On a new approach to the dual symmetric inverse monoid $\mathcal{I}_X^*$

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Abstract

We construct the *inverse partition semigroup* $\mathcal{IP}_X$, isomorphic to the *dual symmetric inverse monoid* $\mathcal{I}_X^*$, introduced in [6]. We give a convenient geometric illustration for elements of $\mathcal{IP}_X$. We describe all maximal subsemigroups of $\mathcal{IP}_X$ and find a generating set for $\mathcal{IP}_X$ when X is finite. We prove that all the automorphisms of $\mathcal{IP}_X$ are inner. We show how to embed the symmetric inverse semigroup into the inverse partition one. For finite sets X, we establish that, up to equivalence, there is a unique faithful effective transitive representation of $\mathcal{IP}_n$, namely to $\mathcal{IS}_{2^n-2}$. Finally, we construct an interesting $\mathcal{H}$-cross-section of $\mathcal{IP}_n$, which is reminiscent of $\mathcal{IO}_n$, the $\mathcal{H}$-cross-section of $\mathcal{IS}_n$, constructed in [4].

1 Introduction

The *dual inverse symmetric monoid* $\mathcal{I}_X^*$ was introduced in [6]. It consists of all *biequivalences* on a set X, i.e. all the binary relations α on X that are both *full*, that is $X\alpha = \alpha X = X$, and *bifunctional*, that is $\alpha \circ \alpha^{-1} \circ \alpha = \alpha$. The multiplication in $\mathcal{I}_X^*$ is given by:

$$\alpha\beta = \alpha \circ \left(\alpha^{-1} \circ \alpha \vee \beta \circ \beta^{-1}\right) \circ \beta, \tag{1}$$

for $\alpha, \beta \in \mathcal{I}_X^*$.

In the present paper we introduce the *inverse partition semigroup* $\mathcal{IP}_X$, isomorphic to $\mathcal{I}_X^*$ (see Theorem 1), and investigate some its properties. The main idea for considering the same semigroup under another point of view as in [6] (see definition of $\mathcal{IP}_X$ below) is to provide a convenient geometric realization for elements of this semigroup, which will enable us to handle them more easily. Besides, the semigroup $\mathcal{IP}_X$ naturally arises as an inverse subsemigroup of the *composition semigroup* $\mathcal{CS}_X$ (see Proposition 11), constructed below, a generalization of the semigroup $\mathcal{CS}_n$, introduced in [3]. The latter semigroup is close to, so called, *Brauer-type semigroups*, which were investigated for different reasons and from different contexts.

The first paper within these investigations, was the work of Brauer, [2], where he introduced the *Brauer semigroup* $\mathcal{B}_n$ in connection with representations of orthogonal groups. One more work, where $\mathcal{B}_n$ was studied in connection with representation theory is [8]. Further work, dedicated to $\mathcal{B}_n$ are [10], [14], [17]. For example, in [10] all the $\mathcal{L}$- and $\mathcal{R}$-cross-sections are described and in [17] a presentation for the singular part of $\mathcal{B}_n$ is given with respect to its minimal generating set. There are several generalizations of the Brauer semigroup: the *partial Brauer semigroup* $\mathcal{PB}_n$, introduced in [18]; the *composition semigroup* $\mathcal{CS}_n$, appeared in [3]; the *dual symmetric inverse monoid* $\mathcal{I}_X^*$, introduced in [6]; the finite *inverse partition semigroup* $\mathcal{IP}_n$, appeared in [16] (which is isomorphic to $\mathcal{I}_n^*$); the *partial inverse partition semigroup* $\mathcal{PIP}_X$, introduced in [9]. For other papers, dedicated to these semigroups we refer reader to [5], [12], [15], [19].

The main purpose of this paper is to investigate some inner semigroup properties of $\mathcal{IP}_X$, as well as to establish some connections of $\mathcal{IP}_X$ with other semigroups.

The paper is organized in the following way. In section 2 we define $\mathcal{IP}_X$. After this, in section 3, we prove that the constructed semigroup $\mathcal{IP}_X$ is isomorphic to $\mathcal{I}_X^*$. In section 4 we characterize the Green's relations and the natural order in $\mathcal{IP}_X$. In section 5 we investigate maximal subsemigroups and ideals of $\mathcal{IP}_X$ and define the *inverse type-preserving semigroup*. In section 6 we describe the automorphism group $\mathrm{Aut}(\mathcal{IP}_X)$. In section 7 we obtain a method how to embed the *symmetric inverse semigroup* $\mathcal{IS}_X$ into the inverse partition one. In section 8 we obtain that $\mathcal{IP}_X$ embeds into $\mathcal{IS}_{2^{|X|}-2}$ when $|X| \in \mathbb{N}\setminus\{1\}$. Finally, in section 9 we define the *inverse ordered partition semigroup* $\mathcal{IOP}_n$, which behaves similar to the $\mathcal{H}$-cross-section $\mathcal{IO}_n$ of $\mathcal{IS}_n$, studied in [4].

Throughout this paper for S a semigroup we denote by $E(S)$ the set of all idempotents of S. The natural order on an inverse semigroup S will be denoted by $\leq$, i.e., $a \leq b$ for $a, b \in S$ if and only if there is an idempotent e of S such that $a = be$ (see [7]). We will also need the notion of the *trace* $\mathrm{tr}(S)$ of an inverse semigroup S: the set S together with the partial multiplication $*$, defined as follows: $a * b$ is defined precisely when $ab \in \mathcal{R}_a \cap \mathcal{L}_b$ and is equal then to ab (see [20] and section XIV.2 of [21]). Finally, we recall one more definition. For any inverse semigroup S, the *inductive groupoid* of S, or *imprint* $\mathrm{im}(S)$ of S, is the triple $\big(\mathrm{tr}(S), \leq, \star\big)$, where $\leq$ is the natural partial order in S, and $\star$ is a partial product defined by: for $e \in E(S)$, $a \in S$,

$e \leq aa^{-1}$, $e \star a = ea$ (see section XIV.3.4 of [21]).

2 Definition of the inverse partition semigroup $\mathcal{IP}_X$

Throughout all the paper let X be an arbitrary set. We consider a map $' : X \to X'$ as a fixed bijection and will denote the inverse bijection by the same symbol, that is $(x')' = x$ for all $x \in X$. We are going to construct a semigroup $\mathcal{CS}_X$.

Let $\mathcal{CS}_X$ be the set of all partitions of $X \cup X'$ into nonempty blocks. If $X \cup X' = \dot{\bigcup_{i \in I}} A_i$ is a partition of $X \cup X'$ into nonempty blocks A_i, $i \in I$, corresponding to an element $a \in \mathcal{CS}_X$, then we will write $a = \bigl(A_i\bigr)_{i \in I}$. In the case when $I = \{i_1, \dots, i_k\}$ is finite, we will also write $a = \bigl\{A_{i_1}, \dots, A_{i_k}\bigr\}$.

For $a \in \mathcal{CS}_X$ and $x, y \in X \cup X'$, we set $x \equiv_a y$ provided that x and y are at the same block of a. Clearly, we can realize $a \in \mathcal{CS}_X$ as the equivalence relation $\equiv_a$. Thus in spite of the fact that elements of $\mathcal{CS}_X$ will be partitions, we will sometimes treat with them as with the associated equivalence relations.

Take now $a, b \in \mathcal{CS}_X$. Define a new equivalence relation, $\equiv$, on $X \cup X'$ as follows:

- for $x, y \in X$ we have $x \equiv y$ if and only if $x \equiv_a y$ or there is a sequence, $c_1, \dots, c_{2s}$, $s \geq 1$, of elements in X, such that $x \equiv_a c_1'$, $c_1 \equiv_b c_2$, $c_2' \equiv_a c_3', \dots, c_{2s-1} \equiv_b c_{2s}$, and $c_{2s}' \equiv_a y$;

- for $x, y \in X$ we have $x' \equiv y'$ if and only if $x' \equiv_b y'$ or there is a sequence, $c_1, \dots, c_{2s}$, $s \geq 1$, of elements in X, such that $x' \equiv_b c_1$, $c_1' \equiv_a c_2'$, $c_2 \equiv_b c_3, \dots, c_{2s-1}' \equiv_a c_{2s}'$, and $c_{2s} \equiv_b y'$;

- for $x, y \in X$ we have $x \equiv y'$ if and only if $y' \equiv x$ if and only if there is a sequence, $c_1, \dots, c_{2s-1}$, $s \geq 1$, of elements in X, such that $x \equiv_a c_1'$, $c_1 \equiv_b c_2$, $c_2' \equiv_a c_3', \dots, c_{2s-2}' \equiv_a c_{2s-1}'$, and $c_{2s-1} \equiv_b y'$.

Proposition 1. *$\equiv$ is an equivalence relation on $X \cup X'$.*

Proof. It follows immediately from the definition of $\equiv$ that this relation is reflexive and symmetric. Let now $x \equiv y$ and $y \equiv z$ for some $x, y, z \in X \cup X'$. We are going to establish that $x \equiv z$. In the rest of the proof we may assume that $y \in X$, the other case is treated analogously. We have four possible cases.

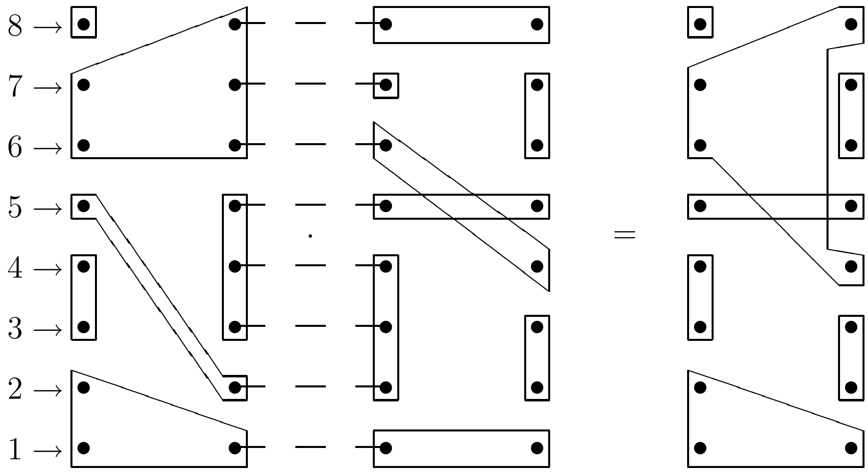

Figure 1: Elements of $\mathcal{CS}_8$ and their multiplication.

Case 1. $x, z \in X$. If $x \equiv_a y$ or $y \equiv_a z$ then since $\equiv_a$ is an equivalence relation, we immediately obtain from the definition of $\equiv$ that $x \equiv z$. Otherwise we have that there exist $c_1, \ldots, c_{2s}, d_1, \ldots, d_{2t}$, elements of X, such that $x \equiv_a c_1'$, $c_1 \equiv_b c_2$, $c_2' \equiv_a c_3', \ldots$, $c_{2s-1} \equiv_b c_{2s}$, $c_{2s}' \equiv_a y$ and $y \equiv_a d_1'$, $d_1 \equiv_b d_2$, $d_2' \equiv_a d_3', \ldots$, $d_{2t-1} \equiv_b d_{2t}$, $d_{2t}' \equiv_a z$. Now, using transitiveness of $\equiv_a$, we can write $c_{2s}' \equiv_a d_1'$ and hence $x \equiv z$.

Case 2. $x, z \in X'$. Then there are $c_1, \ldots, c_{2s-1}$, $d_1, \ldots, d_{2t-1}$, elements of X, such that $x \equiv_b c_{2s-1}$, $c_{2s-1}' \equiv_a c_{2s-2}' \ldots$, $c_3' \equiv_a c_2'$, $c_2 \equiv_b c_1$, $c_1' \equiv_a y$ and $y \equiv_a d_1'$, $d_1 \equiv_b d_2$, $d_2' \equiv_a d_3', \ldots$, $d_{2t-2}' \equiv_a d_{2t-1}'$, $d_{2t-1} \equiv_b z$. Again, using transitiveness of $\equiv_a$, we obtain that $c_1' \equiv_a d_1'$, whence $x \equiv z$.

Case 3. $x \in X$ and $z \in X'$. There exist $d_1, \ldots, d_{2t-1}$, elements of X, such that $y \equiv_a d_1'$, $d_1 \equiv_b d_2$, $d_2' \equiv_a d_3', \ldots$, $d_{2t-2}' \equiv_a d_{2t-1}'$, and $d_{2t-1} \equiv_b z$. If $x \equiv_a y$ then due to transitiveness of $\equiv_a$, we have $x \equiv z$. Otherwise there are $c_1, \ldots, c_{2s}$, elements of X, such that $x \equiv_a c_1'$, $c_1 \equiv_b c_2$, $c_2' \equiv_a c_3', \ldots$, $c_{2s-1} \equiv_b c_{2s}$, and $c_{2s}' \equiv_a y$. Then it remains to notice that $c_{2s}' \equiv_a d_1'$.

