

# Permutation avoidance games on comparability digraphs

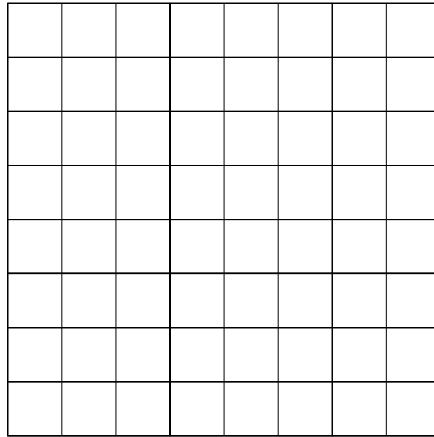
Peter J. Dukes

Kristin Parton

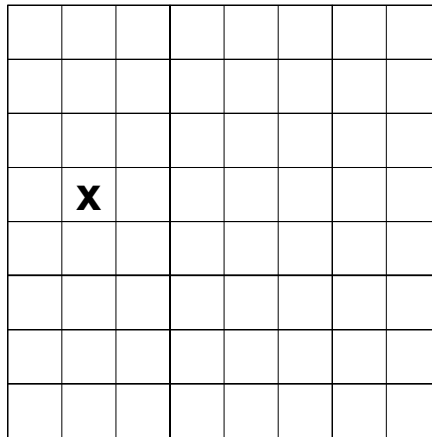
Julian West

University of Victoria

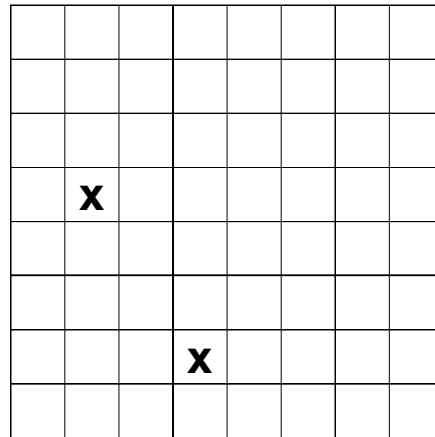
# A game of placing rooks



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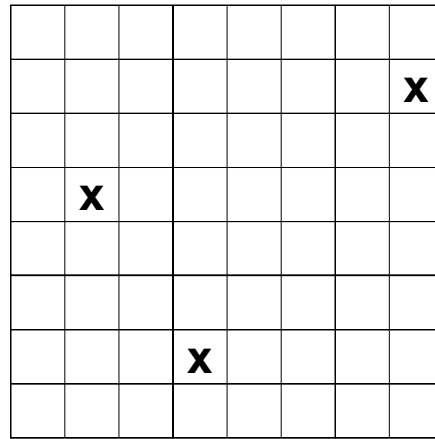
# A game of placing rooks



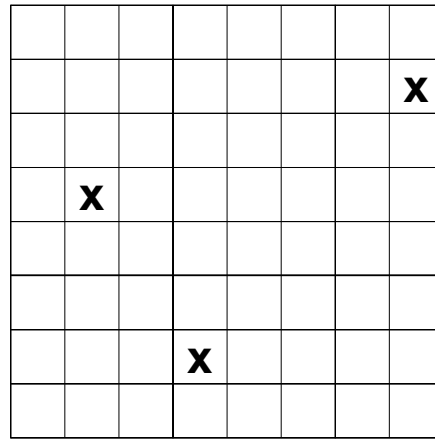
# A game of placing rooks

							<b>x</b>
	<b>x</b>						
			<b>x</b>				

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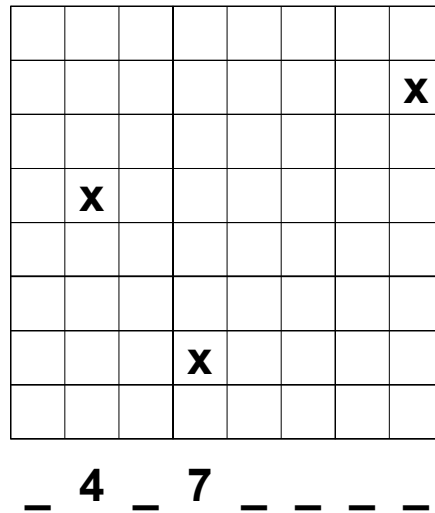


