$(0,2,-2,1),\ (0,-2,2,1),\ (0,0,0,1),\ (1,-1,2,-1),\ (1,1,0,-1)$
$11, 22, 37$	$(32, 28)$	$(0,0,0,1)$
$12, 23, 38$	$(16, 12)$	$(0,0,0,1)$

Finally, in cases 13, 14, 24, 25, 26, 27 we use the additional inequality

$$\mu_0(u,\chi_2,3) = \tfrac{1}{8}(-12\nu_{2a} + 4\nu_{2b} + 4\nu_{4a} - 4\nu_{8a} - 4) \geq 0,$$

and in the case 29 we use two additional inequalities

$$\begin{aligned}
\mu_4(u,\chi_{10},*) &= \tfrac{1}{8}(-40\nu_{2a} - 24\nu_{2b} + 8\nu_{4a} + 8) \geq 0;\\
\mu_0(u,\chi_{10},7) &= \tfrac{1}{8}(36\nu_{2a} + 20\nu_{2b} - 12\nu_{4a} - 4\nu_{8a} + 4) \geq 0,
\end{aligned}$$

to show that in these cases we have no solutions.

Now the union of the solutions obtained above gives us part (vii) of Theorem 2.

- Let u be a unit of order 15. By (1) and Proposition 2 we get

$$\nu_{3a} + \nu_{3b} + \nu_{5a} + \nu_{5b} + \nu_{5c} + \nu_{5d} + \nu_{15a} + \nu_{15b} = 1.$$

We need to consider 30 cases defined by parts (iv) and (vi) of Theorem 2. Only in two cases we will get a system of inequalities that has solutions. Furthermore, each time these solutions are trivial. This will prove that units of order 15 are conjugate to group elements. Now we will show this in detail.

First, in eight cases we obtain the following system of inequalities that has no solutions such that all $\mu_i(u,\chi_j,*)$ are non-negative integers:

$$\begin{aligned}
\mu_0(u,\chi_6,*) &= \tfrac{1}{15}(72\nu_{3a} - 32\nu_{5a} - 32\nu_{5b} + 8\nu_{5c} + 8\nu_{5d} - 8\nu_{15a} - 8\nu_{15b} + \alpha) \geq 0;\\
\mu_5(u,\chi_6,*) &= \tfrac{1}{15}(-36\nu_{3a} + 16\nu_{5a} + 16\nu_{5b} - 4\nu_{5c} - 4\nu_{5d} + 4\nu_{15a} + 4\nu_{15b} + \beta) \geq 0,
\end{aligned}$$

where the tuples (α, β) are given in the following table:

$\chi(u^5)$	$\chi(u^3)$	(α,β)
$\chi(3a)$	$\chi(5a)+\chi(5b)-\chi(5d)$ $\chi(5a)+\chi(5b)-\chi(5c)$	$(18,-9)$
$-\chi(3a)+2\chi(3b)$	$\chi(5a)$ $\chi(5b)$ $\chi(5a)-\chi(5c)+\chi(5d)$ $\chi(5b)+\chi(5c)-\chi(5d)$	$(2,29)$
$-\chi(3a)+2\chi(3b)$	$\chi(5a)+\chi(5b)-\chi(5d)$ $\chi(5b)+\chi(5c)-\chi(5d)$	$(-18,9)$

In four other cases we obtain the following system of inequalities that has no solution such that all $\mu_i(u,\chi_j,*)$ are non-negative integers:

$$\begin{aligned}
\mu_0(u,\chi_2,*) &= \tfrac{1}{15}(40\nu_{3a} - 8\nu_{3b} + 12\nu_{5a} + 12\nu_{5b} + 12\nu_{5c} + 12\nu_{5d} + 6) \geq 0;\\
\mu_1(u,\chi_2,*) &= \tfrac{1}{15}(5\nu_{3a} - \nu_{3b} - 6\nu_{5a} + 9\nu_{5b} + 4\nu_{5c} - \nu_{5d} + \alpha_1) \geq 0;\\
\mu_3(u,\chi_2,*) &= \tfrac{1}{15}(-10\nu_{3a} + 2\nu_{3b} - 18\nu_{5a} + 12\nu_{5b} + 2\nu_{5c} - 8\nu_{5d} + \alpha_7) \geq 0;\\
\mu_5(u,\chi_2,*) &= \tfrac{1}{15}(-20\nu_{3a} + 4\nu_{3b} - 6\nu_{5a} - 6\nu_{5b} - 6\nu_{5c} - 6\nu_{5d} + 27) \geq 0;
\end{aligned}$$

$$\begin{aligned}
\mu_6(u,\chi_2,*) &= \tfrac{1}{15}(-10\nu_{3a}+2\nu_{3b}+12\nu_{5a}-18\nu_{5b}-8\nu_{5c}+2\nu_{5d}+\alpha_2)\geq 0;\\
\mu_0(u,\chi_4,*) &= \tfrac{1}{15}(24\nu_{3a}+28\nu_{5a}+28\nu_{5b}+8\nu_{5c}+8\nu_{5d}+4\nu_{15a}+4\nu_{15b}+19)\geq 0;\\
\mu_1(u,\chi_4,*) &= \tfrac{1}{15}(3\nu_{3a}+6\nu_{5a}+\nu_{5b}-4\nu_{5c}+6\nu_{5d}+3\nu_{15a}-2\nu_{15b}+\alpha_3)\geq 0;\\
\mu_5(u,\chi_4,*) &= \tfrac{1}{15}(-12\nu_{3a}-14\nu_{5a}-14\nu_{5b}-4\nu_{5c}-4\nu_{5d}-2\nu_{15a}-2\nu_{15b}+28)\geq 0;\\
\mu_6(u,\chi_4,*) &= \tfrac{1}{15}(-6\nu_{3a}-12\nu_{5a}-2\nu_{5b}+8\nu_{5c}-12\nu_{5d}-6\nu_{15a}+4\nu_{15b}+\alpha_4)\geq 0;\\
\mu_0(u,\chi_6,*) &= \tfrac{1}{15}(72\nu_{3a}-32\nu_{5a}-32\nu_{5b}+8\nu_{5c}+8\nu_{5d}-8\nu_{15a}-8\nu_{15b}+22)\geq 0;\\
\mu_3(u,\chi_6,*) &= \tfrac{1}{15}(-18\nu_{3a}+8\nu_{5a}+8\nu_{5b}-2\nu_{5c}-2\nu_{5d}+2\nu_{15a}+2\nu_{15b}+17)\geq 0;\\
\mu_0(u,\chi_{11},*) &= \tfrac{1}{15}(-72\nu_{3a}+8\nu_{5a}+8\nu_{5b}+8\nu_{5c}+8\nu_{5d}+8\nu_{15a}+8\nu_{15b}+148)\geq 0;\\
\mu_5(u,\chi_{11},*) &= \tfrac{1}{15}(36\nu_{3a}-4\nu_{5a}-4\nu_{5b}-4\nu_{5c}-4\nu_{5d}-4\nu_{15a}-4\nu_{15b}+121)\geq 0;\\
\mu_1(u,\chi_{16},*) &= \tfrac{1}{15}(8\nu_{3a}-\nu_{3b}+9\nu_{5a}-11\nu_{5b}+4\nu_{5c}-6\nu_{5d}+3\nu_{15a}-2\nu_{15b}+\alpha_5)\geq 0;\\
\mu_6(u,\chi_{16},*) &= \tfrac{1}{15}(-16\nu_{3a}+2\nu_{3b}-18\nu_{5a}+22\nu_{5b}-8\nu_{5c}\\
&\qquad\qquad +12\nu_{5d}-6\nu_{15a}+4\nu_{15b}+\alpha_6)\geq 0,
\end{aligned}$$

where the tuples $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7)$ are given in the following table:

$\chi(u^5)$	$\chi(u^3)$	$(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7)$
$-\chi(3a)+2\chi(3b)$	$\chi(5c)$	$(22,1,18,9,240,210,-4)$
$-\chi(3a)+2\chi(3b)$	$\chi(5d)$	$(17,-4,28,19,230,200,1)$
$-\chi(3a)+2\chi(3b)$	$2\chi(5c)-\chi(5d)$	$(27,6,8,-1,250,220,-9)$
$-\chi(3a)+2\chi(3b)$	$-\chi(5c)+2\chi(5d)$	$(12,-9,38,29,220,190,6)$