Case 4. $x \in X'$ and $z \in X$. Then, since $z \equiv y$ and $y \equiv x$, according to Case 3, we have that $z \equiv x$, whence $x \equiv z$.

The proof is complete. □

Thus $\equiv$ defines a partition of $X \cup X'$ into disjoint blocks and so belongs to $\mathcal{CS}_X$. Set this partition to be a product $a \cdot b$ in $\mathcal{CS}_X$. One can easily show that $(\mathcal{CS}_X, \cdot)$ is a semigroup. We will call this semigroup the *composition semigroup* on the set X.

Let $\mathcal{IP}_X$ be the subset of $\mathcal{CS}_X$, containing those elements $(A_i)_{i \in I} \in \mathcal{CS}_X$ such that $A_i \cap X \neq \varnothing$ and $A_i \cap X' \neq \varnothing$ for all $i \in I$. Since the construction of $\mathcal{CS}_X$, we have that $\mathcal{IP}_X$ is closed under the multiplication in $\mathcal{CS}_X$ and so $\mathcal{IP}_X$

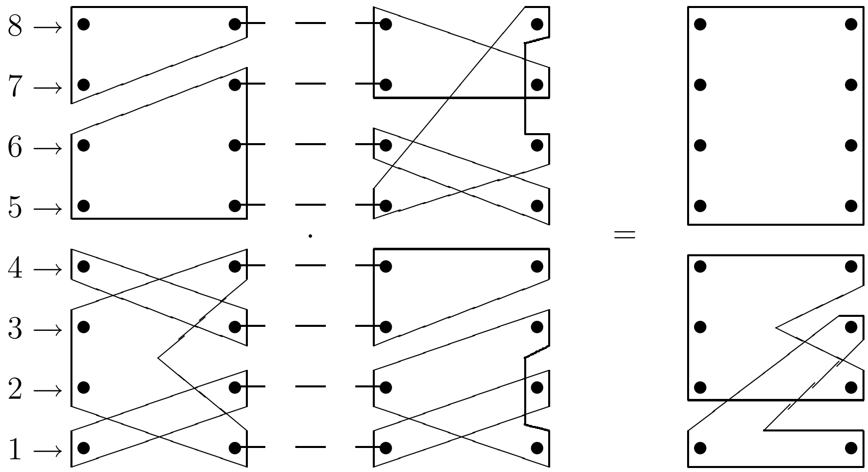

Figure 2: Elements of $\mathcal{IP}_8$ and their multiplication.

is a subsemigroup of $\mathcal{CS}_X$. Observe that $\mathcal{IP}_X$ has the zero element, namely $\{X \cup X'\}$. We will denote this element by 0. Obviously, if $|X|=|Y|$ then $\mathcal{CS}_X \cong \mathcal{CS}_Y$ and $\mathcal{IP}_X \cong \mathcal{IP}_Y$. In the case when $X=\{1,\dots,n\}$, it will be convenient to denote $\mathcal{CS}_X$ and $\mathcal{IP}_X$ by $\mathcal{CS}_n$ and $\mathcal{IP}_n$ respectively. Figures 1 and 2 illustrate the given notions for the case when $X=\{1,\dots,8\}$, where we consider elements of semigroups as couples of vertical rows of points, divided into blocks. More precisely, the left vertical row corresponds to the set X and the right one to X'. The multiplication $a \cdot b$ is just a gluing of elements a and b by dint of identifying the points of X' from a with the corresponding elements of X from b. On Fig. 1 we present the equality

$$\begin{aligned}&\big\{\{1,2,1'\},\{3,4\},\{5,2'\},\{3',4',5'\},\{6,7,6',7',8'\},\{8\}\big\}\cdot\\ &\quad\big\{\{1,1'\},\{2,3,4\},\{2',3'\},\{5,5'\},\{6,4'\},\{7\},\{6',7'\},\{8,8'\}\big\}=\\ &\qquad\big\{\{1,2,1'\},\{3,4\},\{2',3'\},\{5,5'\},\{6,7,4',8'\},\{6',7'\},\{8\}\big\}\end{aligned} \tag{2}$$

and on Fig. 2 we present the following one:

$$\begin{aligned}&\big\{\{1,2'\},\{2,3,1',4'\},\{4,3'\},\{5,6,5',6',7'\},\{7,8,8'\}\big\}\cdot\\ &\quad\big\{\{1,2'\},\{2,1',3'\},\{3,4,4'\},\{5,6',8'\},\{6,5'\},\{7,8,7'\}\big\}=\\ &\qquad\big\{\{1,1',3'\},\{2,3,4,2',4'\},\{5,6,7,8,5',6',7',8'\}\big\}.\end{aligned} \tag{3}$$

Now we move to the proof of the fact that $\mathcal{IP}_X$ is isomorphic to $\mathcal{I}_X^*$.

3 $\mathcal{IP}_X$ is isomorphic to $\mathcal{I}_X^*$

The main goal of this section is to prove the following

Theorem 1. $\mathcal{IP}_X \cong \mathcal{I}_X^*$.

Proof. We begin with recalling one notion from [6]. A *block bijection* of X is a bijection between two quotient sets X/σ and X/τ for certain equivalence relations σ and τ on X such that $|X/\sigma|=|X/\tau|$. We will need the following statement, stated in [6] (one might find it also in [13], Section 4.2).

Lemma 1 (Lemma 2.1 from [6])**.** *If α is a biequivalence on X, then both $\alpha \circ \alpha^{-1}$ and $\alpha^{-1} \circ \alpha$ are equivalence relations on X. Moreover the map $\widetilde{\alpha}$ defined by $\widetilde{\alpha} : x(\alpha \circ \alpha^{-1}) \mapsto x\alpha$ for $x \in X$ is a block bijection of $X/\alpha \circ \alpha^{-1}$ to $X/\alpha^{-1} \circ \alpha$. Conversely, given equivalence relations β and γ on X together with a block bijection $\mu : X/\beta \to X/\gamma$, a unique biequivalence $\widehat{\mu}$ on X inducing μ is given by: $x\widehat{\mu}y$ if and only if $x\beta \mapsto y\gamma$ under the block bijection μ (in which case $\beta = \widehat{\mu} \circ \widehat{\mu}^{-1}$ and $\gamma = \widehat{\mu}^{-1} \circ \widehat{\mu}$). Finally, the two processes are reciprocal: $\widehat{\widetilde{\alpha}} = \alpha$ and $\widetilde{\widehat{\mu}} = \mu$.*

To define an isomorphism between $\mathcal{IP}_X$ and $\mathcal{I}_X^*$, we need some auxiliary notation.

Let $a \in \mathcal{IP}_X$. Define the following relations ρ_a and λ_a on X as follows:

$$x\rho_a y \text{ if and only if } x \equiv_a y, \text{ and } x\lambda_a y \text{ if and only if } x' \equiv_a y', \tag{4}$$

for $x, y \in X$. Since ρ_a is a restriction of the relation $\equiv_a$ to X, we obtain that ρ_a is an equivalence relation on X. From the definition of λ_a and similar arguments it follows that λ_a is an equivalence relation on X as well. Remark that a is not determined by λ_a and ρ_a.

Define a map $\pi : \mathcal{IP}_X \to \mathcal{I}_X^*$ as follows: for $a \in \mathcal{IP}_X$ we put $\pi(a) = \widehat{\mu_a}$, where μ_a is a block bijection from X/ρ_a onto X/λ_a such that the block A of ρ_a is mapped under μ_a to that block B of λ_a, for which $A \cup B'$ is a block of $\equiv_a$. In view of our definition of $\mathcal{IP}_X$ and Lemma 1, we obtain that π is a bijection from $\mathcal{IP}_X$ onto $\mathcal{I}_X^*$.

We are left to prove that π is a morphism from $\mathcal{IP}_X$ to $\mathcal{I}_X^*$. Take $a, b \in \mathcal{IP}_X$. We need to prove that $\widehat{\mu_{ab}} = \widehat{\mu_a}\widehat{\mu_b} = \widehat{\mu_a} \circ \big(\widehat{\mu_a}^{-1} \circ \widehat{\mu_a} \vee \widehat{\mu_b} \circ \widehat{\mu_b}^{-1}\big) \circ \widehat{\mu_b}$. Notice that due to Lemma 1, we have that $\widehat{\mu_b} \circ \widehat{\mu_b}^{-1} = \rho_b$ and $\widehat{\mu_a}^{-1} \circ \widehat{\mu_a} = \lambda_a$ and hence we must establish that $\widehat{\mu_{ab}} = \widehat{\mu_a} \circ \big(\lambda_a \vee \rho_b\big) \circ \widehat{\mu_b}$. Note also that for all $c \in \mathcal{IP}_X$ it follows immediately from the definition of μ_c that for all $x, y \in X$ one has $x\widehat{\mu_c}y$ if and only if $x \equiv_c y'$. Finally, we recall that for equivalence relations λ and ρ on X, the join $\lambda \vee \rho$ coincides with the transitive closure of the relation $\lambda \cup \rho$.

Suppose firstly that $x\widehat{\mu_{ab}}y$, for some $x, y \in X$. Then $x \equiv_{ab} y'$ and so there exist $c_1, \dots, c_{2s-1}$, $s \geq 1$, elements of X, such that $x \equiv_a c_1'$, $c_1 \equiv_b c_2$, $c_2' \equiv_a c_3', \dots, c_{2s-2}' \equiv_a c_{2s-1}'$, and $c_{2s-1} \equiv_b y'$. Then we have $x\widehat{\mu_a}c_1$, $c_1\rho_b c_2$,

$c_2\lambda_a c_3, \ldots, c_{2s-2}\lambda_a c_{2s-1}$, and $c_{2s-1}\widehat{\mu_b}y$. Thus, we have $x\widehat{\mu_a}c_1$, $c_1(\lambda_a \vee \rho_b)c_{2s-1}$ and $c_{2s-1}\widehat{\mu_b}y$, whence $(x,y) \in \widehat{\mu_a} \circ \big(\lambda_a \vee \rho_b\big) \circ \widehat{\mu_b}$.

Conversely, suppose that $(x,y) \in \widehat{\mu_a} \circ \big(\lambda_a \vee \rho_b\big) \circ \widehat{\mu_b}$. Then there exist $c,d \in X$ such that $x\widehat{\mu_a}c$, $c(\lambda_a \vee \rho_b)d$ and $d\widehat{\mu_b}y$. Then we have $x \equiv_a c'$ and $d \equiv_b y'$. Notice that if $c\lambda_a r$ then $x \equiv_a r'$ and if $t\rho_b d$ then $t \equiv_b y'$. Hence, taking to account $c(\lambda_a \vee \rho_b)d$, there exist $c_1, \ldots, c_{2s-1}$, $s \geq 1$, elements of X, such that $x \equiv_a c_1'$, $c_1\rho_b c_2$, $c_2\lambda_a c_3, \ldots, c_{2s-2}\lambda_a c_{2s-1}$, and $c_{2s-1} \equiv_b y'$. These imply $x \equiv_a c_1'$, $c_1 \equiv_b c_2$, $c_2' \equiv_a c_3', \ldots, c_{2s-2}' \equiv_a c_{2s-1}'$, and $c_{2s-1} \equiv_b y'$. Thus $x \equiv_{ab} y'$, whence $x\widehat{\mu_{ab}}y$.

The proof of the theorem is complete. □

As a consequence of Theorem 1 we obtain the following statement.

Proposition 2. *$\mathcal{IP}_X$ is an inverse semigroup.*

Proof. Follows from the fact that $\mathcal{I}_X^*$ is inverse, see [6]. □

Due to what we have already obtained, we can now call $\mathcal{IP}_X$ the *inverse partition semigroup* on the set X.

4 Green's relations and the natural order in $\mathcal{IP}_X$

We begin this section with description of Green's relations on $\mathcal{IP}_X$. But before we need some preparation.

First notice that it follows immediately from the definition of multiplication in $\mathcal{IP}_X$ that

$$\rho_{ab} \supseteq \rho_a \text{ and } \lambda_{ab} \supseteq \lambda_b \text{ for all } a,b \in \mathcal{IP}_X. \tag{5}$$

Then we obtain that every ρ_{ab}-class is a union of some ρ_a-classes and that every λ_{ab}-class is a union of some λ_b-classes.

Note also that the cardinal number of the set of all ρ_a-classes and the cardinal number of the set of all λ_a-classes coincide with the cardinal number of the set of all $\equiv_a$-classes. Denote this common number by $\mathrm{rank}(a)$. We will call this number the *rank* of a. Due to (5), we have

$$\mathrm{rank}(ab) \leq \min\big\{\mathrm{rank}(a), \mathrm{rank}(b)\big\} \text{ for all } a,b \in \mathcal{IP}_X. \tag{6}$$

Note that if $a = \big(A_i \cup B_i'\big)_{i\in I}$ then $\mathrm{rank}(a) =\mid I \mid$. We denote the Green's relations in the standard way: $\mathcal{R}$, $\mathcal{L}$, $\mathcal{H}$, $\mathcal{D}$, and $\mathcal{J}$ (see [7]).