# A game of placing rooks



**4** - - - - -

# A game of placing rooks

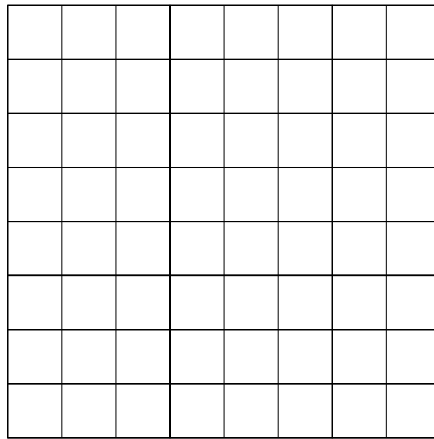




# A game of building permutations

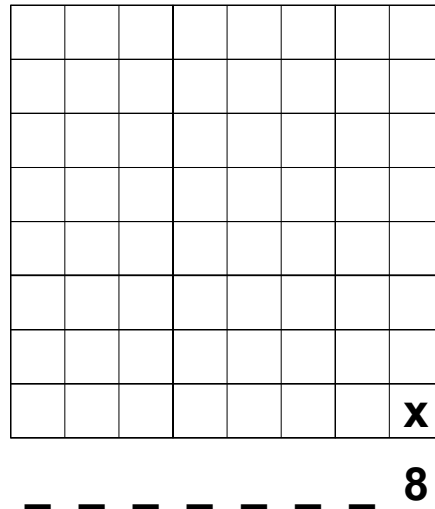
							<b>x</b>
	<b>x</b>						
			<b>x</b>				
<b>_</b>	<b>4</b>	<b>_</b>	<b>7</b>	<b>_</b>	<b>_</b>	<b>_</b>	<b>2</b>

# Avoiding the pattern 12





# Avoiding the pattern 12



# Avoiding the pattern 12

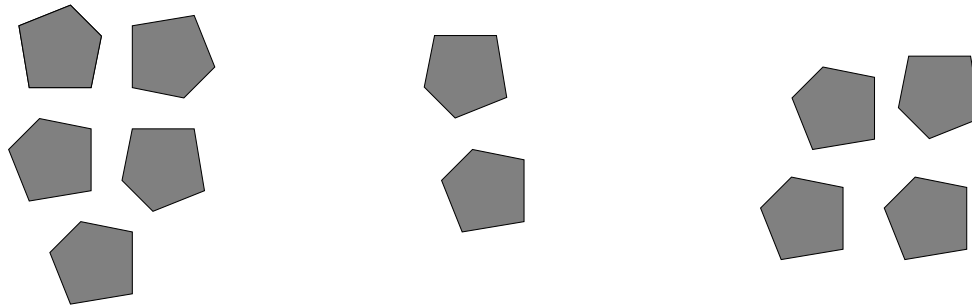
	<b>x</b>						

**Avoiding the pattern 12** What is the best reply?

	<b>x</b>						

## The game of Nim

Recall that in Nim, there are piles containing various numbers of stones. Two players take turns removing some positive number of stones from a single pile of the player's choice. As usual, the last mover wins.



## The game of Nim

The analysis of Nim with piles of sizes  $n_1, \dots, n_t$  is found by computing the *nimber*  $*N$ , where

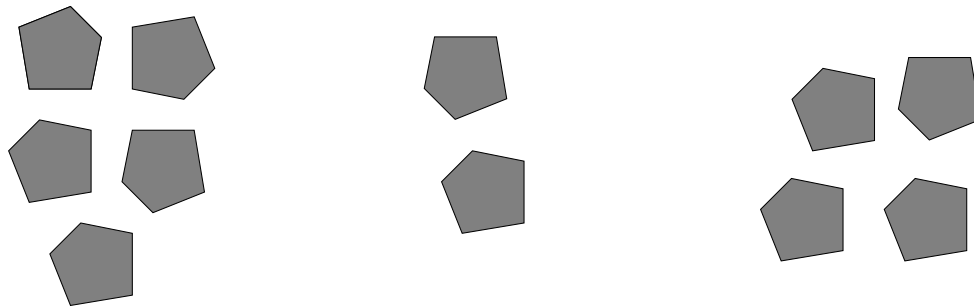
$$N = n_1 \oplus \dots \oplus n_t$$

is the carry-free sum (or *nim sum*) of pile sizes in base two.