In the remaining 18 cases first we consider the following system of inequalities:

$$\begin{aligned}
\mu_0(u,\chi_2,*) &= \tfrac{1}{15}(40\nu_{3a}-8\nu_{3b}+12\nu_{5a}+12\nu_{5b}+12\nu_{5c}+12\nu_{5d}+\alpha_1)\geq 0;\\
\mu_1(u,\chi_2,*) &= \tfrac{1}{15}(5\nu_{3a}-\nu_{3b}-6\nu_{5a}+9\nu_{5b}+4\nu_{5c}-\nu_{5d}+\alpha_2)\geq 0;\\
\mu_5(u,\chi_2,*) &= \tfrac{1}{15}(-20\nu_{3a}+4\nu_{3b}-6\nu_{5a}-6\nu_{5b}-6\nu_{5c}-6\nu_{5d}+\alpha_3)\geq 0;\\
\mu_6(u,\chi_2,*) &= \tfrac{1}{15}(-10\nu_{3a}+2\nu_{3b}+12\nu_{5a}-18\nu_{5b}-8\nu_{5c}+2\nu_{5d}+\alpha_4)\geq 0;\\
\mu_0(u,\chi_4,*) &= \tfrac{1}{15}(24\nu_{3a}+28\nu_{5a}+28\nu_{5b}+8\nu_{5c}+8\nu_{5d}+4\nu_{15a}+4\nu_{15b}+\alpha_5)\geq 0;\\
\mu_1(u,\chi_4,*) &= \tfrac{1}{15}(3\nu_{3a}+6\nu_{5a}+\nu_{5b}-4\nu_{5c}+6\nu_{5d}+3\nu_{15a}-2\nu_{15b}+\alpha_6)\geq 0;\\
\mu_5(u,\chi_4,*) &= \tfrac{1}{15}(-12\nu_{3a}-14\nu_{5a}-14\nu_{5b}-4\nu_{5c}-4\nu_{5d}-2\nu_{15a}-2\nu_{15b}+\alpha_7)\geq 0;\\
\mu_6(u,\chi_4,*) &= \tfrac{1}{15}(-6\nu_{3a}-12\nu_{5a}-2\nu_{5b}+8\nu_{5c}-12\nu_{5d}-6\nu_{15a}+4\nu_{15b}+\alpha_8)\geq 0;\\
\mu_0(u,\chi_6,*) &= \tfrac{1}{15}(72\nu_{3a}-32\nu_{5a}-32\nu_{5b}+8\nu_{5c}+8\nu_{5d}-8\nu_{15a}-8\nu_{15b}+\alpha_9)\geq 0;\\
\mu_5(u,\chi_6,*) &= \tfrac{1}{15}(-36\nu_{3a}+16\nu_{5a}+16\nu_{5b}-4\nu_{5c}-4\nu_{5d}+4\nu_{15a}+4\nu_{15b}+\alpha_{10})\geq 0;\\
\mu_0(u,\chi_{11},*) &= \tfrac{1}{15}(-72\nu_{3a}+8\nu_{5a}+8\nu_{5b}+8\nu_{5c}+8\nu_{5d}+8\nu_{15a}+8\nu_{15b}+\alpha_{11})\geq 0;\\
\mu_5(u,\chi_{11},*) &= \tfrac{1}{15}(36\nu_{3a}-4\nu_{5a}-4\nu_{5b}-4\nu_{5c}-4\nu_{5d}-4\nu_{15a}-4\nu_{15b}+\alpha_{12})\geq 0;\\
\mu_1(u,\chi_{16},*) &= \tfrac{1}{15}(8\nu_{3a}-\nu_{3b}+9\nu_{5a}-11\nu_{5b}+4\nu_{5c}-6\nu_{5d}+3\nu_{15a}-2\nu_{15b}+\alpha_{13})\geq 0;\\
\mu_6(u,\chi_{16},*) &= \tfrac{1}{15}(-16\nu_{3a}+2\nu_{3b}-18\nu_{5a}+22\nu_{5b}-8\nu_{5c}\\
&\qquad\qquad +12\nu_{5d}-6\nu_{15a}+4\nu_{15b}+\alpha_{14})\geq 0,
\end{aligned}$$

where the tuples $(\alpha_1, \ldots, \alpha_{14})$ are given in the following table:

	$\chi(u^5)$	$\chi(u^3)$	$(\alpha_1, \ldots, \alpha_{14})$
1	$\chi(3a)$	$\chi(5a)$	$(30, 0, 15, 15, 41, 17, 32, 26, 38, 11, 112, 139, 227, 251)$
2	$\chi(3a)$	$\chi(5b)$	$(30, 15, 15, 30, 41, 12, 32, 21, 38, 11, 112, 139, 207, 231)$
3	$\chi(3a)$	$\chi(5c)$	$(30, 10, 15, 25, 31, 12, 22, 21, 58, 31, 112, 139, 222, 246)$
4	$\chi(3a)$	$\chi(5d)$	$(30, 5, 15, 20, 31, 22, 22, 31, 58, 31, 112, 139, 212, 236)$
5	$\chi(3b)$	$\chi(5a)$	$(18, 6, 21, 3, 35, 20, 35, 20, 20, 20, 130, 130, 236, 233)$
6	$\chi(3b)$	$\chi(5b)$	$(18, 21, 21, 18, 35, 15, 35, 15, 20, 20, 130, 130, 216, 213)$
7	$\chi(3b)$	$\chi(5c)$	$(18, 16, 21, 13, 25, 15, 25, 15, 40, 40, 130, 130, 231, 228)$
8	$\chi(3b)$	$\chi(5d)$	$(18, 11, 21, 8, 25, 25, 25, 25, 40, 40, 130, 130, 221, 218)$
9	$\chi(3a)$	$2\chi(5c) - \chi(5d)$	$(30, 15, 15, 30, 31, 2, 22, 11, 58, 31, 112, 139, 232, 256)$
10	$\chi(3b)$	$2\chi(5c) - \chi(5d)$	$(18, 21, 21, 18, 25, 5, 25, 5, 40, 40, 130, 130, 241, 238)$
11	$\chi(3a)$	$-\chi(5c) + 2\chi(5d)$	$(30, 0, 15, 15, 31, 32, 22, 41, 58, 31, 112, 139, 202, 226)$
12	$\chi(3b)$	$-\chi(5c) + 2\chi(5d)$	$(18, 6, 21, 3, 25, 35, 25, 35, 40, 40, 130, 130, 211, 208)$
13	$\chi(3a)$	$\chi(5a) - \chi(5c) + \chi(5d)$	$(30, -5, 15, 10, 41, 27, 32, 36, 38, 11, 112, 139, 217, 241)$
14	$\chi(3b)$	$\chi(5a) - \chi(5c) + \chi(5d)$	$(18, 1, 21, -2, 35, 30, 35, 30, 20, 20, 130, 130, 226, 223)$
15	$\chi(3b)$	$\chi(5a) + \chi(5b) - \chi(5d)$	$(18, 16, 21, 13, 45, 10, 45, 10, 0, 0, 130, 130, 231, 228)$
16	$\chi(3b)$	$\chi(5a) + \chi(5b) - \chi(5c)$	$(18, 11, 21, 8, 45, 20, 45, 20, 0, 0, 130, 130, 221, 218)$
17	$\chi(3a)$	$\chi(5b) + \chi(5c) - \chi(5d)$	$(30, 20, 15, 35, 41, 2, 32, 11, 38, 11, 112, 139, 217, 241)$
18	$\chi(3b)$	$\chi(5b) + \chi(5c) - \chi(5d)$	$(18, 26, 21, 23, 35, 5, 35, 5, 20, 20, 130, 130, 226, 223)$

In cases 3, 4, 9 and 11 we use the additional inequalities

$$\begin{aligned}
\mu_0(u, \chi_2, 2) &= \tfrac{1}{15}(-24\nu_{3a} + 8\nu_{5a} + 8\nu_{5b} - 12\nu_{5c} - 12\nu_{5d} - 4\nu_{15a} - 4\nu_{15b} - 6) \geq 0; \\
\mu_0(u, \chi_{17}, 7) &= \tfrac{1}{15}(-64\nu_{3a} - 16\nu_{3b} - 8\nu_{5a} - 8\nu_{5b} \\
&\qquad - 8\nu_{5c} - 8\nu_{5d} + 16\nu_{15a} + 16\nu_{15b} + 179) \geq 0,
\end{aligned}$$

in cases 5, 6, 14, 18 we use the additional inequalities

$$\begin{aligned}
\mu_0(u, \chi_7, *) &= \tfrac{1}{15}(24\nu_{3b} + 24\nu_{5a} + 24\nu_{5b} - 16\nu_{5c} - 16\nu_{5d} + 81) \geq 0; \\
\mu_0(u, \chi_{12}, 7) &= \tfrac{1}{15}(56\nu_{3a} + 8\nu_{3b} - 8\nu_{5a} - 8\nu_{5b} \\
&\qquad - 8\nu_{5c} - 8\nu_{5d} + 16\nu_{15a} + 16\nu_{15b} + 122) \geq 0,
\end{aligned}$$