Theorem 2. *Let $a,b \in \mathcal{IP}_X$. Then*

1. *$a\mathcal{R}b$ if and only if $\rho_a = \rho_b$;*

2. *$a\mathcal{L}b$ if and only if $\lambda_a = \lambda_b$;*

3. *$a\mathcal{H}b$ if and only if $\rho_a = \rho_b$ and $\lambda_a = \lambda_b$ hold simultaneously;*

4. *$a\mathcal{J}b$ if and only if $a\mathcal{D}b$ if and only if $\mathrm{rank}(a) = \mathrm{rank}(b)$;*

5. *$|\mathcal{IP}_n| = \sum_{k=1}^{n} \big(s(n,k)\big)^2 \cdot k!$, where $s(n,k)$ denotes the Stirling number of the second kind;*

6. *$|E(\mathcal{IP}_n)| = \mathrm{B}_n$, where B_n denotes the Bell number.*

Proof. In view of Theorem 1, these statements are just reformulations of those of Theorem 2.2 from [6]. $\square$

Now we move to description of the group of units of $\mathcal{IP}_X$. Denote by $\mathcal{S}_X$ the *symmetric group* on X. Set a map $\eta : \mathcal{S}_X \to \mathcal{IP}_X$ as follows:

$$\eta(g) = \big(\{x, g(x)'\}\big)_{x\in X} \text{ for all } g \in \mathcal{S}_X. \tag{7}$$

Lemma 2. *The map η is an injective homomorphism.*

Proof. That η is a homomorphism, follows from the definition of the multiplication in $\mathcal{IP}_X$. If now $\eta(g_1) = \eta(g_2)$ for some $g_1, g_2 \in \mathcal{S}_X$, then $g_1(x) = g_2(x)$ for all $x \in X$ and so $g_1 = g_2$. This completes the proof. $\square$

As a consequence of Lemma 2 we obtain that $\mathcal{IP}_X$ contains a subgroup $\eta(\mathcal{S}_X)$, isomorphic to $\mathcal{S}_X$. Let us identify this subgroup with $\mathcal{S}_X$. Clearly, the identity element 1 of $\mathcal{S}_X$ is the identity element of $\mathcal{IP}_X$. Using Theorem 2, we obtain now the following corollary.

Proposition 3. *The group of all invertible elements of $\mathcal{IP}_X$ coincides with $\mathcal{S}_X$.*

Proof. Since the maximal subgroup of an arbitrary semigroup coincides with some $\mathcal{H}$-class of this semigroup (see [7]), we obtain that an element g is invertible in $\mathcal{IP}_X$ if and only if $g\mathcal{H}1$. Due to Theorem 2, this is equivalent to $g \in \mathcal{S}_X$. $\square$

Let us now switch to the description of the natural order on $\mathcal{IP}_X$. But before, we need to describe the idempotents of $\mathcal{IP}_X$.

Lemma 3. *Let $e \in \mathcal{IP}_X$. Then e is an idempotent if and only if there is a partition $X = \dot{\bigcup_{i\in I}} E_i$ such that $e = \big(E_i \cup E_i'\big)_{i\in I}$. In addition, for idempotents e and f the elements ef and fe coincide with the minimum equivalence relation on $X \cup X'$, which contains e and f.*

Proof. Let us prove firstly the first part of the statement. The sufficiency of it is obvious.

Let now e be an idempotent of $\mathcal{IP}_X$. Let $A \cup B'$ be some block in e. Suppose that $A \setminus B \neq \varnothing$. Then there is $a \in A$ such that $a \notin B$. Take an arbitrary b of B. Take also $c \in X$ such that $c \equiv_e a'$. Then $c \notin A$. Indeed, otherwise we would have $a \equiv_e c \equiv_e a'$ which implies $a \in B$. Thus, $c \notin A$.

Now due to $c \equiv_e a'$ and $a \equiv_e b'$, we obtain that $c \equiv_{e^2} b'$. But the latter gives us $c \in A$. We get a contradiction. Thus, $A \setminus B = \varnothing$ and so $A \subseteq B$. Analogously, $B \subseteq A$. Thus, every block of e has the form $A \cup A'$ for certain $A \subseteq X$. This completes the proof of the first part of the statement. The second one now follows immediately from the definition of the multiplication in $\mathcal{IP}_X$. □

Proposition 4. *Let $a, b \in \mathcal{IP}_X$. Then $a \leq b$ if and only if $\equiv_a \supseteq \equiv_b$.*

Proof. Let $a = \big(A_i \cup B_i'\big)_{i\in I}$ and $b = \big(C_j \cup D_j'\big)_{j\in J}$.

Suppose first that $\equiv_b \subseteq \equiv_a$. Then we have that for all $i \in I$, $A_i \cup B_i'$ is a union of some blocks $C_j \cup D_j'$, $j \in J$. Put $f = \big(B_i \cup B_i'\big)_{i\in I}$. Then we obtain that $a = bf$. It remains to note that, due to Lemma 3, f is an idempotent.

Suppose now that there is an idempotent e of $\mathcal{IP}_X$ such that $a = be$. Due to Lemma 3, we have that $e = \big(E_k \cup E_k'\big)_{k\in K}$ for some partition $X = \dot{\bigcup_{k\in K}} E_k$. Take now $(x, y) \in \equiv_b$. There is z of X such that z' is $\equiv_b$–equivalent to x and y. Then, since $z \equiv_e z'$, we obtain that $(x, y) \in \equiv_{be}$ or just that $(x, y) \in \equiv_a$. This completes the proof. □

Now we are able to characterize the trace of $\mathcal{IP}_X$.

Proposition 5. *Let $a, b \in \mathrm{tr}(\mathcal{IP}_X)$. The product $a * b$ is defined if $\lambda_a = \rho_b$ and in this case $\pi(a) \circ \pi(b) \in \mathcal{I}_X^*$ and $a * b = \pi^{-1}(\pi(a) \circ \pi(b))$.*

Proof. It is known that for $x, y \in \mathrm{tr}(S)$, where S is an inverse semigroup, the product $x * y$ is defined if and only if $x^{-1}x = yy^{-1}$ (see [20]). Note also that, using Lemma 3, we have that for every $x \in \mathcal{IP}_X$ the condition $\rho_x = \lambda_x$ holds if and only if $x \in E(\mathcal{IP}_X)$. In addition, for $e, f \in E(\mathcal{IP}_X)$ we have that $\lambda_e = \lambda_f$ if and only if $e = f$. Hence, $a * b$ is defined if and only if $a^{-1}a = bb^{-1}$ if and only if $\lambda_{a^{-1}a} = \rho_{bb^{-1}}$. It remains to notice that since $a^{-1}a\mathcal{L}a$ and $bb^{-1}\mathcal{R}b$, using Theorem 2, we have $\lambda_{a^{-1}a} = \lambda_a$ and $\rho_{bb^{-1}} = \rho_b$.

If now $a*b$ is defined then $\pi(a)*\pi(b)$ is defined in $\mathcal{I}_X^*$ and then $\pi(a)*\pi(b) = \pi(a) \circ \pi(b)$ (see [13]). The statement follows. $\square$

The following proposition is concerned with $\mathrm{im}(\mathcal{IP}_X)$, the imprint of $\mathcal{IP}_X$.

Proposition 6. *Let $e \in E(\mathcal{IP}_X)$ and $a \in \mathcal{IP}_X$. The product $e \star a$ is defined if and only if $\rho_a \subseteq \rho_e$.*

Proof. By the definition of imprint, we have that $e \star a$ is defined if and only if $e \leq aa^{-1}$, which, in view of Proposition 4, holds if and only if $\equiv_{aa^{-1}} \subseteq \equiv_e$ which is equivalent to $\rho_{aa^{-1}} \subseteq \rho_e$. It remains to notice that $\rho_a = \rho_{aa^{-1}}$. $\square$

5 Generating set, ideals and maximal subsemigroups of $\mathcal{IP}_n$

To begin this section, we put some auxiliary notations. Let $A \subseteq X$. Define an element τ_A of $\mathcal{IP}_X$ as follows:

$$\tau_A = \big\{A \cup A', \{x, x'\}_{x \in X \setminus A}\big\}. \tag{8}$$

Clearly, τ_X is the zero element of $\mathcal{IP}_X$. If x and y are distinct elements of X, we will use the notation $\tau_{x,y} = \tau_{\{x,y\}}$.

Suppose that $|X| \geq 3$. For pairwise distinct elements x, y, z of X define an element $\xi_{x,y,z}$ as follows:

$$\xi_{x,y,z} = \big\{\{x, y, x'\}, \{z, y', z'\}, \{t, t'\}_{t \in X \setminus \{x,y,z\}}\big\}. \tag{9}$$

If necessary, we will write $\xi^X_{x,y,z}$ instead of $\xi_{x,y,z}$ to stress on that $\xi_{x,y,z} \in \mathcal{IP}_X$.

Lemma 4. *Let $|X| \geq 3$. Then*

$$g^{-1}\xi_{x,y,z}g = \xi_{g(x),g(y),g(z)},\ \ g^{-1}\tau_{x,y}g = \tau_{g(x),g(y)},$$
$$\xi^2_{x,y,z} = \tau_{\{x,y,z\}} \text{ and } \xi_{x,y,z}\xi_{z,y,x} = \tau_{x,y} \tag{10}$$

for all pairwise distinct $x, y, z \in X$ and $g \in \mathcal{S}_X$.

Proof. Direct calculation. $\square$

Now our local goal is to provide a generating set for $\mathcal{IP}_n$ (see Proposition 8). In order to do this we will construct an inverse subsemigroup $\mathcal{IT}_n$ of $\mathcal{IP}_n$ (see below), which is interesting itself as a semigroup. In addition, the notion of $\mathcal{IT}_n$ will help us to describe all the maximal subsemigroups of $\mathcal{IP}_n$. So we are starting with putting some auxiliary notations.

Let $n \geq 2$. Set $\mathcal{IT}_n = \langle \mathcal{S}_n, \tau_{1,2} \rangle$. Set also $\mathcal{IT}_1 = \mathcal{IP}_1$. Let ρ be some equivalence relation on $\{1, \dots, n\}$. Define a *type* of the relation ρ as a tuple $(t_1, \dots, t_n)$, where t_i denotes the number of all i-element ρ-classes, $1 \leq i \leq n$. The following proposition shows that $\mathcal{IT}_n$ is an inverse subsemigroup of $\mathcal{IP}_n$. But before, we give one more definition: an element a of $\mathcal{IP}_n$ is said to be *special* if

$$x \equiv_a y' \text{ implies } |x\rho_a| = |y\lambda_a| \text{ for all } x, y \in \{1, \dots, n\}. \tag{11}$$

Proposition 7. *The following statements hold:*

1. $\mathcal{IT}_n$ *is an inverse subsemigroup of* $\mathcal{IP}_n$*;*
2. $\tau_A \in \mathcal{IT}_n$ *for all* $A \subseteq \{1, \dots, n\}$*;*
3. *the elements of* $\mathcal{IT}_n$ *are precisely all special elements of* $\mathcal{IP}_n$*;*
4. *if* $a \in \mathcal{IT}_n$ *then the types of* ρ_a *and* λ_a *coincide.*

Proof. We will assume that $n \geq 2$ as all the statements hold in the case when $n = 1$.

Since $\mathcal{S}_n$ is a subgroup of $\mathcal{IP}_n$ and $\tau_{1,2}$ is an idempotent in $\mathcal{IP}_n$, we obtain that $\mathcal{IT}_n$ is an inverse subsemigroup of $\mathcal{IP}_n$. This completes the proof of 1).

Note that, due to Lemma 4, we have that $\tau_{x,y} \in \mathcal{IT}_n$ for all distinct x and y of $\{1, \dots, n\}$. Now the statement 2) follows from the equality $\tau_{\{x\}} = 1$, for all $x \in \{1, \dots, n\}$, and the fact that if $A = \{x_1, \dots, x_k\}$, $k \geq 2$, then

$$\tau_A = \prod_{i=1}^{k-1} \tau_{x_k, x_{k+1}}. \tag{12}$$

Let us prove 3). Let $a = \big(A_i \cup B_i'\big)_{i \in I}$ be an element of $\mathcal{IP}_n$ such that $x \equiv_a y'$ implies $|x\rho_a| = |y\lambda_a|$ for all $x, y \in \{1, \dots, n\}$. Then $|A_i| = |B_i|$ for all $i \in I$ and so there exists $g \in \mathcal{S}_n$ such that $ga = \big(B_i \cup B_i'\big)_{i \in I}$. Now due to 2), we have that

$$a = g^{-1} \cdot \prod_{i \in I} \tau_{B_i} \in \mathcal{IT}_n. \tag{13}$$

Conversely, suppose that $a \in \mathcal{IT}_n$. Note that $\tau_{1,2}$ is special and all the elements of $\mathcal{S}_n$ are special, too. Hence, to prove that a is special, it is enough to prove that if $b \in \mathcal{IP}_n$ is special then $b\tau_{1,2}$ is special and bg is special for all $g \in \mathcal{S}_n$. Suppose that $b = \big(C_i \cup D_i'\big)_{i \in K} \in \mathcal{IP}_n$ is special. Then, obviously, bg is also special for all $g \in \mathcal{S}_n$. We have two cases.