Bob has a winning strategy if  $N = 0$ , since there is always a response to ‘cancel’ Alice’s move. Alice has a winning strategy if  $N > 0$ , namely a move leaving  $*0$ .

## The game of Nim

**Example.** Here, Alice wins since  $5 \oplus 2 \oplus 4 = 3$ . The unique winning move takes 1 stone from the middle pile, leaving nim sum  $*0 = 5 \oplus 1 \oplus 4$ .



## The Sprague-Grundy Theorem

*Every* impartial game is equivalent to some number  $*N$ ; i.e. to a single nim heap of size  $N$ .

Furthermore, this number can be calculated recursively:

- A position with no options has value  $*0$ .
- Suppose a position has options corresponding to numbers  $*N_1, \dots, *N_r$ . Then the current position has value  $*N$ , where

$$N = \text{mex}\{N_1, \dots, N_r\},$$

the least nonnegative integer not equal to any  $N_i$ .

## Avoiding $\sigma = (12)$

Alice clearly wins  $P_{(12)}(m, n)$  because she may place a rook in the northwest corner and end the game.

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Alice may also reduce  $P_{(12)}(m, n)$  to any  $P_{(12)}(i, n)$ , for  $i < m$  by placing a rook in the far west, thereby killing the cells south of it.

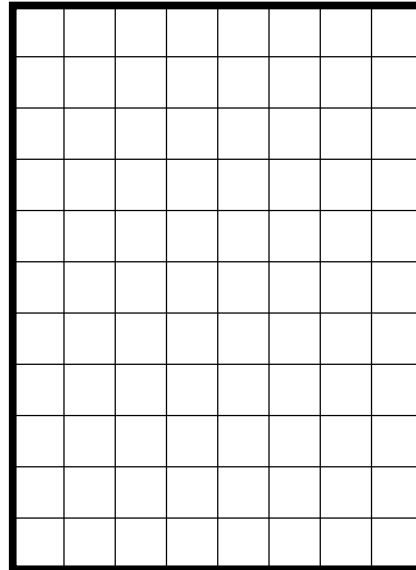
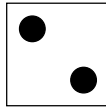
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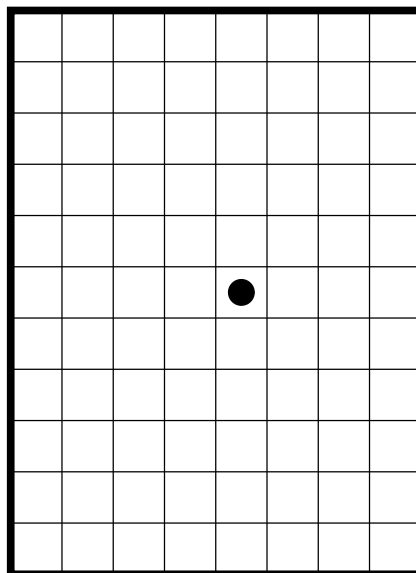
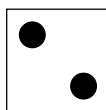
Alice may also reduce  $P_{(12)}(m, n)$  to any  $P_{(12)}(i, n)$ , for  $i < m$  by placing a rook in the far west, thereby killing the cells south of it.

But more generally, the active regions of  $P_{(12)}(m, n)$  are ‘disjoint’ rectangles. They behave as nim heaps, each of whose sizes is the minimum dimension of the corresponding rectangle.