in cases 7, 8, 10, and 12 we use the additional inequalities

$$\begin{aligned}
\mu_0(u, \chi_{11}, 7) &= \tfrac{1}{15}(-56\nu_{3a} + 16\nu_{3b} + 8\nu_{5a} + 8\nu_{5b} \\
&\qquad + 8\nu_{5c} + 8\nu_{5d} - 16\nu_{15a} - 16\nu_{15b} + 109) \geq 0; \\
\mu_0(u, \chi_{12}, 7) &= \tfrac{1}{15}(56\nu_{3a} + 8\nu_{3b} - 8\nu_{5a} - 8\nu_{5b} \\
&\qquad - 8\nu_{5c} - 8\nu_{5d} + 16\nu_{15a} + 16\nu_{15b} + 122) \geq 0,
\end{aligned}$$

in case 13 we use the additional inequalities

$$\begin{aligned}
\mu_3(u, \chi_4, *) &= \tfrac{1}{15}(-6\nu_{3a} - 2\nu_{5a} - 12\nu_{5b} - 12\nu_{5c} + 8\nu_{5d} + 4\nu_{15a} - 6\nu_{15b} + 11) \geq 0; \\
\mu_6(u, \chi_2, 2) &= \tfrac{1}{15}(6\nu_{3a} + 8\nu_{5a} - 12\nu_{5b} + 8\nu_{5c} - 2\nu_{5d} - 4\nu_{15a} + 6\nu_{15b} - 1) \geq 0; \\
\mu_3(u, \chi_7, 2) &= \tfrac{1}{15}(16\nu_{3a} + 4\nu_{3b} + 12\nu_{5a} - 28\nu_{5b} \\
&\qquad + 12\nu_{5c} - 8\nu_{5d} + 6\nu_{15a} - 4\nu_{15b} + 44) \geq 0; \\
\mu_0(u, \chi_{17}, 7) &= \tfrac{1}{15}(-64\nu_{3a} - 16\nu_{3b} - 8\nu_{5a} - 8\nu_{5b} \\
&\qquad - 8\nu_{5c} - 8\nu_{5d} + 16\nu_{15a} + 16\nu_{15b} + 179) \geq 0,
\end{aligned}$$

in case 15 we use the additional inequalities

$$\begin{aligned}
\mu_0(u,\chi_{12},*) &= \tfrac{1}{15}(128\nu_{3a}+8\nu_{3b}-40\nu_{5a}-40\nu_{5b}+8\nu_{15a}+8\nu_{15b}+122)\geq 0;\\
\mu_3(u,\chi_2,2) &= \tfrac{1}{15}(6\nu_{3a}-12\nu_{5a}+8\nu_{5b}-2\nu_{5c}+8\nu_{5d}+6\nu_{15a}-4\nu_{15b}+5)\geq 0;\\
\mu_0(u,\chi_{11},7) &= \tfrac{1}{15}(-56\nu_{3a}+16\nu_{3b}+8\nu_{5a}+8\nu_{5b}\\
&\qquad +8\nu_{5c}+8\nu_{5d}-16\nu_{15a}-16\nu_{15b}+109)\geq 0,
\end{aligned}$$

in case 16 we use the additional inequalities

$$\begin{aligned}
\mu_0(u,\chi_{12},*) &= \tfrac{1}{15}(128\nu_{3a}+8\nu_{3b}-40\nu_{5a}-40\nu_{5b}+8\nu_{15a}+8\nu_{15b}+122)\geq 0;\\
\mu_0(u,\chi_{11},7) &= \tfrac{1}{15}(-56\nu_{3a}+16\nu_{3b}+8\nu_{5a}+8\nu_{5b}\\
&\qquad +8\nu_{5c}+8\nu_{5d}-16\nu_{15a}-16\nu_{15b}+109)\geq 0,
\end{aligned}$$

and in case 17 we use the additional inequalities

$$\begin{aligned}
\mu_2(u,\chi_2,*) &= \tfrac{1}{15}(5\nu_{3a}-\nu_{3b}+9\nu_{5a}-6\nu_{5b}-\nu_{5c}+4\nu_{5d}-5)\geq 0;\\
\mu_3(u,\chi_2,2) &= \tfrac{1}{15}(6\nu_{3a}-12\nu_{5a}+8\nu_{5b}-2\nu_{5c}+8\nu_{5d}+6\nu_{15a}-4\nu_{15b}-1)\geq 0.
\end{aligned}$$

It follows that in all these cases we have no integral solutions such that all $\mu_i(u,\chi_i,p)$ are non-negative integers.

In case 1 we use the additional inequalities

$$\begin{aligned}
\mu_3(u,\chi_4,*) &= \tfrac{1}{15}(-6\nu_{3a}-2\nu_{5a}-12\nu_{5b}-12\nu_{5c}+8\nu_{5d}+4\nu_{15a}-6\nu_{15b}+21)\geq 0;\\
\mu_5(u,\chi_{12},*) &= \tfrac{1}{15}(-64\nu_{3a}-4\nu_{3b}+20\nu_{5a}+20\nu_{5b}-4\nu_{15a}-4\nu_{15b}+124)\geq 0;\\
\mu_6(u,\chi_2,2) &= \tfrac{1}{15}(6\nu_{3a}+8\nu_{5a}-12\nu_{5b}+8\nu_{5c}-2\nu_{5d}-4\nu_{15a}+6\nu_{15b}-6)\geq 0;\\
\mu_3(u,\chi_7,2) &= \tfrac{1}{15}(16\nu_{3a}+4\nu_{3b}+12\nu_{5a}-28\nu_{5b}\\
&\qquad +12\nu_{5c}-8\nu_{5d}+6\nu_{15a}-4\nu_{15b}+34)\geq 0;\\
\mu_6(u,\chi_7,2) &= \tfrac{1}{15}(16\nu_{3a}+4\nu_{3b}-28\nu_{5a}+12\nu_{5b}\\
&\qquad -8\nu_{5c}+12\nu_{5d}-4\nu_{15a}+6\nu_{15b}+54)\geq 0;\\
\mu_0(u,\chi_{17},7) &= \tfrac{1}{15}(-64\nu_{3a}-16\nu_{3b}-8\nu_{5a}-8\nu_{5b}\\
&\qquad -8\nu_{5c}-8\nu_{5d}+16\nu_{15a}+16\nu_{15b}+179)\geq 0,
\end{aligned}$$

to obtain only one trivial solution with $\nu_{15b}=1$.

In case 2 we will use the additional inequalities

$$\begin{aligned}
\mu_2(u,\chi_2,*) &= \tfrac{1}{15}(5\nu_{3a}-\nu_{3b}+9\nu_{5a}-6\nu_{5b}-\nu_{5c}+4\nu_{5d})\geq 0;\\
\mu_5(u,\chi_{12},*) &= \tfrac{1}{15}(-64\nu_{3a}-4\nu_{3b}+20\nu_{5a}+20\nu_{5b}-4\nu_{15a}-4\nu_{15b}+124)\geq 0;\\
\mu_3(u,\chi_2,2) &= \tfrac{1}{15}(6\nu_{3a}-12\nu_{5a}+8\nu_{5b}-2\nu_{5c}+8\nu_{5d}+6\nu_{15a}-4\nu_{15b}-6)\geq 0;\\
\mu_3(u,\chi_7,2) &= \tfrac{1}{15}(16\nu_{3a}+4\nu_{3b}+12\nu_{5a}-28\nu_{5b}\\
&\qquad +12\nu_{5c}-8\nu_{5d}+6\nu_{15a}-4\nu_{15b}+54)\geq 0;\\
\mu_6(u,\chi_7,2) &= \tfrac{1}{15}(16\nu_{3a}+4\nu_{3b}-28\nu_{5a}+12\nu_{5b}\\
&\qquad -8\nu_{5c}+12\nu_{5d}-4\nu_{15a}+6\nu_{15b}+34)\geq 0;\\
\mu_0(u,\chi_{17},7) &= \tfrac{1}{15}(-64\nu_{3a}-16\nu_{3b}-8\nu_{5a}-8\nu_{5b}\\
&\qquad -8\nu_{5c}-8\nu_{5d}+16\nu_{15a}+16\nu_{15b}+179)\geq 0,
\end{aligned}$$

to show that it has the unique trivial solution with $\nu_{15a}=1$.

• Let u be a unit of order 14. By (1) and Proposition 2 we have that

$$\nu_{2a} + \nu_{2b} + \nu_{7a} = 1.$$

Since u^7 has order two, part (iii) of Theorem 2 says that we need to consider six cases.