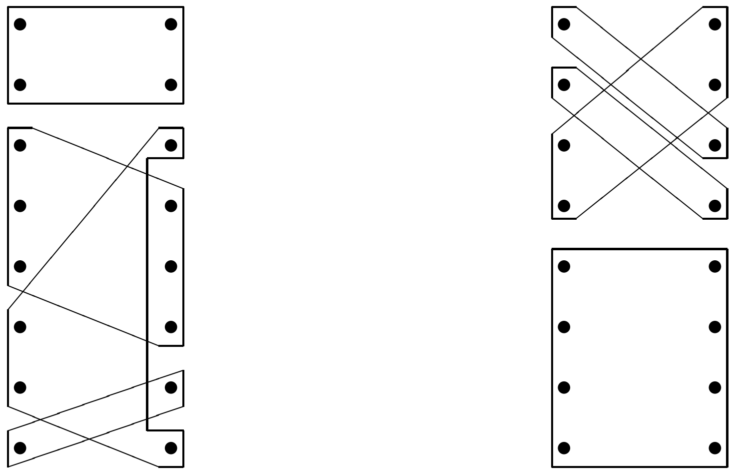

Figure 3: Elements of $\mathcal{IT}_8$.

Case 1. There is $i \in K$ such that $D_i \supseteq \{1, 2\}$. Then $b\tau_{1,2} = b$ is special.

Case 2. There are distinct i and j of K such that $1 \in D_i$ and $2 \in D_j$. Then $b\tau_{1,2} = \Big\{\big(C_i \cup C_j\big) \bigcup \big(D_i \cup D_j\big)', \big(C_k \cup D'_k\big)_{k \in K\setminus\{i,j\}}\Big\}$ is, obviously, special. This completes the proof of 3).

The statement 4) follows immediately from 3). $\square$

As a consequence of 4) of Proposition 7, we can now call $\mathcal{IT}_n$ the *inverse type-preserving semigroup* of degree n. We give an illustration of elements of $\mathcal{IT}_8$ on Fig. 3. It also follows from Proposition 7 that $\mathcal{IT}_n = \mathcal{S}_n E(\mathcal{IP}_n)$, that is $\mathcal{IT}_n$ is the greatest factorizable inverse submonoid of $\mathcal{IP}_n$. Remark that $\mathcal{IT}_n$ (more precisely, $\pi(\mathcal{IT}_n)$, the greatest factorizable inverse submonoid of $\mathcal{I}_X^*$) appeared in [5], [6] and [1] under the name of the *monoid of uniform block permutations*.

The following proposition gives us an example of a generating system of $\mathcal{IP}_n$. But to prove this proposition, we need some auxiliary facts.

Lemma 5. *Let $n \geq 3$, $a \in \mathcal{IP}_n$ and $\operatorname{rank}(a) = n-1$. Then either $a \in \xi_{x,y,z}\mathcal{S}_n$ or $a \in \tau_{x,y}\mathcal{S}_n$ for some pairwise distinct $x, y, z \in \{1, \ldots, n\}$.*

Proof. Straightforward. $\square$

Take $n \in \mathbb{N}$. Set $\Pi_n = \Big\{q \in \mathcal{IP}_{n+1} : \ q \text{ contains the block } \{n+1, (n+1)'\}\Big\}$.

Lemma 6. *Let $n \in \mathbb{N}$. Then the map $a \mapsto a \cup \{n+1, (n+1)'\}$, $a \in \mathcal{IP}_n$, is an isomorphism from $\mathcal{IP}_n$ onto Π_n, which maps $\xi_{1,2,3}^{\{1,\ldots,n\}}$ to $\xi_{1,2,3}^{\{1,\ldots,n+1\}}$.*

Proof. Obvious. $\square$

Proposition 8. *Let $n \geq 3$. Then $\mathcal{IP}_n = \langle \mathcal{S}_n, \xi_{1,2,3} \rangle$. Moreover, for $u \in \mathcal{IP}_n$, $\mathcal{IP}_n = \langle \mathcal{S}_n, u \rangle$ if and only if $u \in \mathcal{S}_n \xi_{1,2,3} \mathcal{S}_n$.*

Proof. We will prove the statement that $\mathcal{IP}_n = \langle \mathcal{S}_n, \xi_{1,2,3} \rangle$ for all $n \geq 3$ by the complete induction on n.

First, let us verify that the basis of the induction, the case when $n = 3$, holds. We are to prove that $\mathcal{IP}_3 = \langle \mathcal{S}_3, \xi_{1,2,3} \rangle$. Note that, due to Lemma 4, $0 = \xi_{1,2,3}^2$. Thus, we are left to prove that every element v of $\mathcal{IP}_3$ such that $\mathrm{rank}(v) = 2$, belongs to $\langle \mathcal{S}_3, \xi_{1,2,3} \rangle$. But this follows from Lemmas 4 and 5. Thus, the basis of induction holds.

Assume now that the proposition of induction holds for all numbers k, $3 \leq k \leq n$. We are going to prove now that $\mathcal{IP}_{n+1} = \langle \mathcal{S}_{n+1}, \xi_{1,2,3} \rangle$. Let $a \in \mathcal{IP}_{n+1}$. Then there is $g \in \mathcal{S}_{n+1}$ such that $b = ag$ contains a block $\big(E \cup \{n+1\}\big) \bigcup \big(F \cup \{n+1\}\big)'$ for certain subsets E and F of $\{1, \dots, n\}$. Note that, due to Lemma 4, $\tau_{x,y}$ and $\xi_{x,y,z}$ are both elements of $\langle \mathcal{S}_{n+1}, \xi_{1,2,3} \rangle$ for all pairwise distinct $x, y, z \in \{1, \dots, n\}$. Then taking to account Proposition 7, we obtain that $\mathcal{IT}_n \subseteq \langle \mathcal{S}_{n+1}, \xi_{1,2,3} \rangle$. In particular, $0 \in \langle \mathcal{S}_{n+1}, \xi_{1,2,3} \rangle$. Thus, without loss of generality we may assume that $a \neq 0$, which implies $b \neq 0$. Suppose that all the blocks of b, except $\big(E \cup \{n+1\}\big) \bigcup \big(F \cup \{n+1\}\big)'$, are precisely $E_i \cup F_i'$, $1 \leq i \leq k$. By the proposition of induction and Lemma 6, we obtain that

$$c = \Big\{ \big(E \cup E_1\big) \cup \big(F \cup F_1\big)', E_2 \cup {F_2}', \dots, E_k \cup {F_k}', \{n+1, (n+1)'\} \Big\} \quad (14)$$

is an element of $\langle \mathcal{S}_{n+1}, \xi_{1,2,3} \rangle$. We have four possibilities.

Case 1. $E = \varnothing$ and $F = \varnothing$. Then $b = c \in \langle \mathcal{S}_{n+1}, \xi_{1,2,3} \rangle$.

Case 2. $E = \varnothing$ and $F = \{f_1, \dots, f_m\} \neq \varnothing$. Fix an element $f \in F_1$. Then $b = c \cdot \prod\limits_{i=1}^{m} \xi_{f, f_i, n+1}$ and so $b \in \langle \mathcal{S}_{n+1}, \xi_{1,2,3} \rangle$.

Case 3. $E = \{e_1, \dots, e_l\} \neq \varnothing$ and $F = \varnothing$. Fix an element $e \in E_1$. Then $b = \prod\limits_{i=1}^{l} \xi_{n+1, f_i, e} \cdot c$, whence $b \in \langle \mathcal{S}_{n+1}, \xi_{1,2,3} \rangle$.

Case 4. $E \neq \varnothing$ and $F \neq \varnothing$. Put $d = \big\{ E \cup F', E_1 \cup {F_1}', \dots, E_k \cup {F_k}', \{n+1, (n+1)'\} \big\}$. Due to proposition of induction and Lemma 6, we have that $d \in \langle \mathcal{S}_{n+1}, \xi_{1,2,3} \rangle$. Then $b = \tau_{E \cup \{n+1\}} d \tau_{F \cup \{n+1\}} \in \langle \mathcal{S}_{n+1}, \xi_{1,2,3} \rangle$.

In all these cases we obtained that b belongs to $\langle \mathcal{S}_{n+1}, \xi_{1,2,3} \rangle$ and so does a.

Thus, we have just proved that $\mathcal{IP}_n = \langle \mathcal{S}_n, \xi_{1,2,3} \rangle$ for all $n \geq 3$. This implies that $\mathcal{IP}_n = \langle \mathcal{S}_n, u \rangle$ for all $u \in \mathcal{S}_n \xi_{1,2,3} \mathcal{S}_n$. Conversely, suppose that $\mathcal{IP}_n = \langle \mathcal{S}_n, u \rangle$ for some $u \in \mathcal{IP}_n$. Then, due to (6), we obtain that $\mathrm{rank}(u) = n - 1$. Now taking to account Lemmas 5 and 4, we have that either $u \in \mathcal{S}_n \xi_{1,2,3} \mathcal{S}_n$ or $u \in \mathcal{S}_n \tau_{1,2} \mathcal{S}_n$. But $u \in \mathcal{S}_n \tau_{1,2} \mathcal{S}_n$ is impossible. Indeed, otherwise we would have $\langle \mathcal{S}_n, \xi_{1,2,3} \rangle = \mathcal{IT}_n$ and it remains to note that, due to 3) of Proposition 7, $\xi_{1,2,3} \notin \mathcal{IT}_n$ when $n \geq 3$. Hence, $u \in \mathcal{S}_n \xi_{1,2,3} \mathcal{S}_n$ holds, as was required. This completes the proof. □

Let $k \in \mathbb{N}$, $k \leq n$. Set $I_k = \{a \in \mathcal{IP}_n : \operatorname{rank}(a) \leq k\}$. Note that

$$\{0\} = I_1 \subset I_2 \subset \ldots \subset I_n = \mathcal{IP}_n. \tag{15}$$

We will prove in the following proposition that these sets exhaust all the double-sided ideals (or just ideals) of $\mathcal{IP}_n$.

Proposition 9. *Let I be an ideal of $\mathcal{IP}_n$ and $k \in \mathbb{N}$ such that $k \leq n$. Then*

1. *for all $b \in \mathcal{IP}_n$, $I_k = \mathcal{IP}_n b \mathcal{IP}_n$ if and only if $\operatorname{rank}(b) = k$;*
2. *$I = I_m$ for some $m \in \mathbb{N}$, $m \leq n$;*
3. *$I = \mathcal{IP}_n a \mathcal{IP}_n$ for certain $a \in \mathcal{IP}_n$.*

Proof. Let us prove first that 1) holds. Take $b \in \mathcal{IP}_n$. Suppose that $I_k = \mathcal{IP}_n b \mathcal{IP}_n$. Then due to (6), we obtain that $\operatorname{rank}(b) \geq k$. From the other hand, $b = 1 \cdot b \cdot 1 \in I_k$ and so $\operatorname{rank}(b) \leq k$. Thus, $\operatorname{rank}(b) = k$. Conversely, suppose that $\operatorname{rank}(b) = k$. Then $b = \left(A_i \cup B_i'\right)_{1 \leq i \leq k}$ for some partitions $\{1, \ldots, n\} = \dot{\bigcup\limits_{1 \leq i \leq k}} A_i$ and $\{1, \ldots, n\} = \dot{\bigcup\limits_{1 \leq i \leq k}} B_i$. Take $c \in I_k$ and let $\operatorname{rank}(c) = m \leq k$. Since

$$d = b\tau_{B_1 \cup \ldots \cup B_{k+1-m}} = \Big\{ \big(A_1 \cup \ldots \cup A_{k+1-m}\big) \cup \big(B_1 \cup \ldots \cup B_{k+1-m}\big)', \\ A_{k+2-m} \cup B'_{k+2-m}, \ldots, A_k \cup B'_k \Big\} \tag{16}$$

is an element of the rank m, then due to 4) of Theorem 2, we obtain that there are $u, v \in \mathcal{IP}_n$ such that $c = udv = ub\tau_{B_1 \cup \ldots \cup B_{k+1-m}} v \in \mathcal{IP}_n b \mathcal{IP}_n$. Thus, $I_k = \mathcal{IP}_n b \mathcal{IP}_n$ and the proof of 1) is complete.

