Avoidance of Partially Ordered Generalized Patterns of the form k-σ-k

Marteinn Þór Harðarson

Reykjavík University

11. júní 2007

What does the Title mean

Partially Ordered Generalized Patterns of the form k-σ-k

$$3 - 1 - 2$$

$$5\,3\,1\,2\,4$$

Partially Ordered Generalized Patterns of the form k-σ-k

312

5 3 1 2 4

What does the Title mean

Partially Ordered Generalized Patterns of the form k-σ-k

$3-12$

5 3 1 2 4

What does the Title mean

Partially Ordered Generalized Patterns of the form k-σ-k

121

132 or 231

14352

What does the Title mean

Partially Ordered Generalized Patterns of the form k-σ-k

1 2 1

1 3 2 or 2 3 1

1 4 3 5 2

What does the Title mean

Partially Ordered Generalized Patterns of the form k-σ-k

σ is contiguous

k is largest

3 - 1 2 - 3

What does the Title mean

Partially Ordered Generalized Patterns of the form k-σ-k

σ is contiguous

k is largest

3 - 1 2 - 3

Avoidance of the pattern 3-12

$\pi = \pi_L n \pi_R$

π avoids 3-12 iff

π_L avoids 3-12 and π_R avoids 12

6 3 1 8 7 9 5 4 2

$\{6, 3, 1\}\ \{8, 7\}\ \{9, 5, 4, 2\}$

Avoidance of the pattern 3-12

$\pi = \pi_L n \pi_R$

π avoids 3-12 iff

π_L avoids 3-12 and π_R avoids 12

6 3 1 8 7 9 5 4 2

$\{6, 3, 1\}\ \{8, 7\}\ \{9, 5, 4, 2\}$

Avoidance of the pattern 3-12

$\pi = \pi_L n \pi_R$

π avoids 3-12 iff

π_L avoids 3-12 and π_R avoids 12

6 3 1 8 7 9 5 4 2

$\{6, 3, 1\}$ $\{8, 7\}$ $\{9, 5, 4, 2\}$

Avoidance of the pattern 3-12

$\pi = \pi_L n \pi_R$

π avoids 3-12 iff

π_L avoids 3-12 and π_R avoids 12

6 3 1 8 7 9 5 4 2

$\{6,3,1\}$ $\{8,7\}$ $\{9,5,4,2\}$

Avoidance of the pattern 3-12

$\pi = \pi_L n \pi_R$

π avoids 3-12 iff

π_L avoids 3-12 and π_R avoids 12

6 3 1 8 7 9 5 4 2

$\{6, 3, 1\}\ \{8, 7\}\ \{9, 5, 4, 2\}$

Avoidance of the pattern 3-12

$\pi = \pi_L n \pi_R$

π avoids 3-12 iff

π_L avoids 3-12 and π_R avoids 12

6 3 1 8 7 9 5 4 2

$\{6, 3, 1\}\,\{8, 7\}\,\{9, 5, 4, 2\}$

Avoidance of the pattern 3-12

$\pi = \pi_L n \pi_R$

$$\pi \text{ avoids 3-12 iff}$$
$$\pi_L \text{ avoids 3-12 and } \pi_R \text{ avoids 12}$$

[Claesson 2001] So permutations avoiding 3-12 are in one-to-one correspondence with set partitions

Avoidance of the pattern k-σ

$\pi = \pi_L n \pi_R$

$$\pi \text{ avoids } k\text{-}\sigma \text{ iff}$$

$$\pi_L \text{ avoids } k\text{-}\sigma \text{ and } \pi_R \text{ avoids } \sigma$$

[Kitaev 2003, Elizalde 2006]
Let $f(x)$ be the EGF for avoiders of σ
and $g(x)$ be the EGF for avoiders of k-σ
then

$$Dg(x) = g(x)f(x)$$

Avoidance of the pattern k-σ

$\pi = \pi_L n \pi_R$

π avoids k-σ iff

π_L avoids k-σ and π_R avoids σ

[Kitaev 2003, Elizalde 2006]
Let $f(x)$ be the EGF for avoiders of σ
and $g(x)$ be the EGF for avoiders of k-σ
then

$$Dg(x) = g(x)f(x)$$

Avoidance of the pattern k-σ

$\pi = \pi_L n \pi_R$

$$\pi \text{ avoids } \{k_i\text{-}\sigma_i\}_{i\in I} \text{ iff}$$

$$\pi_L \text{ avoids } \{k_i\text{-}\sigma_i\}_{i\in I} \text{ and } \pi_R \text{ avoids } \{\sigma_i\}_{i\in I}$$

Let $f(x)$ be the EGF for avoiders of $\{\sigma_i\}_{i\in I}$
and $g(x)$ be the EGF for avoiders of $\{k_i\text{-}\sigma_i\}_{i\in I}$
then

