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Polynomial and quasipolynomial counting

(Conference Permutation Patterns 2007, June 11–15, St Andrews, Scotland)

- joint work with Vít Jelínek

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$

$\mathcal{S}$ = class of enumerative problems

$S \in \mathcal{S}$, $f_S : \mathbb{N} \to \mathbb{N}$ counting function, e.g.,

$f_S(n)$ = the # of $A \in S$ with $\mathrm{size}(A) = n$.

We are interested in $\mathcal{S}$ such that

$f_S(n)$ = polynomial(n)

or = quasipolynomial(n) (for $n > n_0$) $\forall S \in \mathcal{S}$

$f: \mathbb{Z} \to \mathbb{C}$ is a quasipolynomial: $\exists$ d polynomials $p_1, \dots, p_d$ s.t. $n \equiv i \pmod{d} \Rightarrow f(n) = p_i(n)$.

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equivalently, $f(n) = a_r(n) \cdot n^r + a_{r-1}(n) \cdot n^{r-1} + \dots + a_0(n)$, where $a_i : \mathbb{Z} \to \mathbb{C}$ are periodic functions

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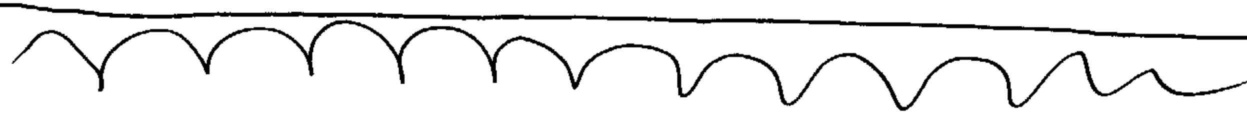
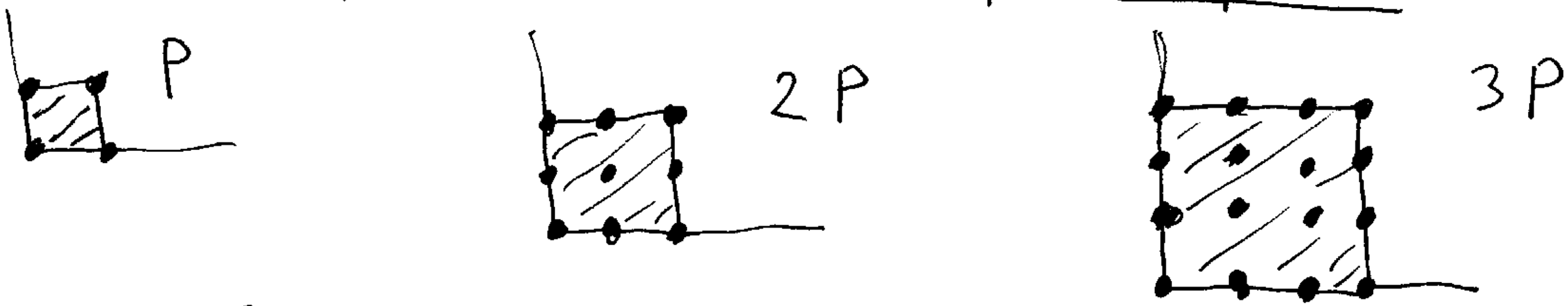
e.g. $f(n) = \left\lceil \frac{n^2-1}{4} \right\rceil$



Overview of the talk:

- Four (quasi)polynomial classes
- Their connections (simple sets, ...)
- Some more polynomial classes

① Lattice points in lattice polytopes



$P \subset \mathbb{R}^2$, $P = \mathrm{conv}(X)$ with finite $X \subset \mathbb{Z}^2$, P is a lattice polytope

$i(P,n) := \#(nP \cap \mathbb{Z}^2)$ where $nP = \{nx : x \in P\}$.