Let now a be an arbitrary element of I such that $\operatorname{rank}(a)$ has the maximum value among the numbers $\operatorname{rank}(x)$, $x \in I$. Then due to the statement 1), condition (15) and the fact that $I = \bigcup\limits_{x \in I} \mathcal{IP}_n x \mathcal{IP}_n$, we have that $I = I_{\operatorname{rank}(a)} = \mathcal{IP}_n a \mathcal{IP}_n$. Thus, statements 2) and 3) hold. □

As a corollary we obtain now the following proposition.

Proposition 10. *All the ideals of $\mathcal{IP}_n$ are principal and form the chain* (15).

Proof. Follows from Proposition 9. □

Set $\mathcal{D}_k = \{a \in \mathcal{IP}_n : \operatorname{rank}(a) = k\}$ for all $k \in \mathbb{N}$, $1 \leq k \leq n$. Due to 4) of Theorem 2, we have that all these sets exhaust all the $\mathcal{D}$-classes of $\mathcal{IP}_n$. Now we are able to formulate a result on the structure of maximal subsemigroups of $\mathcal{IP}_n$.

Theorem 3. *Let $n \geq 3$ and S be a subset of $\mathcal{IP}_n$. Then the following statements are equivalent:*

1. *S is a maximal subsemigroup of $\mathcal{IP}_n$;*

2. *either $S = \mathcal{IT}_n \cup I_{n-2}$ or $S = G \cup I_{n-1}$ for some maximal subgroup G of $\mathcal{S}_n$.*

In addition, every maximal subsemigroup of $\mathcal{IP}_n$ is an inverse subsemigroup of $\mathcal{IP}_n$.

Proof. Let us prove first that 2) implies 1). If S coincides with the subsemigroup $G \cup I_{n-1}$ of $\mathcal{IP}_n$ for some maximal subgroup G of $\mathcal{S}_n$ then since the condition (15), we have that S is a maximal subsemigroup of $\mathcal{IP}_n$. Note that $\mathcal{IT}_n \cup I_{n-2}$ is a subsemigroup of $\mathcal{IP}_n$, as $\mathcal{IT}_n$ is a subsemigroup of $\mathcal{IP}_n$ and I_{n-2} is an ideal of $\mathcal{IP}_n$. If now $\mathcal{IT}_n \cup I_{n-2}$ is a proper subsemigroup of T, where T is a subsemigroup of $\mathcal{IP}_n$, then, due to Lemma 5, T contains an element of $\mathcal{S}_n \xi_{1,2,3} \mathcal{S}_n$ and so, taking to account Proposition 8 and the fact that $\mathcal{S}_n \subseteq \mathcal{IT}_n$, we obtain that $T = \mathcal{IP}_n$. Thus, 2) implies 1).

Let now S be a maximal subsemigroup in $\mathcal{IP}_n$. Note that $S \cup I_{n-2}$ is a subsemigroup of $\mathcal{IP}_n$. Besides, $S \cup I_{n-2}$ is a proper subset of $\mathcal{IP}_n$. Indeed, otherwise we would have $S \cup I_{n-2} = \mathcal{IP}_n$, whence $\mathcal{S}_n \cup \mathcal{D}_{n-1} \subseteq S$ and so due to Proposition 8, we would obtain that $S = \mathcal{IP}_n$. Thus, $S \cup I_{n-2} = S$ and so $I_{n-2} \subseteq S$. Since $S \cup \{1\}$ is a proper subsemigroup of $\mathcal{IP}_n$, we have that $S = S \cup \{1\}$ and $G = S \cap \mathcal{S}_n \neq \varnothing$. Obviously, G is a subgroup of $\mathcal{S}_n$. Now we have two possibilities.

Case 1. G is a proper subgroup of $\mathcal{S}_n$. Then $S \subseteq G \cup I_{n-1}$ and due to the fact that $G \cup I_{n-1}$ is a proper subsemigroup of $\mathcal{IP}_n$, we obtain that $S = G \cup I_{n-1}$. It remains to note that the latter implies that G is a maximal subgroup of $\mathcal{S}_n$.

Case 2. $G = \mathcal{S}_n$. Then $\mathcal{S}_n \cup I_{n-2} \subseteq S$. Since $\mathcal{S}_n \cup I_{n-2}$ is a proper subsemigroup of $\mathcal{IT}_n \cup I_{n-2}$, we have that S contains an element a of $\mathcal{D}_{n-1}$. Then due to Lemma 5 and Proposition 8, we obtain that $S \subseteq \mathcal{IT}_n \cup I_{n-2}$. But $\mathcal{IT}_n \cup I_{n-2}$ is a maximal subsemigroup of $\mathcal{IP}_n$ and so $S = \mathcal{IT}_n \cup I_{n-2}$. This completes the proof of that 1) implies 2).

That every maximal subsemigroup of $\mathcal{IP}_n$ is an inverse subsemigroup of $\mathcal{IP}_n$, follows from what we already have done and the fact that $\mathcal{IT}_n \cup I_{n-2}$ and $G \cup I_{n-1}$ are inverse subsemigroups of $\mathcal{IP}_n$ for all subgroups G of $\mathcal{S}_n$. $\square$

6 Automorphism group $\mathrm{Aut}(\mathcal{IP}_X)$

Let $g \in \mathcal{S}_X$. Denote by φ_g the map from $\mathcal{IP}_X$ to $\mathcal{IP}_X$, given by

$$\varphi_g(a) = g^{-1}ag \text{ for every } a \in \mathcal{IP}_X. \tag{17}$$

Clearly, φ_g belongs to $\mathrm{Aut}(\mathcal{IP}_X)$, automorphism group of $\mathcal{IP}_X$. Throughout this section, denote by id the identity map of the set X to itself.

The main result of this section is the following theorem.

Theorem 4. *Let $\varphi \in \mathrm{Aut}(\mathcal{IP}_X)$. Then $\varphi = \varphi_g$ for some $g \in \mathcal{S}_X$. In particular,* $\mathrm{Aut}(\mathcal{IP}_X) \cong \mathcal{S}_X$ *when* $|X| \neq 2$ *and* $\mathrm{Aut}(\mathcal{IP}_2) = \{\mathrm{id}\}$.

We will divide the proof of this theorem into few lemmas.

Naturally, φ induces an automorphism $\chi = \varphi|_{E(\mathcal{IP}_X)}$ of the semilattice $E(\mathcal{IP}_X)$. Set $\zeta_x = \tau_{X\setminus\{x\}}$ for all $x \in X$. Set also $\Phi = \big\{\zeta_x \in \mathcal{IP}_X : x \in X\big\}$. Recall that if $(E, \leq)$ is a semilattice with the zero element 0, then an element f of E is said to be *primitive* if $g \leq f$ implies either $g = f$ or $g = 0$, for all $g \in E$. For all $n \geq 2$ set

$$\begin{aligned}\Theta^n_{\max} &= \big\{\tau_{i,j} \in \mathcal{IP}_n : i, j \in \{1, \ldots, n\},\ i \neq j\big\} \text{ and}\\ \Theta^n_{\mathrm{pr}} &= \big\{\tau_F\tau_{\{1,\ldots,n\}\setminus F} \in \mathcal{IP}_n : F \text{ is a proper subset of } \{1, \ldots, n\}\big\} =\\ &\qquad \mathcal{D}_2 \cap E(\mathcal{IP}_n).\end{aligned} \tag{18}$$

Notice that $\Phi \subseteq \Theta^n_{\mathrm{pr}}$.

Lemma 7. *Let $n \geq 2$. Then the set of all primitive elements of the semilattice $E(\mathcal{IP}_n)$ coincides with Θ^n_{pr}. Also then the set of all maximal elements of the semilattice $E(\mathcal{IP}_n) \setminus \{1\}$ coincides with $\Theta^n_{\max}$.*

Proof. Follows from Proposition 4. □

Lemma 8. *Take $\theta \in \mathrm{Aut}\big(E(\mathcal{IP}_n)\big)$. Then there is $g \in \mathcal{S}_n$ such that $\theta(e) = g^{-1}eg$ for all $e \in E(\mathcal{IP}_n)$.*

Proof. Clearly, the statement holds when $n = 1$. Thus, let us assume that $n \geq 2$.

Obviously, $\theta(1) = 1$. Then $\theta\big(E(\mathcal{IP}_n) \setminus \{1\}\big) = E(\mathcal{IP}_n) \setminus \{1\}$. Hence, due to Lemma 7, we obtain that $\theta\big(\Theta^n_{\max}\big) = \Theta^n_{\max}$ and $\theta\big(\Theta^n_{\mathrm{pr}}\big) = \Theta^n_{\mathrm{pr}}$. Take $f = \tau_F\tau_{\{1,\ldots,n\}\setminus F} \in \Theta^n_{\mathrm{pr}}$. Set $\Lambda_f = \big\{a \in \Theta^n_{\max} : fa = f\big\}$. Then $\theta(\Lambda_f) = \Lambda_{\theta(f)}$. If $f \notin \Phi$ then $2 \leq |F| \leq n - 2$. Thus,

$$|\Lambda_f| = \binom{|F|}{2} + \binom{n - |F|}{2}, \text{ if } f \notin \Phi. \tag{19}$$

Otherwise, we have the following:

$$|\Lambda_f| = \binom{n-1}{2}, \text{ if } f \in \Phi. \tag{20}$$

Let us prove now that for all $n \geq 4$ and for all k, $2 \leq k \leq n-2$, the following holds:

$$\binom{k}{2} + \binom{n-k}{2} < \binom{n-1}{2}. \tag{21}$$

Indeed, the inequality

$$k(k-n) = (k^2-1) + 1 - kn = (k-1)(k+1) + 1 - kn < (k-1)n + 1 - kn = 1 - n \text{ implies} \tag{22}$$

$$\binom{k}{2} + \binom{n-k}{2} = \frac{1}{2}\big(k(k-1) + (n-k)(n-k-1)\big) = \frac{1}{2}\big(2k^2 - 2kn + n^2 - n\big) = k(k-n) + \frac{1}{2}\big(n^2-n\big) < \frac{1}{2}\big(n^2-n\big) + 1 - n = \frac{1}{2}(n-1)(n-2) = \binom{n-1}{2}. \tag{23}$$

Now due to (19), (20), (21) and the equality $\theta(\Lambda_f) = \Lambda_{\theta(f)}$, we obtain that $\theta(\Phi) = \Phi$. Then there is an element g of $\mathcal{S}_n$ such that $\theta(\zeta_x) = \zeta_{g(x)}$ for all $x \in \{1, \ldots, n\}$.

Take now distinct x and y of $\{1, \ldots, n\}$. Since $\zeta_x \tau_{x,y} = 0$ and $\zeta_y \tau_{x,y} = 0$, we have that $\zeta_{g(x)}\theta(\tau_{x,y}) = 0$ and $\zeta_{g(y)}\theta(\tau_{x,y}) = 0$. The latter, taking to account $\theta\big(\Theta^n_{\max}\big) = \Theta^n_{\max}$, implies that $\theta(\tau_{x,y}) = \tau_{g(x),g(y)} = g^{-1}\tau_{x,y}g$.

Let now $e = (E_i \cup E_i')_{i \in I}$ be a nonidentity idempotent element of $E(\mathcal{IP}_n)$. Then

$$e = \prod\big\{\tau_{x,y} :\ x \neq y,\ \{x,y\} \subseteq E_i \text{ for some } i \in I\big\} \tag{24}$$

implies

$$\theta(e) = \prod\big\{\tau_{g(x),g(y)} :\ x \neq y,\ \{x,y\} \subseteq E_i \text{ for some } i \in I\big\} = \prod\big\{g^{-1}\tau_{x,y}g :\ x \neq y,\ \{x,y\} \subseteq E_i \text{ for some } i \in I\big\} = g^{-1}eg. \tag{25}$$

This completes the proof. □

Take distinct x and y of X. Define an element $\varepsilon_{x,y}$ of $\mathcal{S}_X$ as follows:

$$\varepsilon_{x,y}(x) = y,\ \varepsilon_{x,y}(y) = x \text{ and } \varepsilon_{x,y}(t) = t \text{ for all } t \in X \setminus \{x,y\}. \tag{26}$$

Corollary 1. *Let $|X|=6$. Then there is $g \in \mathcal{S}_6$ such that $\varphi(h)=\varphi_g(h)$ for all $h \in \mathcal{S}_6$.*

Proof. If we put $\chi=\theta$ and $n=6$ in the statement of Lemma 8, we will obtain that there is $g \in \mathcal{S}_6$ such that $\chi(e)=g^{-1}eg$ for all $e \in E(\mathcal{IP}_6)$. Take distinct x and y of $\{1,\dots,6\}$. Then

$$g^{-1}\tau_{x,y}g=\varphi(\tau_{x,y})=\varphi(\tau_{x,y}\varepsilon_{x,y})=g^{-1}\tau_{x,y}g\varphi(\varepsilon_{x,y}), \tag{27}$$

whence

$$\tau_{x,y}=\tau_{x,y}g\varphi(\varepsilon_{x,y})g^{-1}. \tag{28}$$

The latter implies that either $g\varphi(\varepsilon_{x,y})g^{-1}=1$ or $g\varphi(\varepsilon_{x,y})g^{-1}=\varepsilon_{x,y}$. But since the order of $g\varphi(\varepsilon_{x,y})g^{-1}$ equals 2, we have that $g\varphi(\varepsilon_{x,y})g^{-1}=\varepsilon_{x,y}$, which is equivalent to $\varphi(\varepsilon_{x,y})=g^{-1}\varepsilon_{x,y}g$. Now, taking to account the known fact that $\langle \varepsilon_{x,y}:\ x\neq y\rangle=\mathcal{S}_n$ (see [11]), we obtain that $\varphi(h)=\varphi_g(h)$ for all $h\in\mathcal{S}_6$. ◻

Lemma 9. *There is $g\in\mathcal{S}_X$ such that $\varphi(h)=\varphi_g(h)$ for all $h\in\mathcal{S}_X$.*

Proof. Due to Corollary 1, we have that the statement holds when $|X|=6$. Assume now that $|X|\neq 6$.