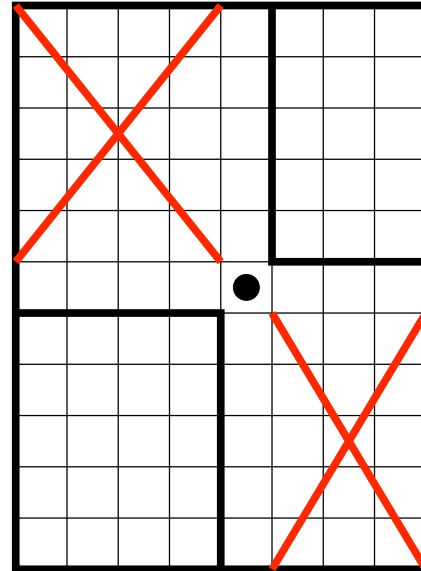
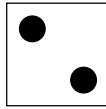
$$P_{(12)}(10, 8)$$



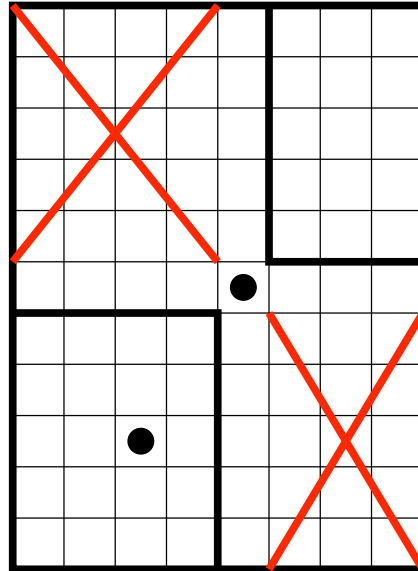
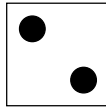
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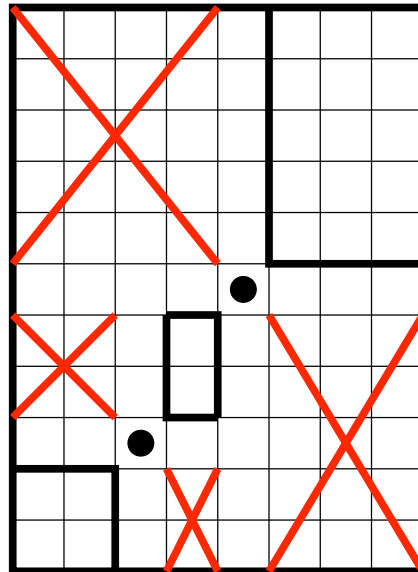
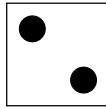
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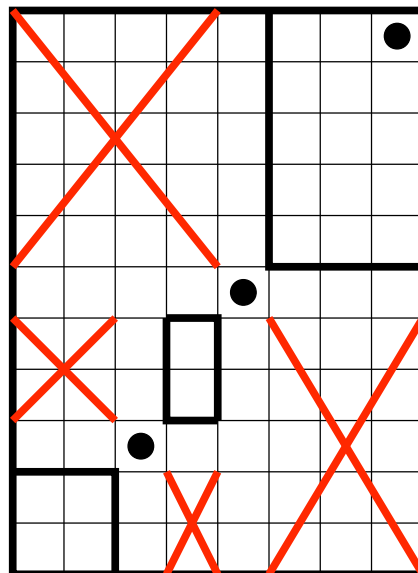
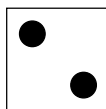
$$P_{(12)}(10, 8)$$



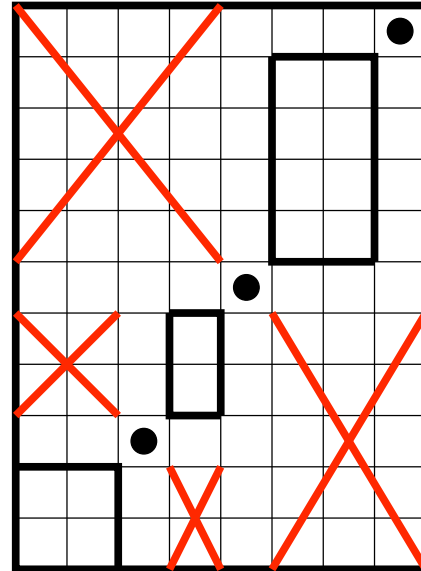
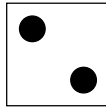
$$P_{(12)}(10, 8)$$