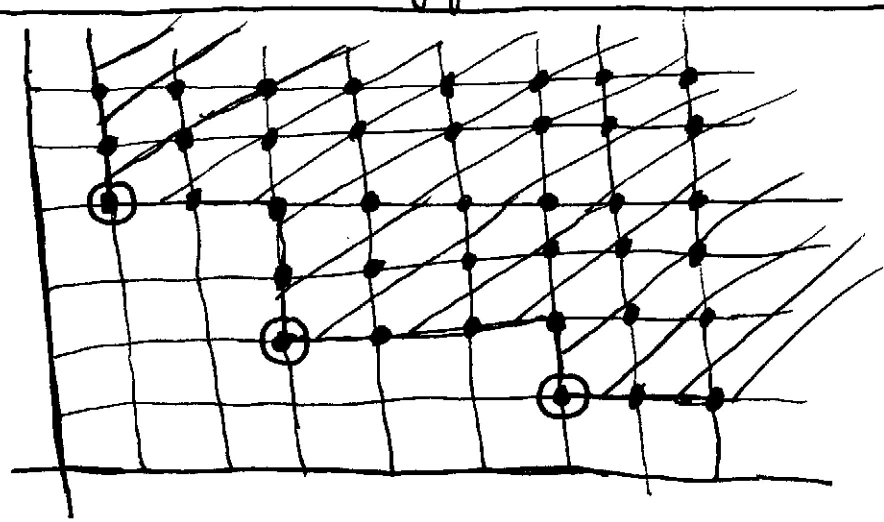
$\boxed{\text{Thm.}}$ (Ehrhart '62) $i(P,n) = \mathrm{poly}(n)$ $\forall n \in \mathbb{N}$.

More generally, for rational polytope P the function $i(P,n)$ is a quasi polynomial $\forall n \in \mathbb{N}$.

Further ramifications: Stanley, Beck, Sottile, ...

(2) Lower and upper ideals in a poset

$k=2$



$X \subset \mathbb{N}^k$ $\quad (\mathbb{N} = \{0,1,2,\dots\})$

is an upper ideal if

$a \le b,\ a \in X \Longrightarrow b \in X$;

here $a=(a_1,\dots,a_k) \le b=(b_1,\dots,b_k) \Longleftrightarrow a_i \le b_i \ \forall i$.

Thm. (Stanley '75) If X is an upper ideal in $(\mathbb{N}^k, \le)$ then

$$f_X(n) := \#(a \in X : \|a\|_1 = a_1+a_2+\dots+a_k = n) = \mathrm{poly}(n)$$

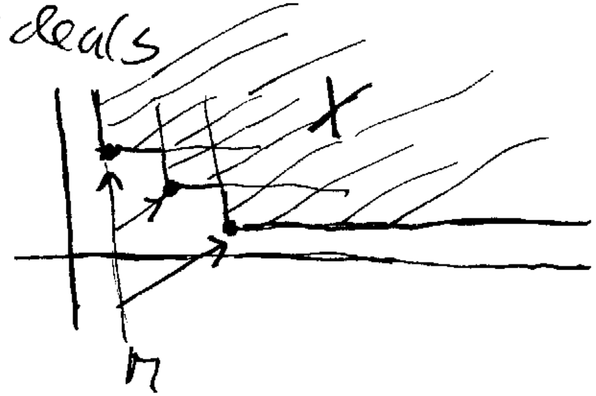
for $n > n_0$. (Same for lower ideals via complements.)

(Same for finite sets.)

$\boxed{\text{Proof.}}$ $X \subset \mathbb{N}^2$ – an upper ideal.