Since φ preserves the set of all invertible elements of $\mathcal{IP}_X$, we have, due to Proposition 3, that $\varphi(\mathcal{S}_X)=\mathcal{S}_X$. Hence, φ induces an automorphism of $\mathcal{S}_X$. Then due to known fact, which claims that if $|X|\neq 6$ then every automorphism of $\mathcal{S}_X$ is inner (see [11]), we have that there is $g\in\mathcal{S}_X$ such that $\varphi(h)=g^{-1}hg=\varphi_g(h)$ for all $h\in\mathcal{S}_X$. This completes the proof. ◻

Set now $\psi=\varphi\varphi_g$. Then ψ is, obviously, an automorphism of $\mathcal{IP}_X$ and, due to Lemma 9, $\psi|_{\mathcal{S}_X}$ is the identity map of $\mathcal{S}_X$ to itself. For all $M\subseteq X$ set

$$\widetilde{\mathcal{S}}_M=\big\{h\in\mathcal{S}_X:\ h(x)=x \text{ for all } x\in X\setminus M\big\}. \tag{29}$$

For all $a\in\mathcal{IP}_X$ set

$$\mathrm{Fix}_l(a)=\big\{h\in\mathcal{S}_X:\ ha=a\big\} \text{ and } \mathrm{Fix}_r(a)=\big\{h\in\mathcal{S}_X:\ ah=a\big\}. \tag{30}$$

Lemma 10. *Let $a\in\mathcal{IP}_X$. Let also $X=\dot{\bigcup\limits_{i\in I}}A_i=\dot{\bigcup\limits_{i\in I}}B_i$. Then*

1. $\mathrm{Fix}_l(a)=\bigoplus\limits_{i\in I}\widetilde{\mathcal{S}}_{A_i}$ *if and only if* $a=\big(A_i\cup U_i'\big)_{i\in I}$ *for some partition* $X=\dot{\bigcup\limits_{i\in I}}U_i$*;*

2. $\mathrm{Fix}_r(a) = \bigoplus_{i\in I} \widetilde{\mathcal{S}}_{B_i}$ *if and only if* $a = \big(V_i \cup B_i'\big)_{i\in I}$ *for some partition* $X = \dot{\bigcup_{i\in I}} V_i$.

Proof. Straightforward. □

Corollary 2. *$a\mathcal{H}\psi(a)$ for all $a \in \mathcal{IP}_X$. In particular, $\psi(e) = e$ for all $e \in E(\mathcal{IP}_X)$.*

Proof. That $a\mathcal{H}\psi(a)$ for all $a \in \mathcal{IP}_X$ follows from Lemma 10 and Theorem 2. Then $\psi(e) = e$ for all $e \in E(\mathcal{IP}_X)$, due to the fact that every $\mathcal{H}$-class of an arbitrary semigroup contains at most one idempotent (see Corollary 2.2.6 from [7]). □

Lemma 11. *Let $a \in \mathcal{IP}_X$ and* $\mathrm{rank}(a) \geq 3$*. Then $\psi(a) = a$.*

Proof. Let $a = \big(A_i \cup B_i'\big)_{i\in I}$, $\mid I \mid \geq 3$. Due to Corollary 2, we have that $a\mathcal{H}\psi(a)$ and so $\psi(a) = \big(A_i \cup B_{\alpha(i)}'\big)_{i\in I}$ for some bijective map $\alpha : I \to I$. Due to Corollary 2, we also have that $ea\mathcal{H}e\psi(a)$ for all $e \in E(\mathcal{IP}_X)$.

Take arbitrary distinct i and j of I. Since $\tau_{A_i\cup A_j} a\mathcal{H}\tau_{A_i\cup A_j}\psi(a)$, we have that

$$\Big\{\big(A_i \cup A_j\big) \cup \big(B_i \cup B_j\big)', \big(A_l \cup B_l'\big)_{l\in I\setminus\{i,j\}}\Big\} \text{ and }$$
$$\Big\{\big(A_i \cup A_j\big) \cup \big(B_{\alpha(i)} \cup B_{\alpha(j)}\big)', \big(A_l \cup B_l'\big)_{l\in I\setminus\{i,j\}}\Big\} \quad (31)$$

are $\mathcal{H}$-equivalent, whence $\{i, j\} = \{\alpha(i), \alpha(j)\}$. Let now $k \in I$. Then $\alpha(k) = k$. Suppose the contrary. Then $\{k, m\} = \{\alpha(k), \alpha(m)\}$ for all $m \in I \setminus \{k\}$ implies that $\alpha(k) = m$ for all $m \in I \setminus \{k\}$. But $\mid I \mid\geq 3$ and we get a contradiction. Thus, α is an identity map of I to itself, which is equivalent to $\psi(a) = a$. This completes the proof. □

Note that since $\mathcal{IP}_1$ is isomorphic to the unit group and since $\mathcal{IP}_2 \cong \mathbb{Z}_2^0$, where $\mathbb{Z}_2^0$ denotes the group $\mathbb{Z}_2$ with adjoint zero, we have that $\mathrm{Aut}(\mathcal{IP}_X) = \{\mathrm{id}\}$ when $|X| \leq 2$.

Lemma 12. *Let $a \in \mathcal{IP}_X$ and* $\mathrm{rank}(a) \leq 2$*. Then $\psi(a) = a$.*

Proof. If $\mathrm{rank}(a) = 1$ then $a = 0$ and, obviously, $\psi(a) = a$. So let us suppose that $\mathrm{rank}(a) = 2$. Assume that $a = \big\{A \cup B', C \cup D'\big\}$. Fix $x \in A$ and $y \in B$.

Suppose that $|A| \geq 2$ and $|B| \geq 2$. Then $\psi(a) = a$. Indeed, we have that $A \setminus \{x\} \neq \varnothing$ and $B \setminus \{y\} \neq \varnothing$, so if $y_1 \in B \setminus \{y\}$ then we can consider the equality $a = \big\{\{x, y'\}, \big(A \setminus \{x\}\big) \bigcup \big(B \setminus \{y\}\big)', C \cup D'\big\} \cdot \tau_{y,y_1}$, whence, due to Corollary 2 and Lemma 11, we will have that $\psi(a) = a$.

Analogously, if $|C| \geq 2$ and $|D| \geq 2$ then $\psi(a) = a$.

Thus, we may assume that either $|A| = 1$ or $|B| = 1$, and that either $|C| = 1$ or $|D| = 1$. Without loss of generality we may suppose that $|A| = 1$. Then we will have two possibilities.

Case 1. $|C| = 1$. Then $|X| = 2$ and we obtain $\psi = \mathrm{id}$.

Case 2. $|D| = 1$. Then $\psi(a) = a$. Suppose the contrary. Then we would obtain that $\psi(a) = \{A \cup D', C \cup B'\} = \zeta_x h$ for some $h \in \mathcal{S}_X$. But $\zeta_x h = \psi(\zeta_x h)$ and so $a = \zeta_x h$, whence $B = D$, which leads to a contradiction.

Thus, we proved that $\psi(a) = a$, which was required. □

As a consequence of that we have from Lemmas 11 and 12, we have that $\psi = \mathrm{id}$, whence $\varphi = \varphi_g$. It remains to prove that $\mathrm{Aut}(\mathcal{IP}_X) \cong \mathcal{S}_X$ when $|X| \neq 2$. This follows from the following lemma.

Lemma 13. *Suppose that $|X| \geq 3$. Then a map $\vartheta : \mathcal{S}_X \to \mathrm{Aut}(\mathcal{IP}_X)$, given by*

$$\vartheta(h) = \varphi_h \text{ for all } h \in \mathcal{S}_X, \tag{32}$$

is an isomorphism from $\mathcal{S}_X$ onto $\mathrm{Aut}(\mathcal{IP}_X)$.

Proof. We have already proved that ϑ is an onto homomorphism from $\mathcal{S}_X$ to $\mathrm{Aut}(\mathcal{IP}_X)$. But, besides, ϑ is an injective map. Indeed, $\vartheta(h_1) = \vartheta(h_2)$ implies that $h_1^{-1} h h_1 = h_2^{-1} h h_2$ or just that $(h_1 h_2^{-1})^{-1} h (h_1 h_2^{-1}) = h$ for all $h \in \mathcal{S}_X$ and it remains to note that $\mathcal{S}_X$ is a center-free group when $|X| \geq 3$ (see [11]). Thus, ϑ is an isomorphism. □

The proof of theorem is complete.

7 Connections between $\mathcal{IP}_X$ and other semigroups

Set $\Upsilon = \{X, X'\}$. Then $\Upsilon \in \mathcal{CS}_X$. The following proposition shows that $\mathcal{IP}_X \cup \{\Upsilon\}$ is a maximal inverse subsemigroup of $\mathcal{CS}_X$ when $|X| \geq 2$.

Proposition 11. *Let $|X| \geq 2$. Then $\mathcal{IP}_X \cup \{\Upsilon\}$ is a maximal inverse subsemigroup of $\mathcal{CS}_X$.*

Proof. Since $\mathcal{IP}_X$ is an inverse subsemigroup of $\mathcal{CS}_X$ and $a\Upsilon = \Upsilon a = \Upsilon$ for all $a \in \mathcal{IP}_X \cup \{\Upsilon\}$, we obtain that $\mathcal{IP}_X \cup \{\Upsilon\}$ is a proper inverse subsemigroup of $\mathcal{CS}_X$.

Suppose now that S is an inverse subsemigroup of $\mathcal{CS}_X$ such that $\mathcal{IP}_X \cup \{\Upsilon\}$ is a subsemigroup of S. Take $s \in S \setminus \mathcal{IP}_X$. Then there is a nonempty subset A of X such that either s contains a block A or s contains a block A'. Without loss of generality we may assume that s contains the block A. Let t be the inverse of s in S. Then st is an idempotent in S and so, due to the fact that idempotents of inverse semigroup commute, we obtain that $u = st \cdot \Upsilon = \Upsilon \cdot st$. The latter implies that u contains both blocks A and X, whence $A = X$. Then s is an idempotent and due to equalities $s = \Upsilon s$ and $\Upsilon s = s\Upsilon$, we have that s contains the block X' and so $s = \Upsilon$. That is, $S = \mathcal{IP}_X \cup \{\Upsilon\}$. This implies that $\mathcal{IP}_X \cup \{\Upsilon\}$ is a maximal inverse subsemigroup of $\mathcal{CS}_X$ which was required. □

Denote by $\mathcal{IS}_X$ the *symmetric inverse* semigroup on the set X. Let $s \in \mathcal{IS}_X$. Denote by $\mathrm{dom}(s)$ and $\mathrm{ran}(s)$ the *domain* and the *range* of s respectively. The following theorem shows how one can embed the symmetric inverse semigroup into the inverse partition one.