First we consider the following four cases:

$$\chi(u^7) \in \{\ \chi(2a), \quad \chi(2b), \quad 2\chi(2a) - \chi(2b), \quad -\chi(2a) + 2\chi(2b)\ \}.$$

Put $\quad t_1 = \nu_{2a} - \nu_{2b}, \quad t_2 = 5\nu_{2a} - 3\nu_{2b} \quad$ and

$$(\alpha, \beta, \gamma, \delta) = \begin{cases} (16, 12, 26, 16), & \text{if} \quad \chi(u^7) = \chi(2a); \\ (12, 16, 18, 24), & \text{if} \quad \chi(u^7) = \chi(2b); \\ (20, 8, 34, 8), & \text{if} \quad \chi(u^7) = 3\chi(2a) - 2\chi(2b); \\ (8, 20, 10, 32), & \text{if} \quad \chi(u^7) = -\chi(2a) + 2\chi(2b), \end{cases}$$

for any character χ of G. Applying Proposition 3 to characters χ_2, χ_4 we get the following system of inequalities

$$\begin{aligned} \mu_7(u, \chi_2, *) &= \tfrac{1}{14}(12\nu_{2a} - 12\nu_{2b} + \alpha) = \tfrac{1}{14}(12t_1 + \alpha) \geq 0; \\ \mu_0(u, \chi_2, *) &= \tfrac{1}{14}(-12\nu_{2a} + 12\nu_{2b} + \beta) = \tfrac{1}{14}(-12t_1 + \beta) \geq 0; \\ \mu_0(u, \chi_4, *) &= \tfrac{1}{14}(30\nu_{2a} - 18\nu_{2b} + \gamma) = \tfrac{1}{14}(6t_1 + \gamma) \geq 0; \\ \mu_7(u, \chi_4, *) &= \tfrac{1}{14}(-30\nu_{2a} + 18\nu_{2b} + \delta) = \tfrac{1}{14}(-6t_2 + \delta) \geq 0. \end{aligned}$$

If either $\chi(u^7) = 2\chi(2a) - \chi(2b)$ or $\chi(u^7) = -\chi(2a) + 2\chi(2b)$ from the last system we obtain that there is no integer t_1 such that $\mu_7(u, \chi_2, *)$ and $\mu_0(u, \chi_2, *)$ are nonnegative integers.

If $\chi(u^7) = \chi(2a)$, then $t_1 = \nu_{2a} - \nu_{2b} = 1$ and $t_2 = 5\nu_{2a} - 3\nu_{2b} = -2$, which has no integral solutions (ν_{2a}, ν_{2b}). Similarly, if $\chi(u^7) = \chi(2b)$, then $t_1 = \nu_{2a} - \nu_{2b} = -1$ and $t_2 = 5\nu_{2a} - 3\nu_{2b} \in \{-3, 4\}$, which also has no integral solutions (ν_{2a}, ν_{2b}).

If $\chi(u^7) = -2\chi(2a) + 3\chi(2b)$, then applying Proposition 3 to the characters χ_2, χ_4 we obtain the following system

$$\begin{aligned} \mu_7(u, \chi_2, *) &= \tfrac{1}{14}(12\nu_{2a} - 12\nu_{2b} + 4) = \tfrac{1}{14}(12t_1 + 4) \geq 0; \\ \mu_0(u, \chi_2, *) &= \tfrac{1}{14}(-12\nu_{2a} + 12\nu_{2b} + 24) = \tfrac{1}{14}(-12t_1 + 24) \geq 0; \\ \mu_0(u, \chi_4, *) &= \tfrac{1}{14}(30\nu_{2a} - 18\nu_{2b} + 2) = \tfrac{1}{14}(6t_2 + 2) \geq 0; \\ \mu_2(u, \chi_4, *) &= \tfrac{1}{14}(-5\nu_{2a} + 3\nu_{2b} + 2) = \tfrac{1}{14}(-5t_2 + 2) \geq 0, \end{aligned}$$

where $t_1 = \nu_{2a} - \nu_{2b}$, and $t_2 = 5\nu_{2a} - 3\nu_{2b}$. It is easy to check that $(t_1, t_2) = (2, 2)$ and we have only a unique integral solution: $(\nu_{2a}, \nu_{2b}, \nu_{7a}) = (-2, -4, 7)$. Since $\mu_1(u, \chi_4, *) = \frac{1}{14}(5\nu_{2a} - 3\nu_{2b} + 24) = \frac{13}{7}$, we get a contradiction.

Finally, if $\chi(u^7) = 3\chi(2a) - 2\chi(2b)$, then applying Proposition 3 to the characters χ_2, χ_4 we obtain the following system

$$\begin{aligned} \mu_2(u, \chi_2, *) &= \tfrac{1}{14}(2\nu_{2a} - 2\nu_{2b} + 4) = \tfrac{1}{14}(2t_1 + 4) \geq 0; \\ \mu_0(u, \chi_2, *) &= \tfrac{1}{14}(-12\nu_{2a} + 12\nu_{2b} + 4) = \tfrac{1}{14}(-12t_1 + 4) \geq 0; \\ \mu_1(u, \chi_4, *) &= \tfrac{1}{14}(5\nu_{2a} - 3\nu_{2b}) = \tfrac{1}{14}(t_2) \geq 0; \\ \mu_7(u, \chi_4, *) &= \tfrac{1}{14}(-30\nu_{2a} + 18\nu_{2b}) = \tfrac{1}{14}(-6t_2) \geq 0, \end{aligned}$$

where $t_1 = \nu_{2a} - \nu_{2b}$, and $t_2 = 5\nu_{2a} - 3\nu_{2b}$. It is easy to check that $(t_1, t_2) = (-2, *)$ and we have only a unique integral solution: $(\nu_{2a}, \nu_{2b}, \nu_{7a}) = (3, 5, -7)$. Since $\mu_0(u, \chi_7, *) = \frac{1}{14}(90\nu_{2a} - 6\nu_{2b} + 30) = \frac{270}{14}$, we again get a contradiction.

• Let u be a unit of order 21. By (1) and Proposition 2 we have that

$$\nu_{3a} + \nu_{3b} + \nu_{7a} = 1.$$

Since u^7 has order 3, because of part (iv) of Theorem 2 we need to consider the following cases:

$$\chi(u^7) \in \{\ \chi(3a), \quad \chi(3b), \quad -\chi(3a) + 2\chi(3b)\ \}.$$

Put $\quad t_1 = 5\nu_{3a} - \nu_{3b}, \quad t_2 = \nu_{3a} \quad$ and

$$(\alpha, \beta, \gamma, \delta) = \begin{cases} (29, 9, 27, 54), & \text{if} \quad \chi(u^7) = \chi(3a); \\ (12, 15, 21, 84), & \text{if} \quad \chi(u^7) = \chi(3b); \\ (0, 0, 15, 114), & \text{if} \quad \chi(u^7) = -\chi(3a) + 2\chi(2b), \end{cases}$$

for any character χ of G. Applying Proposition 3 to the characters χ_2, χ_4, χ_9 we get the following system of inequalities

$$\begin{aligned} &\mu_0(u, \chi_2, *) = \tfrac{1}{21}(60\nu_{3a} - 12\nu_{3b} + \alpha) \geq 0; \\ &\mu_7(u, \chi_2, *) = \tfrac{1}{21}(-30\nu_{3a} + 6\nu_{3b} + \beta) \geq 0; \\ &\mu_0(u, \chi_4, *) = \tfrac{1}{21}(36\nu_{3a} + \gamma) \geq 0; \quad \mu_0(u, \chi_9, 2) = \tfrac{1}{21}(-180\nu_{3a} + \delta) \geq 0. \end{aligned}$$

If $\chi(u^7) \in \{\chi(3a), -\chi(3a) + 2\chi(3b)\}$, then it is easy to check that there is no integer t_2, such that $\mu_0(u, \chi_4, *)$ and $\mu_0(u, \chi_9, 2)$ are nonnegative integers.

If $\chi(u^7) = \chi(3b)$, then the last system on inequalities has only one integral solution $(\nu_{3a}, \nu_{3b}, \nu_{7a}) = (0, 1, 0)$ such that $\mu_i(u, \chi_2, *)$, $\mu_0(u, \chi_j, p)$ are nonnegative integers. Now by (3) we have that $\mu_1(u, \chi_2, *) = \frac{1}{21}(5\nu_{3a} - \nu_{3b} + 15) = \frac{14}{21}$, a contradiction.