$a \in \mathbb{N}^2$, $\sigma_a := \{ b \in \mathbb{N}^2 : b \geq a \}$ – main u. ideals

$M := \min(X) \rightsquigarrow X = \bigcup_{a \in M} \sigma_a$

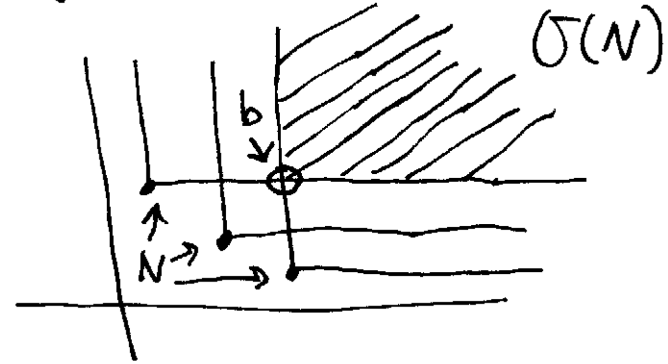


M is an antichain in $(\mathbb{N}^2, \leq)$ $\rightsquigarrow$ M is finite

(Dickson's lemma, 1912)

$P_n := \{ a \in \mathbb{N}^2 : \|a\|_1 = n \}$

PIE : $f_X(n) = \#(X \cap P_n) = \sum_{N \subset M} (-1)^{|N|} \, \#(\sigma(N) \cap P_n)$



where $\sigma(N) = \bigcap_{a \in N} \sigma_a$. But $\sigma(N) = \sigma_b$

with $b = \max(a : a \in N)$. ← poly(n), $n \geq \|b\|_1$
pointwise

So $f_X(n) = \sum_{b \in T} \pm \#(\sigma_b \cap P_n) = \mathrm{poly}(n)$. T ... finite

$\boxtimes$

③ Sumsets in semigroups

all semigroups considered are commutative | ● $=3\mathbb{N}+4\mathbb{N}$.

$(G,+)$ – (comm.) semigroup, i.e., $+$ is commutative and associative.

$A,B\subset G$, $A+B := \{a+b : a\in A,\ b\in B\}$

$A\subset G,\ n\in\mathbb{N}$, $n*A := \{a_1+a_2+\dots+a_n : a_i\in A\}$

$$= \underbrace{A+A+\dots+A}_{n}$$

? $n\mapsto \#(n*A)$?

$\boxed{\text{Thm.}}$ (Khovanskiĭ, '92) If $A,B\subset G$ are finite sets in a semigroup, then $n\mapsto\#(n*A)$ and $n\mapsto\#(n*A+B)$ are polynomials for $n>n_0$.

④ Lower ideals of permutations

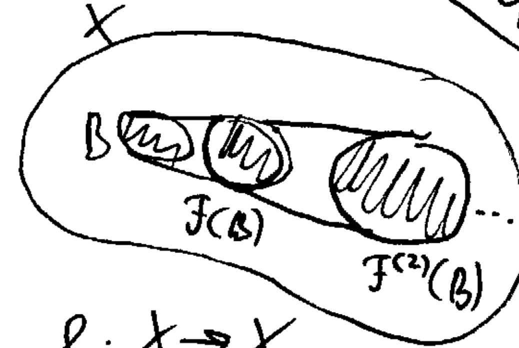
Let $\mathcal{S}=\bigcup_{n=0}^{\infty}\mathcal{S}_n$ be finite perm-ms, so $\mathcal{S}_n=\{a_1a_2\cdots a_n : \{a_1,\dots,a_n\}=\{1,2,\dots,n\}\}$.

$(\mathcal{S},\preceq)$ Containment of permutations: $\pi=a_1a_2\cdots a_m\preceq\sigma=b_1b_2\cdots b_n \iff \sigma$ has a subsequence order-isomorphic to π.

$X\subset\mathcal{S}$ a lower ideal, $f_X(n):=\#(X\cap\mathcal{S}_n)$.

$\boxed{\text{Thm.}}$ (Kaiser & Klazar, '03) Either $f_X(n)\geq F_n$ $\forall n\in\mathbb{N}$ or $f_X(n)=\mathrm{poly}(n)$ for $n>n_0$; here F_n are the Fibonacci numbers, $(F_n)_{n\geq 0}=(1,1,2,3,5,8,13,21,\dots)$, $F_n\sim 1.6^n$.

Generalizations, unification, strengthenings...



Khovanskiĭ himself proved a stronger result.