Theorem 5. *Let $\overline{x} \notin X$. Then $\mathcal{IS}_X$ isomorphically embeds into $\mathcal{IP}_{X\cup\{\overline{x}\}}$.*

Proof. For all $s \in \mathcal{IS}_X$, set

$$\Omega_s = \big(X \cup \{\overline{x}\} \setminus \mathrm{dom}(s)\big) \bigcup \big(X \cup \{\overline{x}\} \setminus \mathrm{ran}(s)\big)'. \tag{33}$$

Set a map $\kappa : \mathcal{IS}_X \to \mathcal{IP}_{X\cup\{\overline{x}\}}$ as follows:

$$\kappa(s) = \big\{\Omega_s, \big(\{x, s(x)'\}\big)_{x\in\mathrm{dom}(s)}\big\} \text{ for all } s \in \mathcal{IS}_X. \tag{34}$$

Take an arbitrary s of $\mathcal{IS}_X$. Then we have the following condition:

$$x \equiv_{\kappa(s)} \overline{x} \equiv_{\kappa(s)} \overline{x}' \equiv_{\kappa(s)} y' \text{ for all } x \in X \setminus \mathrm{dom}(s) \text{ and } y \in X \setminus \mathrm{ran}(s). \tag{35}$$

Take $s, t \in \mathcal{IS}_X$. Then due to (33) and (35), we obtain that

$$x \equiv_{\kappa(s)\kappa(t)} \overline{x} \equiv_{\kappa(s)\kappa(t)} \overline{x}' \equiv_{\kappa(s)\kappa(t)} y' \text{ for all } x, y \in X \text{ such that } x \notin s^{-1}\big(\mathrm{ran}(s) \cap \mathrm{dom}(t)\big) \text{ and } y \notin t\big(\mathrm{dom}(t) \cap \mathrm{ran}(s)\big). \tag{36}$$

Notice that

$$s^{-1}\big(\mathrm{ran}(s) \cap \mathrm{dom}(t)\big) = \mathrm{dom}(st) \text{ and } t\big(\mathrm{dom}(t) \cap \mathrm{ran}(s)\big) = \mathrm{ran}(st). \tag{37}$$

If now $x \in \mathrm{dom}(st)$ then $x \equiv_{\kappa(s)} s(x)'$ and $s(x) \equiv_{\kappa(t)} st(x)'$, whence

$$x \equiv_{\kappa(s)\kappa(t)} st(x)' \text{ for all } x \in \mathrm{dom}(st). \tag{38}$$

The conditions (36), (37) and (38) imply that

$$\kappa(s)\kappa(t) = \big\{\Omega_{st}, \big(\{x, st(x)'\}\big)_{x\in\mathrm{dom}(st)}\big\} = \kappa(st). \tag{39}$$

Thus, κ is a homomorphism from $\mathcal{IS}_X$ to $\mathcal{IP}_{X\cup\{\overline{x}\}}$. It remains to prove that κ is an injective map.

Suppose that $\kappa(s) = \kappa(t)$ for some $s, t \in \mathcal{IS}_X$. Then it follows from (34) that $\mathrm{dom}(s) \subseteq \mathrm{dom}(t)$ and $\mathrm{dom}(t) \subseteq \mathrm{dom}(s)$, whence $\mathrm{dom}(s) = \mathrm{dom}(t)$. Then (34) implies that $s(x) = t(x)$ for all $x \in \mathrm{dom}(s) = \mathrm{dom}(t)$. Hence, $s = t$ and so κ is injective. The proof is complete. $\square$

It follows immediately from Theorem 5 that $\mathcal{IS}_n$ embeds into $\mathcal{IP}_{n+1}$ for all $n \in \mathbb{N}$. Surprisingly, the following theorem shows that one can not construct an embedding map from $\mathcal{IS}_n$ to $\mathcal{IP}_n$.

Theorem 6. *Let $n \in \mathbb{N}$. There is no an injective homomorphism from $\mathcal{IS}_n$ to $\mathcal{IP}_n$.*

Proof. Suppose the contrary. Then there is a subsemigroup U of $\mathcal{IP}_n$ such that $U \cong \mathcal{IS}_n$. Then we have that U is a regular subsemigroup of $\mathcal{IP}_n$, whence, due to Proposition 2.4.2 from [7], we obtain that $\mathcal{D}^U = \mathcal{D} \cap (U \times U)$, where $\mathcal{D}^U$ denotes the Green's $\mathcal{D}$-relation on U. Note that $\mathcal{IP}_n$ contains exactly n different $\mathcal{D}$-classes. This implies that U contains at most n different $\mathcal{D}^U$-classes. But since $U \cong \mathcal{IS}_n$, we have that U contains exactly $n+1$ different $\mathcal{D}^U$-classes. We get a contradiction. This completes the proof. $\square$

8 $\mathcal{IP}_n$ embeds into $\mathcal{IS}_{2^n-2}$

Let S be an inverse semigroup with the natural partial order $\leq$ on it. For $A \subseteq S$ denote by $[A]$ the order ideal of S with respect to $\leq$, i.e., $[A] = \big\{b : \; a \leq b \text{ for some } a \in A\big\}$. Let also H be a *closed inverse subsemigroup* of S, i.e., H is an inverse subsemigroup of S and $[H] = H$ (see [7]). Recall (see [7]) that one can define the set of all *right $\leq$-cosets* of H as follows:

$$\mathcal{C} = \mathcal{C}_H = \big\{[Hs] : \; ss^{-1} \in H\big\}. \tag{40}$$

Further, one can define the *effective transitive representation* $\phi_H : S \to \mathcal{IS}_{\mathcal{C}}$, given by

$$\phi_H(s) = \big\{\big([Hx], [Hxs]\big) : \; [Hx], [Hxs] \in \mathcal{C}\big\}. \tag{41}$$

Let now K and H be arbitrary closed inverse subsemigroups of S. For a definition of the *equivalence* of representations ϕ_K and ϕ_H, we refer reader to [7]. But we note that due to Proposition IV.4.13 from [21], one has that ϕ_K and ϕ_H are equivalent if and only if there exists $a \in S$ such that $a^{-1}Ha \subseteq K$ and $aKa^{-1} \subseteq H$. We will need the following well-known fact.

Theorem 7 (Proposition 5.8.3 from [7]). *Let H be a closed inverse subsemigroup of an inverse semigroup S and let $a, b \in S$. Then $[Ha] = [Hb]$ if and only if $ab^{-1} \in H$.*

The main result of this section is the following theorem.

Theorem 8. *Let $n \geq 2$. Up to equivalence, there is only one faithful effective transitive representation of $\mathcal{IP}_n$, namely to $\mathcal{IS}_{2^n-2}$. In particular, $\mathcal{IP}_n$ isomorphically embeds into $\mathcal{IS}_{2^n-2}$.*

We divide the proof of this theorem into lemmas. Throughout all further text of this section we suppose that H is a closed inverse subsemigroup of $\mathcal{IP}_n$.

Lemma 14. *$H = [G]$ for some subgroup G of $\mathcal{IP}_n$.*

Proof. Since $\mathcal{IP}_n$ is finite, we have that $E(H)$ contains a zero element. It remains to use Proposition IV.5.5 from [21], which claims that if the set of idempotents of a closed inverse subsemigroup contains a zero element, then this subsemigroup is a closure of some subgroup of the original semigroup. □

Denote by e the identity element of G.

Lemma 15. *If $e = 0$ then ϕ_H is not faithful.*

Proof. We have $G = \{0\}$, whence $H = [0] = \mathcal{IP}_n$ and so $[Hx] \supseteq [0] = \mathcal{IP}_n$ for all $x \in \mathcal{IP}_n$. Thus, $[Hx] = \mathcal{IP}_n$ for all $x \in \mathcal{IP}_n$. Then $|\,\phi_H(\mathcal{IP}_n)\,| = 1$, whence we obtain that ϕ_H is not faithful. □

Lemma 16. *Let $\operatorname{rank}(e) \geq 3$. Then ϕ_H is not faithful.*

Proof. Take $b \in \mathcal{D}_2$. Since $bb^{-1} \in \mathcal{D}_2$, we have that $bb^{-1} \notin H$ and so $[Hb] \notin \mathcal{C}$. The latter gives us that $\phi_H(b)$ equals the zero element of $\mathcal{IS}_{\mathcal{C}}$. Then, due to $|\,\mathcal{D}_2\,| \geq 2$, we obtain that ϕ_H is not faithful. □

Lemma 17. *Let $\operatorname{rank}(e) = 2$ and $G \cong \mathbb{Z}_2$. Then ϕ_H is not faithful.*

Proof. Let $G = \{e, q\}$. We are going to prove that $\phi_H(e) = \phi_H(q)$.

Let us prove first that $\operatorname{dom}\big(\phi_H(e)\big) = \operatorname{dom}\big(\phi_H(q)\big)$. Indeed, take $[Hx] \in \mathcal{C}$. Then, due to the equality $(xe)(xe)^{-1} = xex^{-1} = xqq^{-1}x^{-1} = (xq)(xq)^{-1}$, we obtain that $[Hxe] \in \mathcal{C}$ if and only if $(xe)(xe)^{-1} \in H$ if and only if $(xq)(xq)^{-1} \in H$ if and only if $[Hxq] \in \mathcal{C}$. Thus, $\operatorname{dom}\big(\phi_H(e)\big) = \operatorname{dom}\big(\phi_H(q)\big)$.

Take now $x \in \operatorname{dom}\big(\phi_H(e)\big)$. Then $xex^{-1} \in H = [\{e, q\}]$. But since xex^{-1} is an idempotent and $\operatorname{rank}(xex^{-1}) \leq \operatorname{rank}(e) = 2$, we obtain, taking to account Proposition 4, that $xex^{-1} = e$. Hence, $(xe)(xe)^{-1} = ee^{-1}$ and so, due to Proposition 2.4.1 from [7], we obtain that $xe\mathcal{R}e$. But then we have that $\mathrm{rank}(xe) = \mathrm{rank}(e)$ and due to $\lambda_{xe} \supseteq \lambda_e$ (which follows, in turn, from (5)), we deduce that $\lambda_{xe} = \lambda_e$, whence due to Theorem 2, we have that $xe\mathcal{L}e$. Thus, $xe\mathcal{H}e$, whence $xe \in G$ and so $xq = xe \cdot q \in G$. But then $(xq)(xe)^{-1} \in G \subseteq H$, whence, due to Theorem 7, we have that $[Hxe] = [Hxq]$. The latter implies that $\phi_H(e)(x) = \phi_H(q)(x)$. Thus, $\phi_H(e) = \phi_H(q)$ and so ϕ is not faithful. □

Lemma 18. *Let $f \in \Theta^n_{\mathrm{pr}}$ and $T = [f]$. Take $[Tx] \in \mathcal{C}_T$. Then $\mathrm{rank}(fx) = 2$ and $[Tx] = [fx]$.*

Proof. Clearly, $[Tx] \in \mathcal{C}_T$ is equivalent to $f \leq xx^{-1}$.

Obviously, $\mathrm{rank}(fx) \leq \mathrm{rank}(f) = 2$. But $\mathrm{rank}(fx) = 1$ is impossible. Indeed, otherwise we would have $fx = 0$, whence $0 = fxx^{-1} = f$, which does not hold. Thus, $\mathrm{rank}(fx) = 2$.

Note that $[fx] \subseteq [Tx]$. It remains to prove that $[Tx] \subseteq [fx]$. Take $t \in T$. Then $f \leq t$ and due to the fact that the natural partial order on an arbitrary inverse semigroup is compatible (see [7]), we obtain that $fx \leq tx$. That is, $tx \in [fx]$. Hence, $Tx \subseteq [fx]$, whence $[Tx] \subseteq \big[[fx]\big] = [fx]$.

The proof is complete. □

Lemma 19. *Let $\mathrm{rank}(e) = 2$ and $G = \{e\}$. Then ϕ_H is faithful.*

Proof. Note that $H = [e]$. Let $e = \tau_E\tau_{E_1}$, where E and E_1 are nonempty subsets of $\{1, \dots, n\}$ such that $\{1, \dots, n\} = E \dot{\bigcup} E_1$. Suppose that $\phi_H(s) = \phi_H(t)$ for some s and t of $\mathcal{IP}_n$. Let A be an arbitrary ρ_t-class. Set $\overline{A} = \{1, \dots, n\} \setminus A$.

Suppose first that $s = 0$. We are going to prove that $t = 0$. Suppose the contrary. We have that $\mathrm{rank}(e \cdot xs) = 1$ for all $x \in \mathcal{IP}_n$ such that $xx^{-1} \in [e]$. So, due to Lemma 18, we obtain that $\mathrm{dom}\big(\phi_H(s)\big) = \varnothing$. Then, again by Lemma 18, we have that $\mathrm{rank}(e \cdot xt) = 1$, or just that $ext = 0$, for all $x \in \mathcal{IP}_n$ such that $xx^{-1} \in [e]$. Put now $u = \big\{E \cup A', E_1 \cup \overline{A}'\big\}$ (note that, due to assumption, $\overline{A} \neq \varnothing$). Then $uu^{-1} = e \in [e]$ and $eut \neq 0$. Thus, we get a contradiction and so $s = 0$ implies $t = 0$. Analogously, $t = 0$ implies $s = 0$.