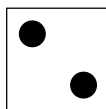
$$P_{(12)}(10, 8)$$



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$$P_{(12)}(10, 8)$$



0	1	2	3	4	5	6	7
1	0	3	2	5	4	7	7
2	3	0	1	6	7	7	7
3	2	1	0	7	7	7	7
4	5	6	7	7	7	7	6
5	4	7	7	7	7	4	5
6	7	7	7	7	6	5	4
7	7	7	7	0	1	2	3
7	7	7	6	1	0	3	2
7	7	4	5	2	3	0	1
7	6	5	4	3	2	1	0

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## Avoiding $\sigma = (123)$

Here, players alternate placing nonattacking rooks on an  $m \times n$  board subject to no three rooks ever being one southeast of another, and a third southeast of both.

A recursive program easily finds  $P_{(123)}(m, n)$  for all values of  $m, n$  with  $m + n \leq 20$ .

**Aside.** Checking if a sequence avoids  $\sigma = (123)$  can be done in linear time with constant memory. Scan the sequence left-to-right and place elements into two stacks, never putting a larger element on top of a smaller element.

## Square boards avoiding $\sigma = (123)$

$1 \times 1$ : first player win

$2 \times 2$ : second player win

$3 \times 3$ : first player win

$4 \times 4$ : ??

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## Square boards avoiding $\sigma = (123)$

$1 \times 1$ : first player win

$2 \times 2$ : second player win

$3 \times 3$ : first player win

$4 \times 4$ : second player win

$5 \times 5$ : first player win

$6 \times 6$ : second player win

$7 \times 7$ : first player win

## Square boards avoiding $\sigma = (123)$

1 × 1: first player win

2 × 2: second player win

3 × 3: first player win

4 × 4: second player win

5 × 5: first player win

6 × 6: second player win

7 × 7: first player win

8 × 8: first player win!!

## Square boards avoiding $\sigma = (123)$

$1 \times 1$ : first player win

$2 \times 2$ : second player win

$3 \times 3$ : first player win

$4 \times 4$ : second player win

$5 \times 5$ : first player win

$6 \times 6$ : second player win

$7 \times 7$ : first player win

$8 \times 8$ : first player win!!

$9 \times 9$ : second player win!!

## Square boards avoiding $\sigma = (123)$

$1 \times 1$ : first player win

$2 \times 2$ : second player win

$3 \times 3$ : first player win

$4 \times 4$ : second player win

$5 \times 5$ : first player win

$6 \times 6$ : second player win

$7 \times 7$ : first player win

$8 \times 8$ : first player win

$9 \times 9$ : second player win

$10 \times 10$ : ??

## Square boards avoiding $\sigma = (123)$

$1 \times 1$ : first player win

$2 \times 2$ : second player win

$3 \times 3$ : first player win

$4 \times 4$ : second player win

$5 \times 5$ : first player win

$6 \times 6$ : second player win

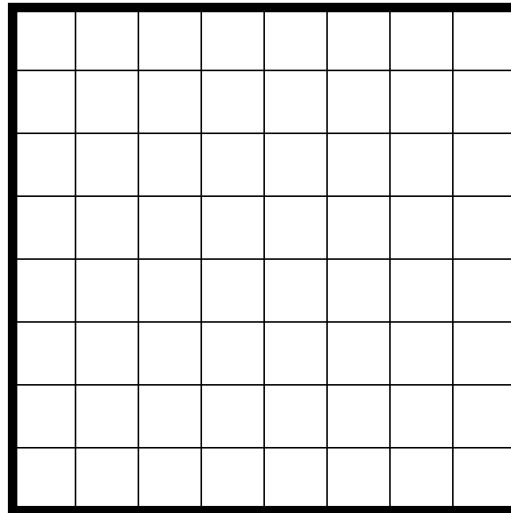
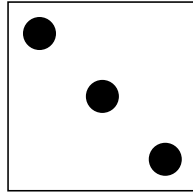
$7 \times 7$ : first player win