X, $\mathcal{F}$... a family of mutually commuting mappings $f: X \to X$

$$(f \circ g = g \circ f \quad \forall f, g \in \mathcal{F})$$

$B \subset X$, $f: X \to X$, $f(B) = \{f(b) : b \in B\}$.

$n \in \mathbb{N}$, $\mathcal{F}^{(n)}(B) := \bigcup_{f_i \in \mathcal{F}} (f_1 \circ f_2 \circ \cdots \circ f_n)(B) = \{f_1(f_2(\cdots f_n(b)\cdots)) : b \in B, f_i \in \mathcal{F}\}$.

iterated image

$\boxed{\text{Thm.}}$ (Khovanskiĭ, '92) Let $B \subset X$ be a finite subset and $\mathcal{F}$ be a finite family of <u>commuting</u> mapping $f: X \to X$. Then $\#(\mathcal{F}^{(n)}(B)) = \mathrm{poly}(n)$ for $n > n_0$.

<u>Examples</u>

- Clear if $\mathcal{F} = \{f\}$, because $|B| \geq |f(B)| \geq |f(f(B))| \geq \cdots$.
- $(G, +)$, $B \subset G$, $A \subset G$, $f = f_a : x \mapsto x + a$. Then $\mathcal{F}^{(n)}(B) = n * A + B$.

  X "= G"; B, A finite; $a \in A$

Multivariate generalizations of Khovanskii's semigroup theorem.

$(G, +)$ ... (comm.) semigroup

$\boxed{\text{Thm.}}$ (Nathanson, '00) If $A_1, A_2, \dots, A_\ell, B \subset G$ are finite, ~~and~~ then

$$|n_1 * A_1 + n_2 * A_2 + \cdots + n_\ell * A_\ell + B| =$$

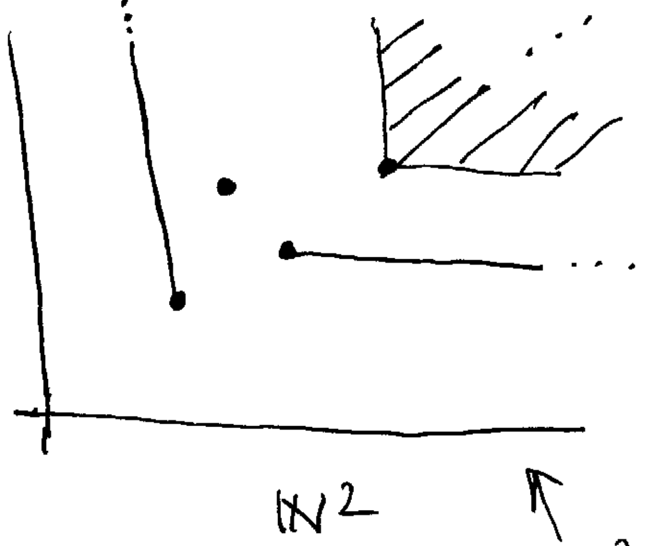
$$= \mathrm{poly}(n_1, n_2, \dots, n_\ell)$$

for all $n_1, \dots, n_\ell \in \mathbb{N}$ bigger than a constant $c > 0$. | Proof? algebraic!

$\boxed{\text{Thm.}}$ (Nathanson & Ruzsa, '02) If $A_1, \dots, A_\ell \subset G$ are finite, then $|n_1 * A_1 + \cdots + n_\ell * A_\ell| = \mathrm{poly}(n_1, \dots, n_\ell)$ $\forall n_1, \dots, n_\ell > c$.

Proof? combinatorial! Based on Stanley's thm. (2 above).

But what if some n_i are $> c$ and some are $\leq c$?