Assume now that $s \neq 0$, then $t \neq 0$ and so $\overline{A} \neq \varnothing$. Put again $u = \big\{E \cup A', E_1 \cup \overline{A}'\big\}$. Due to Theorem 7 and the equality $\phi_H(s) = \phi_H(t)$, we have that $(xt)(xs)^{-1} \in H$ for all $x \in \mathrm{dom}\big(\phi_H(t)\big)$. Note that $u \in \mathrm{dom}\big(\phi_H(t)\big)$. Indeed, we have $uu^{-1} = e \in [e]$ and since A is a $\rho_{tt^{-1}}$-class, we have that

$$(ut)(ut)^{-1} = utt^{-1}u^{-1} = e \in [e]. \tag{42}$$

This implies that $u \cdot ts^{-1} \cdot u^{-1} \in [e]$. Moreover, since $\mathrm{rank}(uts^{-1}u^{-1}) \leq \mathrm{rank}(u) = 2$, we obtain that $\mathrm{rank}(uts^{-1}u^{-1}) = 2$, whence $(ut)(us)^{-1} =$

$uts^{-1}u^{-1} = e$. In particular, we have that $us \neq 0$. But then A is a union of some ρ_s-classes. Since A was an arbitrary chosen ρ_t-class, we obtain that $\rho_s \subseteq \rho_t$. Analogously, one can prove that $\rho_t \subseteq \rho_s$. Thus, $\rho_s = \rho_t$. Further, if s contains a block $A \cup B'$ then $us = \{E \cup B', E_1 \cup \overline{B}'\}$, where $\overline{B} = \{1, \dots, n\} \setminus B$. But $ut = \{E \cup A', E_1 \cup \overline{A}'\}$ and so

$$\{E \cup E', E_1 \cup E_1'\} = e = (ut)(us)^{-1} = \{E \cup A', E_1 \cup \overline{A}'\} \cdot \{E \cup B', E_1 \cup \overline{B}'\}^{-1} = \{E \cup A', E_1 \cup \overline{A}'\} \cdot \{B \cup E', \overline{B} \cup E_1'\}. \quad (43)$$

This implies $A = B$. Indeed, otherwise we would have $B \subseteq \overline{A}$ and so $A \subseteq \overline{B}$, whence $e = \{E \cup E_1', E_1 \cup E'\}$, which is not true. Again, since A was an arbitrary chosen ρ_t-class, we have that $\equiv_s = \equiv_t$. Thus, $s = t$. The proof is complete. □

Lemma 20. *Let $f \in \Theta_{\mathrm{pr}}^n$. Then $|\mathcal{C}_{[f]}| = 2^n - 2$.*

Proof. Take $[Hx]$ and $[Hy]$ of $\mathcal{C}_{[f]}$. Then due to Lemma 18, we have that $[fx] = [fy]$ and $\mathrm{rank}(fx) = \mathrm{rank}(fy)$, whence $fx = fy$. Conversely, if $fx = fy$ then $[Hx] = [fx] = [fy] = [Hy]$. Thus, since $\mathrm{rank}(fx) = \mathrm{rank}(f)$ and $fx = f$ hold simultaneously if and only if $f \mathcal{L} fx$, we obtain that $|\mathcal{C}_{[f]}|$ equals the cardinality of $\mathcal{L}$-class, which contains f, which, in turn, equals the number of all partitions of $\{1, \dots, n\}$ into two nonempty blocks. The latter number is equal to $2^n - 2$. □

Lemma 21. *Let $f_1, f_2 \in \Theta_{\mathrm{pr}}^n$. Then $\phi_{[f_1]}$ and $\phi_{[f_2]}$ are equivalent.*

Proof. Let $f_1 = \tau_{F_1}\tau_{\{1,\dots,n\}\setminus F_1}$ and $f_2 = \tau_{F_2}\tau_{\{1,\dots,n\}\setminus F_2}$ for certain proper subsets F_1 and F_2 of $\{1, \dots, n\}$. Put $a = \{F_1 \cup F_2', (\{1, \dots, n\} \setminus F_1) \bigcup (\{1, \dots, n\} \setminus F_2)'\}$. Then, taking to account Proposition 4, we have that $a^{-1}[f_1]a = \{f_2\} \subseteq [f_2]$ and $a[f_2]a^{-1} = \{f_1\} \subseteq [f_1]$, whence $\phi_{[f_1]}$ and $\phi_{[f_2]}$ are equivalent. This completes the proof. □

Lemmas 15, 16, 17, 19, 20, 21 imply the statement of our theorem. We are done.

9 Definition of the ordered partition semigroup $\mathcal{IOP}_n$

Let $n \in \mathbb{N}$. Consider the natural linear order on the set $\{1, \dots, n\}$. Take $A \subseteq \{1, \dots, n\}$. Denote by $\min_A$ the minimum element of A with respect to this order.

Denote by $\mathcal{IOP}_n$ the set of all elements $a = \big(A_i \cup B'_i\big)_{i \in I}$ of $\mathcal{IP}_n$ such that

$$\min_{A_i} \leq \min_{A_j} \Rightarrow \min_{B_i} \leq \min_{B_j} \text{ for all } i, j \in I. \tag{44}$$

The following theorem shows that $\mathcal{IOP}_n$ is an inverse subsemigroup of $\mathcal{IP}_n$.

Theorem 9. *$\mathcal{IOP}_n$ is an inverse subsemigroup of $\mathcal{IP}_n$.*

Proof. That $a \in \mathcal{IOP}_n$ implies $a^{-1} \in \mathcal{IOP}_n$, follows immediately from (44). It remains to prove that $\mathcal{IOP}_n$ is a subsemigroup of $\mathcal{IP}_n$.

Take $a, b \in \mathcal{IOP}_n$. Set $c = ab$. Let $a = \big(A_i \cup B'_i\big)_{i \in I}$, $b = \big(C_j \cup D'_j\big)_{j \in J}$. Obviously, $0 \in \mathcal{IOP}_n$, so we may assume that $c \neq 0$. Let also $c = \big(E_k \cup F'_k\big)_{k \in K}$ and set a linear order $\preceq$ on K, given by

$$\min_{E_k} \leq \min_{E_l} \text{ if and only if } k \preceq l \text{ for all } k, l \in K. \tag{45}$$

Let now $K = \{k_1, \dots, k_m\}$ and $k_1 \preceq k_2 \preceq \dots \preceq k_m$. Set $P_i = E_{k_i}$ and $Q_i = F_{k_i}$ for all i, $1 \leq i \leq m$. Then we have

$$\min_{P_1} \leq \dots \leq \min_{P_m}. \tag{46}$$

Obviously, we have that $1 \equiv_a 1'$ and $1 \equiv_b 1'$. So $1 \equiv_c 1'$. Due to this fact, we obtain that $\{1, 1'\}$ is a subset of the block $P_1 \cup Q'_1$ of the element c. This implies that $\min_{Q_1} = 1$. So $\min_{Q_1} \leq \min_{Q_2}$ and $\min_{Q_1}$ is the first number among the numbers $\min_{Q_1}, \dots, \min_{Q_m}$.

Suppose now that $\min_{Q_1} \leq \dots \leq \min_{Q_t}$ and that $\min_{Q_1}, \dots, \min_{Q_t}$ are the first t numbers among the numbers $\min_{Q_1}, \dots, \min_{Q_m}$, for some t, $t < m$. Then $\min_{Q_t} \leq \min_{Q_{t+1}}$. Since $Q_1, Q_2, \dots, Q_t$ are all λ_{ab}-classes, we obtain that each Q_i, $i \leq t$, is a union of some λ_b-classes and so $Z = Q_1 \cup Q_2 \cup \dots \cup Q_t$ is a union of the sets D_r, $r \in R \subseteq J$. Further, we have that there is a subset U of I such that $\bigcup\limits_{u \in U} B_u = \bigcup\limits_{r \in R} C_r = W$. There is r_0 of R such that $\min_{Q_t} \in D_{r_0}$. Then, obviously, $\min_{D_{r_0}} = \min_{Q_t}$. Since $\min_{Q_1}, \dots, \min_{Q_t}$ are the first t numbers among $\min_{Q_1}, \dots, \min_{Q_m}$, we obtain that

$$\{1, \dots, \min_{D_{r_0}}\} = \{1, \dots, \min_{Q_1}\} \subseteq \bigcup_{i=1}^{t} Q_i = \bigcup_{r \in R} D_r. \tag{47}$$

The latter implies that $\min_{D_r}$, $r \in R$, are the first $|R|$ numbers among the numbers $\min_{D_j}$, $j \in J$. Besides, $\min_{D_r} \leq \min_{D_{r_0}}$ for all $r \in R$. Then, taking to account that $b \in \mathcal{IOP}_n$, we obtain that $\{1, \dots, \min_{C_{r_0}}\} \subseteq \bigcup\limits_{r \in R} C_r$ and $\min_{C_r} \leq \min_{C_{r_0}}$ for all $r \in R$. Then, taking to account $\bigcup\limits_{u \in U} B_u = \bigcup\limits_{r \in R} C_r$, we

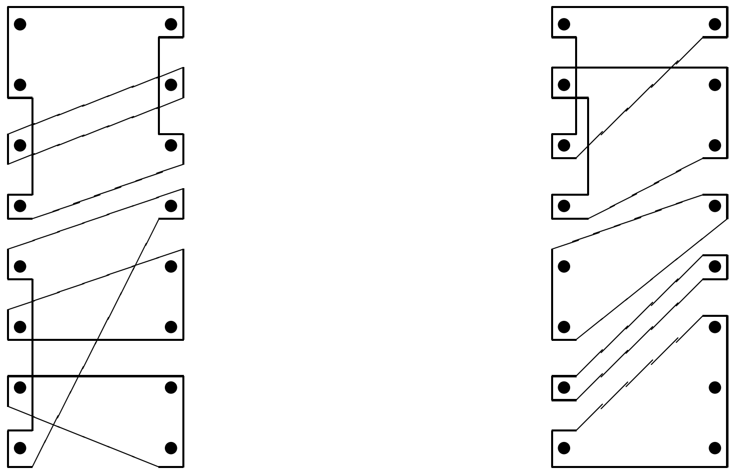

Figure 4: Elements of $\mathcal{IOP}_8$.

obtain that $\min_{B_u}$, $u \in U$, are the first $|\, U \,|$ numbers among the numbers $\min_{B_i}$, $i \in I$. Then, applying $a \in \mathcal{IOP}_n$, we obtain that $\min_{A_u}$, $u \in U$, are the first $|\, U \,|$ numbers among the numbers $\min_{A_i}$, $i \in I$. Note that $\bigcup\limits_{u\in U} A_u = \bigcup\limits_{i=1}^{t} P_t = Y$. Put $y = \min_{\{1,\dots,n\}\setminus Y}$, $w = \min_{\{1,\dots,n\}\setminus W}$ and $y = \min_{\{1,\dots,n\}\setminus Z}$. Then due to what we have already obtained and due to (46), we have that $y = \min_{P_{t+1}}$. Suppose now that $z = \min_{Q_g}$, $g > t$. Then due to our assumption, we have that

$$\begin{aligned}&\min_{Q_1}, \dots, \min_{Q_t}, z \text{ are the first } t+1 \text{ numbers}\\ &\qquad\qquad\qquad\qquad \text{among the numbers } \min_{Q_1}, \dots, \min_{Q_m}. \quad (48)\end{aligned}$$

Due to $a, b \in \mathcal{IOP}_n$, we have that $y \equiv_a w'$ and $w \equiv_b z'$, whence $y \equiv_c z'$. This implies that $z \in Q_{t+1}$, whence $z = \min_{Q_{t+1}}$.

Thus, due to (48), we obtain that inductive arguments lead us to

$$\min_{Q_1} \leq \ldots \leq \min_{Q_m}. \tag{49}$$

The conditions (46) and (49) complete the proof. $\square$

Thus, due to Theorem 9, we can name $\mathcal{IOP}_n$ as the *inverse ordered partition semigroup* of degree n. On Fig. 4 we give some examples of elements of $\mathcal{IOP}_8$.

Recall that a subsemigroup T of a semigroup S is said to be an *$\mathcal{H}$-cross-section* of S if T contains exactly one representative from each $\mathcal{H}$-class of S. In the following proposition we show that $\mathcal{IOP}_n$ is an $\mathcal{H}$-cross-section of $\mathcal{IP}_n$.

Proposition 12. *$\mathcal{IOP}_n$ is an $\mathcal{H}$-cross-section of $\mathcal{IP}_n$.*

Proof. Follows from (44), Theorem 2 and Theorem 9. □

As a consequence of Proposition 12, we obtain the following corollary.

Corollary 3. *Let $n \in \mathbb{N}$. Then $E(\mathcal{IOP}_n) = E(\mathcal{IP}_n)$.*

Proof. Recall that every maximal subgroup of an arbitrary semigroup S coincides with some $\mathcal{H}$-class of S, which contains an idempotent (see [7]). Then every $\mathcal{H}$-cross-section of $\mathcal{IP}_n$ contains all the idempotents of $\mathcal{IP}_n$. In particular, $E(\mathcal{IOP}_n) = E(\mathcal{IP}_n)$, which was required. □

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