• Let u be a unit of order 35. By (1) and Proposition 2 we have that

$$\nu_{5a} + \nu_{5b} + \nu_{5c} + \nu_{5d} + \nu_{7a} = 1.$$

Since u^7 has order 5, because of part (v) of this theorem we need to consider ten cases. Put

$$\alpha = \begin{cases} 5, & \text{if} \quad \chi(u^7) \in \{\ \chi(5a), \quad -\chi(5c) + 2\chi(5d)\}; \\ 20, & \text{if} \quad \chi(u^7) \in \{\ \chi(5b), \quad 2\chi(5c) - \chi(5d)\}; \\ 15, & \text{if} \quad \chi(u^7) \in \{\ \chi(5c), \quad \chi(5a) + \chi(5c) - \chi(5d)\}; \\ 10, & \text{if} \quad \chi(u^7) \in \{\ \chi(5d), \quad \chi(5a) + \chi(5b) - \chi(5c)\}; \\ 0, & \text{if} \quad \chi(u^7) = \chi(5a) - \chi(5c) + \chi(5d); \\ 25, & \text{if} \quad \chi(u^7) = \chi(5b) + \chi(5c) - \chi(5d), \end{cases} \tag{2}$$

where χ is a character of the group G. Now by Proposition 3 we get in all of these ten cases

$$\begin{aligned} \mu_{14}(u, \chi_2, *) &= \tfrac{1}{35}(36\nu_{5a} - 54\nu_{5b} - 24\nu_{5c} + 6\nu_{5d} + \alpha) = \tfrac{1}{35}(6t + \alpha) \geq 0; \\ \mu_1(u, \chi_2, *) &= \tfrac{1}{35}(-6\nu_{5a} + 9\nu_{5b} + 4\nu_{5c} - \nu_{5d} + \alpha) = \tfrac{1}{35}(-t + \alpha) \geq 0, \end{aligned}$$

where $t = 6\nu_{5a} - 9\nu_{5b} - 4\nu_{5c} + \nu_{5d}$ and α from (2). Clearly, this system of inequalities has no integral solutions such that $\mu_{14}(u, \chi_2, *)$ and $\mu_1(u, \chi_2, *)$ are non-negative integers. This finishes the proof of Theorem 2.

5. Proof of Theorem 3

Let G be the third Janko simple group J_3. It is well known [12, 13] that $|G| = 2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$ and $exp(G) = 2^3 \cdot 3^2 \cdot 5 \cdot 17 \cdot 19$.

The group G only possesses elements of orders 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 17 and 19. Hence first we shall investigate units of these orders. By Proposition 4, the order of each torsion unit divides the exponent of G. So, second we consider units of orders 18, 20, 24, 30, 34, 38, 45, 51, 57, 85, 95 and 323. We shall prove that units of all these orders except 18, 20, 24, 30 and 45 do not appear in $V(\mathbb{Z}G)$. We will omit cases of units of orders 18, 20, 24, 30 and 45 since they are not products of two disctinct primes, so they do not contribute to Kimmerle's conjecture.

We consider each case separately.

• Let u be a unit of order 2. Since there is only one conjugacy class in G consisting of elements or order 2, this case immediately follows from Propositions 1 and 2.

• Let u be a unit of order 3. By (1) and Proposition 2 we have $\nu_{3a} + \nu_{3b} = 1$. By Proposition 3 we obtain the system of inequalities

$$\begin{aligned}
\mu_0(u, \chi_2, *) &= \tfrac{1}{3}(-10\nu_{3a} + 8\nu_{3b} + 85) \geq 0;\\
\mu_1(u, \chi_2, *) &= \tfrac{1}{3}(5\nu_{3a} - 4\nu_{3b} + 85) \geq 0;\\
\mu_0(u, \chi_4, 2) &= \tfrac{1}{3}(16\nu_{3a} - 2\nu_{3b} + 80) \geq 0,
\end{aligned}$$

that only has the ten solutions listed in part (iii) of Theorem 3.

• Let u be a unit of order 4. By (1) and Proposition 2 we have $\nu_{2a} + \nu_{4a} = 1$. By Proposition 3 we obtain the system of inequalities

$$\begin{aligned}
\mu_0(u, \chi_2, *) &= \tfrac{1}{4}(10\nu_{2a} + 2\nu_{4a} + 90) \geq 0;\\
\mu_2(u, \chi_2, *) &= \tfrac{1}{4}(-10\nu_{2a} - 2\nu_{4a} + 90) \geq 0;\\
\mu_0(u, \chi_2, 3) &= \tfrac{1}{4}(4\nu_{2a} - 4\nu_{4a} + 20) \geq 0;\\
\mu_2(u, \chi_2, 3) &= \tfrac{1}{4}(-4\nu_{2a} + 4\nu_{4a} + 20) \geq 0.
\end{aligned}$$

Solving this system and applying Proposition 5, only the three solutions listed in part (iv) of Theorem 3 remain.

• Let u be a unit of order 5. Then $\nu_{5a} + \nu_{5b} = 1$, and we have the system

$$\begin{aligned}
\mu_1(u, \chi_8, 2) &= \tfrac{1}{5}(3\nu_{5a} - 2\nu_{5b} + 322) \geq 0;\\
\mu_1(u, \chi_2, 3) &= \tfrac{1}{5}(-3\nu_{5a} + 2\nu_{5b} + 18) \geq 0;\\
\mu_2(u, \chi_2, 3) &= \tfrac{1}{5}(2\nu_{5a} - 3\nu_{5b} + 18) \geq 0,
\end{aligned}$$

that only has the eight solutions listed in part (v) of Theorem 3.

• Let u be a unit of order 8. Then $\nu_{2a} + \nu_{4a} + \nu_{8a} = 1$ and we need to consider three cases defined by part (iv) of Theorem 3.

Case 1. $\chi(u^2) = \chi(4a)$. We have the following system of inequalities:

$$\begin{aligned}
\mu_0(u, \chi_2, *) &= \tfrac{1}{8}(20\nu_{2a} + 4\nu_{4a} - 4\nu_{8a} + 92) \geq 0;\\
\mu_4(u, \chi_2, *) &= \tfrac{1}{8}(-20\nu_{2a} - 4\nu_{4a} + 4\nu_{8a} + 92) \geq 0;\\
\mu_0(u, \chi_4, *) &= \tfrac{1}{8}(12\nu_{2a} + 12\nu_{4a} - 4\nu_{8a} + 332) \geq 0;\\
\mu_4(u, \chi_4, *) &= \tfrac{1}{8}(-12\nu_{2a} - 12\nu_{4a} + 4\nu_{8a} + 332) \geq 0;\\
\mu_0(u, \chi_2, 3) &= \tfrac{1}{8}(8\nu_{2a} - 8\nu_{4a} + 16) \geq 0;\\
\mu_4(u, \chi_2, 3) &= \tfrac{1}{8}(-8\nu_{2a} + 8\nu_{4a} + 16) \geq 0;\\
\mu_0(u, \chi_4, 3) &= \tfrac{1}{8}(16\nu_{2a} - 8\nu_{8a} + 88) \geq 0,
\end{aligned}$$

that only has the following nine solutions satisfying Proposition 5 such that all $\mu_i(u, \chi_j, p)$ are non-negative integers:

$$\begin{aligned}
(\nu_{2a}, \nu_{4a}, \nu_{8a}) \in \{\ &(2, 4, -5),\ (2, 0, -1),\ (-2, 0, 3),\ (0, 0, 1),\ (0, 2, -1),\\
&(2, 2, -3),\ (-2, -4, 7),\ (-2, -2, 5),\ (0, -2, 3)\ \}.
\end{aligned}$$

Case 2. $\chi(u^2)=-2\chi(2a)+3\chi(4a)$. Then we obtain the system

$$\begin{aligned}
&\mu_0(u,\chi_2,*)=\tfrac{1}{8}(20\nu_{2a}+4\nu_{4a}-4\nu_{8a}+76)\geq 0;\\
&\mu_4(u,\chi_2,*)=\tfrac{1}{8}(-20\nu_{2a}-4\nu_{4a}+4\nu_{8a}+76)\geq 0;\\
&\mu_0(u,\chi_4,*)=\tfrac{1}{8}(12\nu_{2a}+12\nu_{4a}-4\nu_{8a}+332)\geq 0;\\
&\mu_4(u,\chi_4,*)=\tfrac{1}{8}(-12\nu_{2a}-12\nu_{4a}+4\nu_{8a}+332)\geq 0;\\
&\mu_0(u,\chi_2,3)=\tfrac{1}{8}(8\nu_{2a}-8\nu_{4a})\geq 0; \quad \mu_4(u,\chi_2,3)=\tfrac{1}{8}(-8\nu_{2a}+8\nu_{4a})\geq 0,
\end{aligned}$$

that only has the following three solutions satisfying Proposition 5 such that all $\mu_i(u,\chi_j,p)$ are non-negative integers:

$$(\nu_{2a},\nu_{4a},\nu_{8a})\in\{\ (0,0,1),\ (-2,-2,5),\ (2,2,-3)\ \}.$$

Case 3. $\chi(u^2)=2\chi(2a)-\chi(4a)$. Then the system

$$\begin{aligned}
\mu_0(u,\chi_2,*)&=\tfrac{1}{8}(20\nu_{2a}+4\nu_{4a}-4\nu_{8a}+108)\geq 0;\\
\mu_4(u,\chi_2,*)&=\tfrac{1}{8}(-20\nu_{2a}-4\nu_{4a}+4\nu_{8a}+108)\geq 0;\\
\mu_0(u,\chi_4,*)&=\tfrac{1}{8}(12\nu_{2a}+12\nu_{4a}-4\nu_{8a}+332)\geq 0;\\
\mu_4(u,\chi_4,*)&=\tfrac{1}{8}(-12\nu_{2a}-12\nu_{4a}+4\nu_{8a}+332)\geq 0;\\
\mu_0(u,\chi_2,3)&=\tfrac{1}{8}(8\nu_{2a}-8\nu_{4a}+32)\geq 0;\\
\mu_4(u,\chi_2,3)&=\tfrac{1}{8}(-8\nu_{2a}+8\nu_{4a}+32)\geq 0;\\
\mu_0(u,\chi_6,3)&=\tfrac{1}{8}(-28\nu_{2a}+4\nu_{4a}+4\nu_{8a}+116)\geq 0;\\
\mu_0(u,\chi_4,3)&=\tfrac{1}{8}(16\nu_{2a}-8\nu_{8a}+104)\geq 0;\\
\mu_4(u,\chi_6,3)&=\tfrac{1}{8}(28\nu_{2a}-4\nu_{4a}-4\nu_{8a}+116)\geq 0,
\end{aligned}$$

only has the following nine solutions satisfying Proposition 5 such that all $\mu_i(u,\chi_j,p)$ are non-negative integers:

$$\begin{aligned}
(\nu_{2a},\nu_{4a},\nu_{8a})\in\{\ &(2,6,-7),\ (2,4,-5),\ (0,4,-3),\ (2,0,-1),\ (-2,0,3),\\
&(0,0,1),\ (-2,-6,9),\ (0,2,-1),\ (2,2,-3),\ (-2,2,1),\\
&(-2,-4,7),\ (0,-4,5),\ (-2,-2,5),\ (0,-2,3),\ (2,-2,1)\ \}.
\end{aligned}$$

The union of the solutions of all three cases gives us part (vi) of Theorem 3.

• Let u be a unit of order 17. Then $\nu_{17a}+\nu_{17b}=1$, and we have the system

$$\begin{aligned}
\mu_1(u,\chi_3,19)&=\tfrac{1}{17}(9\nu_{17a}-8\nu_{17b}+110)\geq 0;\\
\mu_1(u,\chi_9,19)&=\tfrac{1}{17}(-9\nu_{17a}+8\nu_{17b}+706)\geq 0;\\
\mu_1(u,\chi_2,2)&=\tfrac{1}{17}(-10\nu_{17a}+7\nu_{17b}+78)\geq 0;\\
\mu_3(u,\chi_2,2)&=\tfrac{1}{17}(7\nu_{17a}-10\nu_{17b}+78)\geq 0,
\end{aligned}$$

with the ten solutions listed in part (vii) of Theorem 3.

• Let u be a unit of order 19. Then $\nu_{19a}+\nu_{19b}=1$, and we have the system

$$\begin{aligned}
\mu_1(u,\chi_2,*)&=\tfrac{1}{19}(10\nu_{19a}-9\nu_{19b}+85)\geq 0;\\
\mu_2(u,\chi_2,*)&=\tfrac{1}{19}(-9\nu_{19a}+10\nu_{19b}+85)\geq 0;\\
\mu_1(u,\chi_5,2)&=\tfrac{1}{19}(11\nu_{19a}-8\nu_{19b}+84)\geq 0;\\
\mu_2(u,\chi_5,2)&=\tfrac{1}{19}(-8\nu_{19a}+11\nu_{19b}+84)\geq 0,
\end{aligned}$$

with the ten solutions listed in part (viii) of Theorem 3.

It remains to prove part (i) of Theorem 3, showing that there are no elements of orders 34, 38, 51, 57, 85, 95 and 323 in $V(\mathbb{Z}G)$.

• Let u be a unit of order 34. By (1) and Proposition 2 we have $\nu_{2a}+\nu_{17a}+\nu_{17b}=1$. We need to consider ten cases defined by part (vii) of Theorem 3. By Proposition 3 we obtain the system of inequalities

$$\begin{aligned}
&\mu_0(u,\chi_2,*) = \tfrac{1}{34}(80\nu_{2a}+90) \geq 0; \quad \mu_{17}(u,\chi_2,*) = \tfrac{1}{34}(-80\nu_{2a}+80) \geq 0;\\
&\mu_1(u,\chi_{11},*) = \tfrac{1}{34}(15\nu_{2a}-9\nu_{17a}+8\nu_{17b}+\alpha) \geq 0;\\
&\mu_2(u,\chi_{11},*) = \tfrac{1}{34}(-15\nu_{2a}+9\nu_{17a}-8\nu_{17b}+(\alpha+30)) \geq 0;\\
&\mu_1(u,\chi_2,*) = \tfrac{1}{34}(5\nu_{2a}+80) \geq 0,
\end{aligned}$$

$$\text{where} \quad \alpha = \begin{cases}
1209, & \chi(u^2)=\chi(17a),\\
1192, & \chi(u^2)=\chi(17b),\\
1277, & \chi(u^2)=5\chi(17a)-4\chi(17b),\\
1158, & \chi(u^2)=-2\chi(17a)+3\chi(17b),\\
1226, & \chi(u^2)=2\chi(17a)-\chi(17b),\\
1141, & \chi(u^2)=-3\chi(17a)+4\chi(17b),\\
1124, & \chi(u^2)=-4\chi(17a)+5\chi(17b),\\
1243, & \chi(u^2)=3\chi(17a)-2\chi(17b),\\
1175, & \chi(u^2)=-\chi(17a)+2\chi(17b),\\
1260, & \chi(u^2)=4\chi(17a)-3\chi(17b),
\end{cases}$$

that has no solutions such that all $\mu_i(u,\chi_j,*)$ are non-negative integers.

• Let u be a unit of order 38. By (1) and Proposition 2 we have

$$\nu_{2a}+\nu_{19a}+\nu_{19b}=1.$$

We need to consider ten cases defined by part (viii) of Theorem 3. Using Proposition 3 we obtain in each case the system of inequalities

$$\begin{aligned}
\mu_0(u,\chi_2,3) &= \tfrac{1}{38}(36\nu_{2a}-18\nu_{19a}-18\nu_{19b}+2) \geq 0;\\
\mu_{19}(u,\chi_2,3) &= \tfrac{1}{38}(-36\nu_{2a}+18\nu_{19a}+18\nu_{19b}-2) \geq 0,
\end{aligned}$$

that has no integral solutions.

• Let u be a unit of order 51. By (1) and Proposition 2 we have

$$\nu_{3a}+\nu_{3b}+\nu_{17a}+\nu_{17b}=1.$$

We need to consider 100 cases defined by parts (iii) and (vii) of Theorem 3. In all cases by Proposition 3 we obtain systems of inequalities that have no solutions such that all $\mu_i(u,\chi_j,*)$ are non-negative integers. We give a few comments on the calculations. The exact form of these inequalities depends mainly on the choice of the case for $\chi(u^{17})$. In most cases we use that

$$\begin{aligned}
\mu_0(u,\chi_2,*) &= \tfrac{1}{51}(-160\nu_{3a}+128\nu_{3b}+\beta_1) \geq 0;\\
\mu_{17}(u,\chi_2,*) &= \tfrac{1}{51}(80\nu_{3a}-64\nu_{3b}+\beta_2) \geq 0,
\end{aligned}$$

$$\text{where} \quad (\beta_1,\beta_2) = \begin{cases}
(75,90), \text{ when } \chi(u^{17})=\chi(3a);\\
(93,81), \text{ when } \chi(u^{17})=\chi(3b);\\
(129,63), \text{ when } \chi(u^{17})=-2\chi(3a)+3\chi(3b);\\
(165,45), \text{ when } \chi(u^{17})=-4\chi(3a)+5\chi(3b);\\
(39,108), \text{ when } \chi(u^{17})=3\chi(3a)-2\chi(3b);\\
(111,72), \text{ when } \chi(u^{17})=-\chi(3a)+2\chi(3b);\\
(21,117), \text{ when } \chi(u^{17})=4\chi(3a)-3\chi(3b).
\end{cases}$$

When $\chi(u^{17})=5\chi(3a)-4\chi(3b)$, we use the system

$$\begin{aligned}
\mu_0(u,\chi_2,*) &= \tfrac{1}{51}(-160\nu_{3a}+128\nu_{3b}+3) \geq 0;\\
\mu_3(u,\chi_2,*) &= \tfrac{1}{51}(10\nu_{3a}-8\nu_{3b}+3) \geq 0.
\end{aligned}$$