$8 \times 8$ : first player win

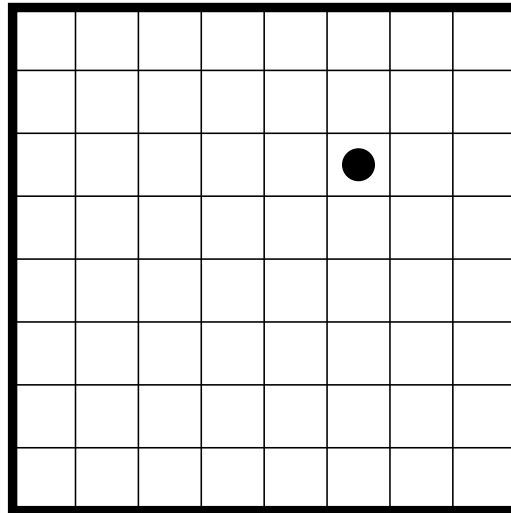
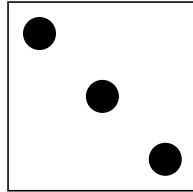
$9 \times 9$ : second player win

$10 \times 10$ : first player win (play in the southwest corner)

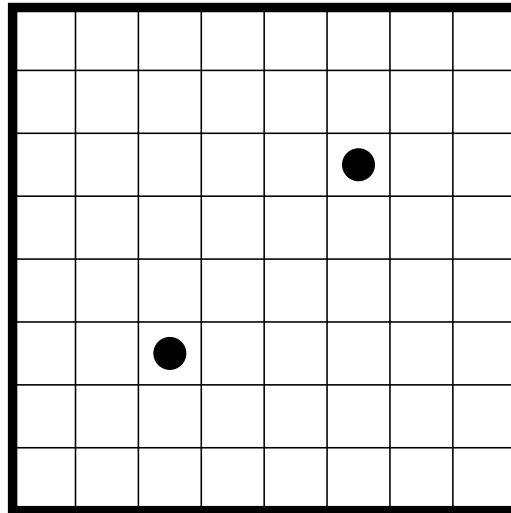
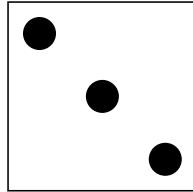
Why can't Bob imitate on  $8 \times 8$ ?



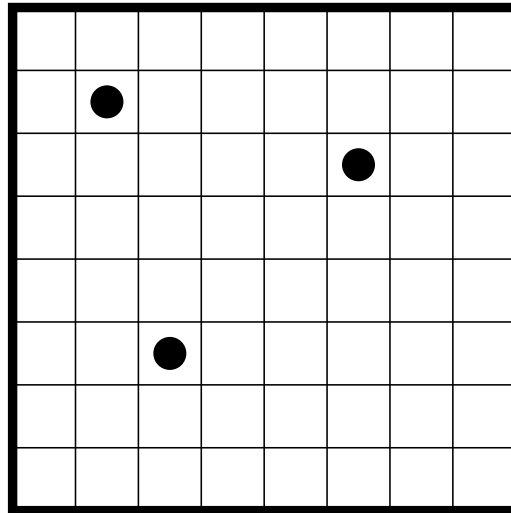
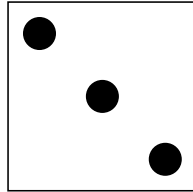
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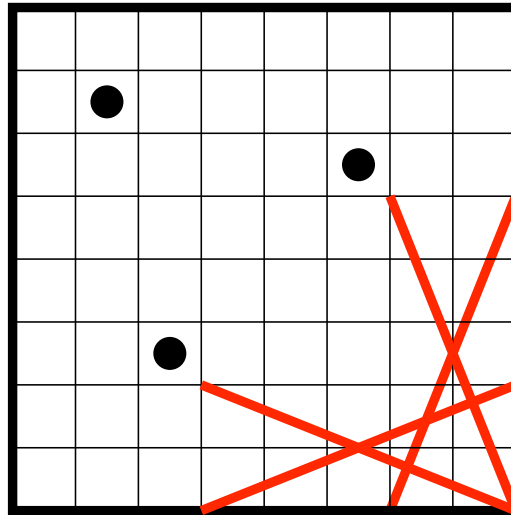
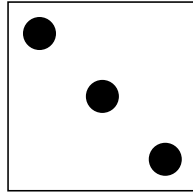
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## Stretching

The numbers for tall (wide) boards of a fixed width (height) seem to stabilize. In fact, the matrix of first-ply options exhibits this behavior as well.