$\mathbb{N}^2$

gen. orthants in $\mathbb{N}^2$

In $\mathbb{N}^q$: $a \in \mathbb{N}^q$, $I \subset [q] = \{1, 2, \dots, q\}$

$$O_{a,I} = \{ b \in \mathbb{N}^q : \begin{array}{l} i \in I \Rightarrow b_i = a_i \\ i \notin I \Rightarrow b_i \geq a_i \end{array} \}.$$

So $O_{a,\emptyset} = O_a = \{ b \in \mathbb{N}^q : b \geq a \}$

$O_{a,[q]} = \{a\}$.

$O_{a,I}$ – <u>generalized orthant</u>.

$c \in \mathbb{N}$, consider an equivalence relation $\sim$ on $\mathbb{N}^q$:

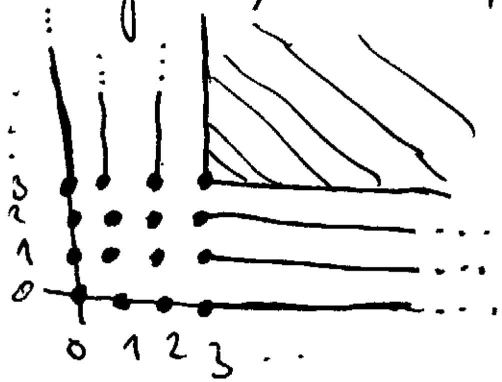
$(a_1, a_2, \dots, a_q) \sim (b_1, b_2, \dots, b_q)$ iff $a_i \leq c \iff b_i \leq c$ and,

moreover, $a_i \leq c \Rightarrow a_i = b_i$.

(In other words, $a_i \neq b_i \Rightarrow a_i, b_i > c$.)

$\mathbb{N}^q/\sim$ has $(c+2)^q$ blocks, which are gen. orthants.

For example, $k=2$, $c=1$ gives this partition into gen. orthants (of $\mathbb{N}^2$):



9: • 0 dim

6: •——— … 1 dim

1:  2 dim

$\boxed{\text{Def.}}$ $f:\mathbb{N}^k \to \mathbb{R}$ is <u>strongly eventually polynomial (SEP)</u> iff $\exists\, c \in \mathbb{N}$ such that f is a polynomial function on every gen. orthant of $\mathbb{N}^k/c$.

For $k=1$ coincides with the usual „$f(n) = \mathrm{Poly}(n)$ for $n > c$".

$\boxed{\text{Thm.}}$ (Jelínek & Klazar, '07) If $A_1, A_2, \dots, A_\ell \subset G$ are finite, where $(G,+)$ is a semigroup, then the function

$(n_1, \dots, n_\ell) \mapsto \#(n_1 * A_1 + \dots + n_\ell * A_\ell)$ is a SEP function from $\mathbb{N}^\ell$ to $\mathbb{N}$.

A set $X \subset \mathbb{N}^k$ is <u>simple</u> if it is a finite union of gen. orthants, $X = \bigcup_{(a,X)\in T} O_{a,X}$ for a finite set T.

<u>Examples.</u> finite sets in $\mathbb{N}^k$, upper ideals, lower ideals, ...

This follows from: $\boxed{\text{Thm.}}$ The class of simple sets in $\mathbb{N}^k$ is closed to finite unions, finite intersections and complements. It is a boolean algebra.

Let $P = \{P_1, \dots, P_e\}$ be a partition of $[k]$. For $a = (a_1, \dots, a_k) \in \mathbb{N}^k$ we define

$$\underline{\|a\|_P := (b_1, b_2, \dots, b_e)}$$

where $\underbrace{b_i = \sum_{j \in P_i} a_j}$. For example, $P = \{\{1\}, \{2\}, \dots, \{k\}\}$ gives $\|a\|_P = a$, and $P = \{[k]\}$ gives $\|a\|_P = \|a\|_1 = (a_1 + a_2 + \dots + a_k)$.

We have the following generalization of Stanley's thm. 2.

$\boxed{\text{Thm.}}$ (J. & K., '07) The following conditions on a set $X \subset \mathbb{N}^q$ are mutually equivalent.