When $\chi(u^{17}) = 2\chi(3a) - \chi(3b)$ or $\chi(u^{17}) = -3\chi(3a) + 4\chi(3b)$, we use the system

$$\begin{aligned}
\mu_0(u, \chi_2, *) &= \tfrac{1}{51}(-160\nu_{3a} + 128\nu_{3b} + 57) \geq 0;\\
\mu_{17}(u, \chi_2, *) &= \tfrac{1}{51}(80\nu_{3a} - 64\nu_{3b} + 99) \geq 0;\\
\mu_0(u, \chi_4, *) &= \tfrac{1}{51}(256\nu_{3a} - 32\nu_{3b} + 357) \geq 0;\\
\mu_{17}(u, \chi_4, *) &= \tfrac{1}{51}(-128\nu_{3a} + 16\nu_{3b} + 306) \geq 0;\\
\mu_1(u, \chi_{11}, *) &= \tfrac{1}{51}(8\nu_{17a} - 9\nu_{17b} + \beta) \geq 0;\\
\mu_9(u, \chi_{11}, *) &= \tfrac{1}{51}(-16\nu_{17a} + 18\nu_{17b} + \beta) \geq 0,
\end{aligned}$$

$$\text{where}\quad \beta = \begin{cases}
1224, \ \text{when } \chi(u^3) = \chi(17a);\\
1207, \ \text{when } \chi(u^3) = \chi(17b);\\
1292, \ \text{when } \chi(u^3) = 5\chi(17a) - 4\chi(17b);\\
1173, \ \text{when } \chi(u^3) = -2\chi(17a) + 3\chi(17b);\\
1241, \ \text{when } \chi(u^3) = 2\chi(17a) - \chi(17b);\\
1156, \ \text{when } \chi(u^3) = -3\chi(17a) + 4\chi(17b);\\
1139, \ \text{when } \chi(u^3) = -4\chi(17a) + 5\chi(17b);\\
1258, \ \text{when } \chi(u^3) = 3\chi(17a) - 2\chi(17b);\\
1190, \ \text{when } \chi(u^3) = -\chi(17a) + 2\chi(17b);\\
1275, \ \text{when } \chi(u^3) = 4\chi(17a) - 3\chi(17b).
\end{cases}$$

- Let u be a unit of order 57. By (1) and Proposition 2 we have

$$\nu_{3a} + \nu_{3b} + \nu_{19a} + \nu_{19b} = 1.$$

We need to consider 100 cases defined by parts (iii) and (viii) of Theorem 3. In all of these cases it is enough to consider the the following system of inequalities

$$\begin{aligned}
\mu_0(u, \chi_2, *) &= \tfrac{1}{57}(-180\nu_{3a} + 144\nu_{3b} - 18\nu_{19a} - 18\nu_{19b} + \alpha_1) \geq 0;\\
\mu_1(u, \chi_2, *) &= \tfrac{1}{57}(-5\nu_{3a} + 4\nu_{3b} + 9\nu_{19a} - 10\nu_{19b} + \beta_1) \geq 0;\\
\mu_6(u, \chi_2, *) &= \tfrac{1}{57}(10\nu_{3a} - 8\nu_{3b} - 18\nu_{19a} + 20\nu_{19b} + \beta_2) \geq 0;\\
\mu_{19}(u, \chi_2, *) &= \tfrac{1}{57}(90\nu_{3a} - 72\nu_{3b} + 9\nu_{19a} + 9\nu_{19b} + \alpha_2) \geq 0;\\
\mu_0(u, \chi_4, *) &= \tfrac{1}{57}(288\nu_{3a} - 36\nu_{3b} + \alpha_3) \geq 0;\\
\mu_{19}(u, \chi_4, *) &= \tfrac{1}{57}(-144\nu_{3a} + 18\nu_{3b} + \alpha_4) \geq 0,
\end{aligned}$$

and in the case when $\chi(u^{19}) = \chi(3b)$, we need to consider one more inequality

$$\mu_1(u, \chi_4, *) = \tfrac{1}{57}(8\nu_{3a} - \nu_{3b} + 324) \geq 0$$

to show that there are no solutions such that all $\mu_i(u, \chi_j, *)$ are non-negative integers.

The coefficients $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ only depend on the case for $\chi(u^{19})$:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \begin{cases}
(66, 81, 339, 315), & \text{when } \chi(u^{19}) = \chi(3a);\\
(84, 72, 321, 324), & \text{when } \chi(u^{19}) = \chi(3b);\\
(-6, 117, 411, 279), & \text{when } \chi(u^{19}) = 5\chi(3a) - 4\chi(3b);\\
(120, 54, 285, 342), & \text{when } \chi(u^{19}) = -2\chi(3a) + 3\chi(3b);\\
(48, 90, 357, 306), & \text{when } \chi(u^{19}) = 2\chi(3a) - \chi(3b);\\
(138, 45, 267, 351), & \text{when } \chi(u^{19}) = -3\chi(3a) + 4\chi(3b);\\
(156, 36, 249, 360), & \text{when } \chi(u^{19}) = -4\chi(3a) + 5\chi(3b);\\
(30, 99, 375, 297), & \text{when } \chi(u^{19}) = 3\chi(3a) - 2\chi(3b);\\
(102, 63, 303, 333), & \text{when } \chi(u^{19}) = -\chi(3a) + 2\chi(3b);\\
(12, 108, 393, 288), & \text{when } \chi(u^{19}) = 4\chi(3a) - 3\chi(3b).
\end{cases}$$

Coefficients (β_1, β_2) depend both on the case for $\chi(u^{19})$ and $\chi(u^3)$, as it is described in the following table:

	$\chi(19a)$	$\chi(19b)$	$5\chi(19a)$ $-4\chi(19b)$	$-2\chi(19a)$ $+3\chi(19b)$	$2\chi(19a)$ $-\chi(19b)$
$\chi(3a)$	$100, 85$	$81, 66$	$176, 161$	$43, 28$	$119, 104$
$\chi(3b)$	$91, 103$	$72, 84$	$167, 179$	$34, 46$	$110, 122$
$5\chi(3a) - 4\chi(3b)$	$136, 13$	$117, -6$	$212, 89$	$79, -44$	$155, 32$
$-2\chi(3a) + 3\chi(3b)$	$73, 139$	$54, 120$	$149, 215$	$16, 82$	$92, 158$
$2\chi(3a) - \chi(3b)$	$109, 67$	$90, 48$	$185, 143$	$52, 10$	$128, 86$
$-3\chi(3a) + 4\chi(3b)$	$64, 157$	$45, 138$	$140, 233$	$7, 100$	$83, 176$
$-4\chi(3a) + 5\chi(3b)$	$55, 175$	$36, 156$	$131, 251$	$-2, 118$	$74, 194$
$3\chi(3a) - 2\chi(3b)$	$118, 49$	$99, 30$	$194, 125$	$61, -8$	$137, 68$
$-\chi(3a) + 2\chi(3b)$	$82, 121$	$63, 102$	$158, 197$	$25, 64$	$101, 140$
$4\chi(3a) - 3\chi(3b)$	$127, 31$	$108, 12$	$203, 107$	$70, -26$	$146, 50$

	$-3\chi(19a)$ $+4\chi(19b)$	$-4\chi(19a)$ $+5\chi(19b)$	$3\chi(19a)$ $-2\chi(19b)$	$-\chi(19a)$ $+2\chi(19b)$	$4\chi(19a)$ $-3\chi(19b)$
$\chi(3a)$	$24, 9$	$5, -10$	$138, 123$	$62, 47$	$157, 142$
$\chi(3b)$	$15, 27$	$-4, 8$	$129, 141$	$53, 65$	$148, 160$
$5\chi(3a) - 4\chi(3b)$	$60, -63$	$41, -82$	$174, 51$	$98, -25$	$193, 70$
$-2\chi(3a) + 3\chi(3b)$	$-3, 63$	$-22, 44$	$111, 177$	$35, 101$	$130, 196$
$2\chi(3a) - \chi(3b)$	$33, -9$	$14, -28$	$147, 105$	$71, 29$	$166, 124$
$-3\chi(3a) + 4\chi(3b)$	$-12, 81$	$-31, 62$	$102, 195$	$26, 119$	$121, 214$
$-4\chi(3a) + 5\chi(3b)$	$-21, 99$	$-40, 80$	$93, 213$	$17, 137$	$112, 232$
$3\chi(3a) - 2\chi(3b)$	$42, -27$	$23, -46$	$156, 87$	$80, 11$	$175, 106$
$-\chi(3a) + 2\chi(3b)$	$6, 45$	$-13, 26$	$120, 159$	$44, 83$	$139, 178$
$4\chi(3a) - 3\chi(3b)$	$51, -45$	$32, -64$	$165, 69$	$89, -7$	$184, 88$