**Example.**  $6 \times n$  boards avoiding (123).

5	5	5	5	3	3
5	5	5	3	2	3
5	5	3	2	3	5
5	3	2	3	5	5
3	2	3	5	5	5
3	3	5	5	5	5

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**Example.**  $6 \times n$  boards avoiding (123).

5	5	5	5	3	1	1
5	5	5	1	5	5	1
5	5	1	5	5	1	5
5	1	5	5	1	5	5
1	5	5	1	5	5	5
1	1	3	5	5	5	5

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5	5	5	1	5	3	5	1
5	5	1	5	3	5	1	5
5	1	5	3	5	1	5	5
1	5	3	5	1	5	5	5
1	1	1	1	5	5	5	5

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5	1	5	3	3	5	1	5	5
1	5	3	1	5	1	5	5	5
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5	5	1	5	3	1	3	5	1	5
5	1	5	3	1	3	5	1	5	5
1	5	3	1	1	5	1	5	5	5
1	1	1	1	1	1	5	5	5	5

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5	5	5	1	5	1	1	1	3	5	1
5	5	1	5	3	1	1	3	5	1	5
5	1	5	3	1	1	3	5	1	5	5
1	5	3	1	1	1	5	1	5	5	5
1	1	1	1	1	1	1	5	5	5	5

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5	5	5	1	5	1	1	1	1	3	5	1
5	5	1	5	3	1	1	1	3	5	1	5
5	1	5	3	1	1	1	3	5	1	5	5
1	5	3	1	1	1	1	5	1	5	5	5
1	1	1	1	1	1	1	1	5	5	5	5

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**Example.**  $6 \times n$  boards avoiding (123).

5	5	5	5	1	1	1	1	1	1	1	1	1
5	5	5	1	5	1	1	1	1	1	3	5	1
5	5	1	5	3	1	1	1	1	3	5	1	5
5	1	5	3	1	1	1	1	3	5	1	5	5
1	5	3	1	1	1	1	1	5	1	5	5	5
1	1	1	1	1	1	1	1	1	5	5	5	5

# Stretching

**Theorem.**

$$\lim_{n \rightarrow \infty} P_{(123)}(m, n)$$

exists

# Stretching

## Theorem.

$$\lim_{n \rightarrow \infty} P_{(123)}(m, n)$$

exists and equals

$$P_{(123)}(m, 2^m).$$

## Stretching

### Theorem.

$$\lim_{n \rightarrow \infty} P_{(123)}(m, n)$$

exists and equals

$$P_{(123)}(m, 2^m).$$

In fact, playing on an infinitely long strip (bounded on the left, say) of width  $n$  makes sense.

### Theorem.

$$P_{(123)}(m, 2^m) = P_{(123)}(m, \infty).$$

## Stretching

The centre is *not* constant in the transverse direction.

**Counterexample (Linton).** avoiding (132, 1234).

2	3	4	5	6	6	6
3	0	5	6	6	6	2
4	5	6	6	6	6	0
5	6	6	2	6	2	4
6	6	6	2	6	6	6
6	6	6	6	2	6	6
6	6	6	2	6	6	6
6	6	0	6	6	6	6

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4	5	6	6	6	6	0
5	6	6	2	6	2	4
6	6	6	2	6	6	6
6	6	6	2	6	6	6
6	6	6	6	2	6	6
6	6	6	2	6	6	6
6	6	0	6	6	6	6

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2	3	4	5	6	6	6
3	0	5	6	6	6	2
4	5	6	6	6	6	0
5	6	6	2	6	2	4
6	6	6	2	6	6	6
6	6	6	2	6	6	6
6	6	6	2	6	6	6
6	6	6	6	2	6	6
6	6	6	2	6	6	6
6	6	0	6	6	6	6