1. X is a simple set (= finite union of gen. orthants).
2. For every partition $P = \{P_1, \dots, P_e\}$ of $[q]$, the function
$$(n_1, \dots, n_e) \mapsto \#(a \in X : \|a\|_P = (n_1, \dots, n_e)) \text{ is SEP.}$$
~~is a SEP function from~~
3. $\chi_X : \mathbb{N}^q \to \{0, 1\}$ is a SEP function.
4. $$F_X(x_1, \dots, x_q) := \sum_{a \in X} x_1^{a_1} \cdots x_q^{a_q} = \frac{r(x_1, \dots, x_q)}{(1-x_1)(1-x_2)\cdots(1-x_q)} \text{ with } r \in \mathbb{Z}[x_1, \dots, x_q].$$

An ~~lower~~ upper ideal is a simple set and $P = \{[q]\}$ gives $\|\cdot\|_1$, thus this indeed generalizes Stanley's theorem in 2.

We have also ~~the~~ a multivariate generalization of Khovanskii's thm. on iterated images.

$B \subset X$, $\mathcal{F}$ = mutually commuting mappings $f: X \to X$, $P = \{P_1, \dots, P_\ell\}$ a partition of $\mathcal{F}$,

$$\mathcal{F}^{(n_1,\dots,n_\ell)}(B) = \{f_1(f_2(\cdots(f_\ell(b)\cdots)) : b \in B,\ f_i \text{ is a composition of } n_i \text{ mappings from } P_i\}.$$

$\boxed{\text{Thm}}$ (J. & Kl., '07) For finite B and $\mathcal{F}$,

$$(n_1,\dots,n_\ell) \longmapsto \#\left(\mathcal{F}^{(n_1,\dots,n_\ell)}(B)\right) \text{ is a SEP function.}$$

Proof. Using the combinatorial argument of Nathanson and Ruzsa, and using simple sets. $\boxtimes$

We have a unification of Ehrhart's thm. in (1) and Khovanskii's thm. on semigroups in (3).

C = (infinite) set of colors.

$\chi : \mathbb{N}^2 \to C$ is an <u>additive</u> coloring iff $\chi(a+b)$ depends only on the colors $\chi(a)$ and $\chi(b)$

(equivalently, χ is a semigroup homomorphism;
— " — , χ defines a congruence on the semigroup $(\mathbb{N}^2, +)$).

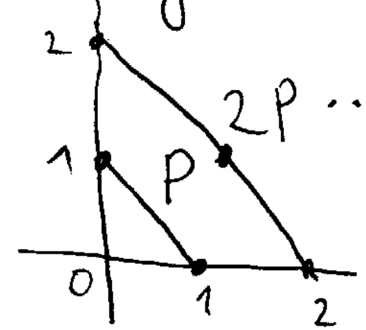
$\boxed{\text{Thm.}}$ (J. & K., '07) Let $P \subset \mathbb{R}^2$ be a lattice polytope with vertices in $\mathbb{N}^2$ and let $\chi : \mathbb{N}^2 \to C$ be additive.

Then $i(P, \chi, n) = \#(\chi(nP \cap \mathbb{N}^2))$ = # of colors appearing on the lattice points in nP

$= \text{poly}(n)$ for $n > n_0$.

P general (lattice) polytope and χ injective $\Rightarrow$ Ehrhart's thm. (1).

$(G,+)$ semigroup, $A=(a_1,a_2,\dots,a_k)\subset G$ finite, we have coloring



$$\chi : \mathbb{N}^k \to G,\ (n_1,\dots,n_k)\mapsto n_1 * a_1+\dots+n_k * a_k.$$

$P=\{x\in\mathbb{R}^k : x_i\geq 0,\ x_1+\dots+x_k=1\}$ - unit simplex.