- Let u be a unit of order 85. By (1) and Proposition 2 we have

$$\nu_{5a} + \nu_{5b} + \nu_{17a} + \nu_{17b} = 1.$$

We need to consider 80 cases defined by parts (v) and (vii) of Theorem 3. In all of these cases we obtain the following system of inequalities that have no integral solutions such that all $\mu_i(u, \chi_j, p)$ are non-negative integers:

$$\begin{aligned}
\mu_{17}(u, \chi_4, *) &= \tfrac{1}{85}(-48\nu_{5a} + 32\nu_{5b} + \alpha_1) \geq 0;\\
\mu_0(u, \chi_{20}, *) &= \tfrac{1}{85}(-64\nu_{5a} - 64\nu_{5b} + 2750) \geq 0;\\
\mu_0(u, \chi_6, 3) &= \tfrac{1}{85}(32\nu_{5a} + 32\nu_{5b} + 155) \geq 0;\\
\mu_1(u, \chi_6, 3) &= \tfrac{1}{85}(-2\nu_{5a} + 3\nu_{5b} + \alpha_2) \geq 0;\\
\mu_{34}(u, \chi_6, 3) &= \tfrac{1}{85}(32\nu_{5a} - 48\nu_{5b} + \alpha_2) \geq 0;\\
\mu_1(u, \chi_3, 19) &= \tfrac{1}{85}(+8\nu_{17a} - 9\nu_{17b} + \alpha_3) \geq 0;\\
\mu_{15}(u, \chi_3, 19) &= \tfrac{1}{85}(-32\nu_{17a} + 36\nu_{17b} + \alpha_3) \geq 0,
\end{aligned}$$

where α_1 and α_2 are determined by the choice of the case for $\chi(u^{17})$ and α_3 is determined by the choice of $\chi(u^5)$ in the following way:

$$(\alpha_1, \alpha_2) = \begin{cases}
(325, 150), & \text{when } \chi(u^{17}) = \chi(5a);\\
(320, 155), & \text{when } \chi(u^{17}) = \chi(5b);\\
(310, 165), & \text{when } \chi(u^{17}) = -2\chi(5a) + 3\chi(5b);\\
(330, 145), & \text{when } \chi(u^{17}) = 2\chi(5a) - \chi(5b);\\
(305, 170), & \text{when } \chi(u^{17}) = -3\chi(5a) + 4\chi(5b);\\
(335, 140), & \text{when } \chi(u^{17}) = 3\chi(5a) - 2\chi(5b);\\
(315, 160), & \text{when } \chi(u^{17}) = -\chi(5a) + 2\chi(5b);\\
(340, 135), & \text{when } \chi(u^{17}) = 4\chi(5a) - 3\chi(5b),
\end{cases}$$

and

$$\alpha_3 = \begin{cases} 119, & \text{when } \chi(u^5) = \chi(17a); \\ 102, & \text{when } \chi(u^5) = \chi(17b); \\ 187, & \text{when } \chi(u^5) = 5\chi(17a) - 4\chi(17b); \\ 68, & \text{when } \chi(u^5) = -2\chi(17a) + 3\chi(17b); \\ 136, & \text{when } \chi(u^5) = 2\chi(17a) - \chi(17b); \\ 51, & \text{when } \chi(u^5) = -3\chi(17a) + 4\chi(17b); \\ 34, & \text{when } \chi(u^5) = -4\chi(17a) + 5\chi(17b); \\ 153, & \text{when } \chi(u^5) = 3\chi(17a) - 2\chi(17b); \\ 85, & \text{when } \chi(u^5) = -\chi(17a) + 2\chi(17b); \\ 170, & \text{when } \chi(u^5) = 4\chi(17a) - 3\chi(17b). \end{cases}$$

• Let u be a unit of order 95. By (1) and Proposition 2 we have

$$\nu_{5a} + \nu_{5b} + \nu_{19a} + \nu_{19b} = 1.$$

We need to consider 80 cases defined by parts (v) and (viii) of Theorem 3. In all of these cases by Proposition 3 we obtain the system of inequalities

$$\mu_0(u, \chi_2, *) = \tfrac{1}{95}(-36\nu_{19a} - 36\nu_{19b} + 76) \geq 0;$$
$$\mu_0(u, \chi_4, 2) = \tfrac{1}{95}(288\nu_{19a} + 288\nu_{19b} + 152) \geq 0,$$

that has no integral solutions such that all $\mu_i(u, \chi_j, p)$ are non-negative integers.

• Let u be a unit of order 323. By (1) and Proposition 2 we have

$$\nu_{17a} + \nu_{17b} + \nu_{19a} + \nu_{19b} = 1.$$

We need to consider 100 cases defined by parts (vii) and (viii) of Theorem 3, but in all of these cases we obtain the same system of inequalities

$$\mu_0(u, \chi_2, *) = \tfrac{1}{323}(-144\nu_{19a} - 144\nu_{19b} + 76) \geq 0;$$
$$\mu_0(u, \chi_6, 3) = \tfrac{1}{323}(288\nu_{19a} + 288\nu_{19b} + 171) \geq 0,$$

that has no solutions such that all $\mu_i(u, \chi_j, p)$ are non-negative integers.

6. INVESTIGATION OF THE GROUP J_4

Let G be the fourth Janko simple group J_4. Then it is well-known that $|G| = 2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$ and $exp(G) = 2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$ (see [12, 13]).

To verify Kimmerle's conjecture, we first need to investigate partial augmentations of units of prime orders for appropriate primes, and then we will need to consider units of orders of the form pq such that G has elements of orders p and q, but no elements of order pq.

We can immediately conclude that units of orders 3, 5, 23 and 29 are rationally conjugate to a group element. However, we will meet serious difficulties with elements of order 31 and this makes it unfeasible to use the same approach as the one we used for J_1, J_2 and J_3.

• Let u be a unit of order 31. By (1) and Proposition 2 we have

$$\nu_{31a} + \nu_{31b} + \nu_{31c} = 1.$$

Then by Proposition 3 we obtain the following system of inequalities

$$\mu_1(u, \chi_{56}, *) = \tfrac{1}{31}(21\nu_{31a} - 10\nu_{31b} - 10\nu_{31c} + 2001151845) \geq 0;$$
$$\mu_3(u, \chi_{56}, *) = \tfrac{1}{31}(-10\nu_{31a} - 10\nu_{31b} + 21\nu_{31c} + 2001151845) \geq 0;$$
$$\mu_5(u, \chi_{56}, *) = \tfrac{1}{31}(-10\nu_{31a} + 21\nu_{31b} - 10\nu_{31c} + 2001151845) \geq 0.$$

Using that $\nu_{31a} = 1 - \nu_{31b} - \nu_{31c}$, we reduce this to the system

$$\begin{aligned}
\mu_1(u, \chi_{56}, *) &= \tfrac{1}{31}31(-\nu_{31b} - \nu_{31c} + 64553286) \geq 0;\\
\mu_3(u, \chi_{56}, *) &= \tfrac{1}{31}(31\nu_{31c} + 31 \cdot 64553285) \geq 0;\\
\mu_5(u, \chi_{56}, *) &= \tfrac{1}{31}(31\nu_{31b} + 31 \cdot 64553285) \geq 0.
\end{aligned}$$

From this follows that all $\mu_i(u, \chi_j, *)$ will be non-negative integers for any ν_{31b} and ν_{31c} such that $\nu_{31b}, \nu_{31c} \geq -64553285$ and $\nu_{31b} + \nu_{31c} \leq 64553286$. Thus, we obtain 18 752 070 203 460 153 tuples $(\nu_{31a}, \nu_{31b}, \nu_{31c})$ that are solutions of this system, what is definitely beyond any reasonable bounds. Using the LAGUNA package [8], we checked that other inequalities for $\mu_i(u, \chi_j, *)$ obtained using the character table of J_4 and p-Brauer character tables for $p \in \{5, 7, 37\}$ do not give us any further restrictions on partial augmentations. Unfortunately, the p-Brauer character tables for $p \in \{2, 3, 11, 23, 29, 31, 43\}$ are not completely known yet (see `http://www.math.rwth-aachen.de/~MOC/work.html`). Thus, in the case of J_4 the Luthar-Passi method is not enough in combination with currently known p-Brauer character tables, and the Kimmerle's conjecture for J_4 still remains open. However, we have a hope that it could be solved using the Luthar-Passi method if p-Brauer character tables for new values of p will be capable of producing new constraints on partial augmentations.

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