This χ and $P \Rightarrow$ Khovanskii's thm. (3), because

$$\#(n * A)=\#(\chi(nP\cap\mathbb{N}^k)).$$

<u>Proof.</u> Uses Khovanskii's result $\#(n * A+B)=\mathrm{poly}(n),\ n>n_0$, and a geometric lemma:

If $P\subset\mathbb{R}^k$, $k\geq 1$, is a lattice polytope and $n\geq k$, then

$$nP\cap\mathbb{Z}^k=(n-k) * (P\cap\mathbb{Z}^k)+(kP\cap\mathbb{Z}^k).$$

⊠

Multivariate generalization?

In the last part of my talk, I turn to the thm. on permutations in (4). Recall that it states: $\#(\pi \in S_n \cap X) = \mathrm{poly}(n)$ for $n > n_0$, whenever $\underbrace{\#(\pi \in S_n \cap X)}_{f_X(n)} < F_n$ for some n; X is a lower ideal of permutations.

Three proofs:

- in Kaiser & Klazar, '03
- simpler by Huczynska & Vatter, '06
- in Balogh, Bollobás and Morris, '06, as a special case of a more general and stronger result for ordered graphs.

Relation to Stanley's thm. in (2):

$\boxed{\text{Thm.}}$ Let $X \subset S$ be a lower ideal of permms. If $f_X(n) < F_n$ for some n, then $\exists\, k \in \mathbb{N}$ and $\exists$ injection $F : X \to \mathbb{N}^k$ s.t.

a) F is size-preserving ($\pi \in S_n \mapsto a$ with $\|a\|_1 = n$), b) $F(X)$ is a simple set.

~~Construction of F.~~ Hence

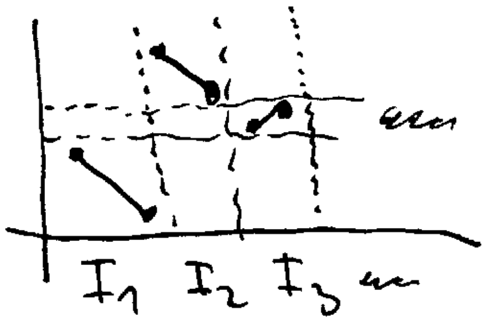
$$f_X(n) = \#(\pi \in X \cap \mathcal{S}_n) = \#(a \in F(x) : \|a\|_1 = n)$$

$$= \mathrm{poly}(n) \text{ for } n > n_0.$$

$\boxed{\text{Proof.}}$

Construction of E. (SRD)

$\pi \in \mathcal{S}_n$, unique strong run decomposition of π: $I_1 < I_2 < \dots < I_r$ int. partition of $[n]$ s.t.

$\pi =$ 

- $\pi | I_i$ monotone
- $\pi(I_i)$ interval in $[n]$
- $I_1, I_2, \dots$ defined greedily to maximize length

$\rho(\pi) =$ 

$z(\pi) = (-1, -1, +1)$

$\rho = \rho(\pi)$ − contract $\pi | I_i$ to a single point $\in \mathcal{S}_r$ | $z = z(\pi) \in \{-1, 1\}^r$ − records sense of monotonicity of $\pi | I_i$

- π is uniquely determined by the triple $(\rho, z, \underbrace{(|I_1|, \dots, |I_r|)}_{\in \mathbb{N}^r})$

⊙ $f_X(n) < F_n$ for some $n \Rightarrow r$ is bounded over $\pi \in X$.

$\Rightarrow$ the set $T = \{(\rho, z) : \underbrace{}_{\pi \text{ runs through } X}\}$ is finite.

For every $(\rho, z) \in T$ take a copy of $\mathbb{N}^q$, $q = |\rho|$. Consider $\pi \in X$ with $(\rho, z, (|I_1|, \dots, |I_q|))$.

F sends π to the point $a = (|I_1|, \dots, |I_q|)$ in the copy of $\mathbb{N}^q$ corresponding to the pair (ρ, z).

The copies can be ~~[illegible]~~ formally glued together to a single $\mathbb{N}^K$ for large K, $K = \sum_{\text{copies}} \dim(\text{copy})$.

F has the two required properties. $\boxtimes$

Thank you