$$Dg(x) = g(x)f(x)$$

The differential equation

$$Dg(x) = g(x)f(x)$$

has the unique solution

$$g(x) = \exp\left(\int_0^x f(t)dt\right)$$

since $g(0) = 1$

Avoidance of the pattern 3-12-3

$\pi = \pi_L n \pi_R$

π avoids 3-12-3 iff

π_L avoids 3-12 and π_R avoids 12-3

6 1 9 4 2 12 3 11 8 5 10 7

$\{6,1\}\,\{9,4,2\}\quad\{3,11\}\,\{8,5,10\}\,\{7\}$

Avoidance of the pattern 3-12-3

$\pi = \pi_L n \pi_R$

π avoids 3-12-3 iff

π_L avoids 3-12 and π_R avoids 12-3

6 1 9 4 2 12 3 11 8 5 10 7

$\{6,1\}\,\{9,4,2\} \quad \{3,11\}\,\{8,5,10\}\,\{7\}$

Avoidance of the pattern 3-12-3

$\pi = \pi_L n \pi_R$

π avoids 3-12-3 iff

π_L avoids 3-12 and π_R avoids 12-3

6 1 9 4 2 12 3 11 8 5 10 7

$\{6,1\}\{9,4,2\}$ $\{3,11\}\{8,5,10\}\{7\}$

Avoidance of the pattern 3-12-3

$\pi = \pi_L n \pi_R$

π avoids 3-12-3 iff

π_L avoids 3-12 and π_R avoids 12-3

6 1 9 4 2 12 3 11 8 5 10 7

$\{6,1\}\ \{9,4,2\}\quad \{3,11\}\ \{8,5,10\}\ \{7\}$

Avoidance of the pattern 3-12-3

$\pi = \pi_L n \pi_R$

π avoids 3-12-3 iff

π_L avoids 3-12 and π_R avoids 12-3

6 1 9 4 2 12 3 11 8 5 10 7

$\{6, 1\}\,\{9, 4, 2\}\quad\{3, 11\}\,\{8, 5, 10\}\,\{7\}$

Avoidance of the pattern 3-12-3

$\pi = \pi_L n \pi_R$

π avoids 3-12-3 iff

π_L avoids 3-12 and π_R avoids 12-3

6 1 9 4 2 12 3 11 8 5 10 7

$\{6, 1\}\ \{9, 4, 2\}\quad \{3, 11\}\ \{8, 5, 10\}\ \{7\}$

Avoidance of the pattern 3-12-3

$\pi = \pi_L n \pi_R$

π avoids 3-12-3 iff

π_L avoids 3-12 and π_R avoids 12-3

6 1 9 4 2 12 3 11 8 5 10 7

$\{6, 1\}\ \{9, 4, 2\}\quad \{3, 11\}\ \{8, 5, 10\}\ \{7\}$

Avoidance of the pattern 3-12-3

$\pi = \pi_L n \pi_R$

π avoids 3-12-3 iff

π_L avoids 3-12 and π_R avoids 12-3

6 1 9 4 2 12 3 11 8 5 10 7

$\{6, 1\}\ \{9, 4, 2\}\quad \{3, 11\}\ \{8, 5, 10\}\ \{7\}$

Avoidance of the pattern 3-12-3

$\pi = \pi_L n \pi_R$

π avoids 3-12-3 iff

π_L avoids 3-12 and π_R avoids 12-3

6 1 9 4 2 12 3 11 8 5 10 7

$\{6, 1\}\{9, 4, 2\} \quad \{3, 11\}\{8, 5, 10\}\{7\}$

Avoidance of the pattern 3-12-3

$\pi = \pi_L n \pi_R$

π avoids 3-12-3 iff

π_L avoids 3-12 and π_R avoids 12-3

6 1 9 4 2 12 3 11 8 5 10 7

$\{6, 1\}\,\{9, 4, 2\} \quad \{3, 11\}\,\{8, 5, 10\}\,\{7\}$

Avoidance of the pattern 3-12-3

$\pi = \pi_L n \pi_R$

π avoids 3-12-3 iff

π_L avoids 3-12 and π_R avoids 12-3

6 1 9 4 2 12 3 11 8 5 10 7

$\{6, 1\}\,\{9, 4, 2\} \quad \{3, 11\}\,\{8, 5, 10\}\,\{7\}$

Avoidance of the pattern 3-12-3

$\pi = \pi_L n \pi_R$

$$\pi \text{ avoids 3-12-3 iff}$$

$$\pi_L \text{ avoids 3-12 and } \pi_R \text{ avoids 12-3}$$

So permutations avoiding 3-12-3 are in one-to-one correspondence with bi-colored set partitions

Avoidance of the pattern 3-12-3

$\pi = \pi_L n \pi_R$

$$\pi \text{ avoids 3-12-3 iff}$$

$$\pi_L \text{ avoids 3-12 and } \pi_R \text{ avoids 12-3}$$

That is, permutations avoiding both the patterns 3-12-4 and 4-12-3 are in one-to-one correspondence with bi-colored set partitions

Avoidance of the pattern k-σ-k

$\pi = \pi_L n \pi_R$

π avoids k-σ-k iff

π_L avoids k-σ and π_R avoids σ-k

Let $g(x)$ be the EGF for avoiders of k-σ and $h(x)$ be the EGF for avoiders of k-σ-k then

$$Dh(x) = g(x)^2$$

Avoidance of the pattern k-σ-k

$\pi = \pi_L n \pi_R$

$$\pi \text{ avoids } k\text{-}\sigma\text{-}k \text{ iff}$$

$$\pi_L \text{ avoids } k\text{-}\sigma \text{ and } \pi_R \text{ avoids } \sigma\text{-}k$$

Let $g(x)$ be the EGF for avoiders of k-σ
and $h(x)$ be the EGF for avoiders of k-σ-k
then

$$Dh(x) = g(x)^2$$

Avoidance of the pattern k-σ-k

$\pi = \pi_L n \pi_R$

$$\pi \text{ avoids } \{k_i\text{-}\sigma_i\text{-}k_i\}_{i \in I} \text{ iff}$$

$$\pi_L \text{ avoids } \{k_i\text{-}\sigma_i\}_{i \in I} \text{ and } \pi_R \text{ avoids } \{\sigma_i\text{-}k_i\}_{i \in I}$$

Let $g(x)$ be the EGF for avoiders of $\{k_i\text{-}\sigma_i\}_{i \in I}$
and $h(x)$ be the EGF for avoiders of $\{k_i\text{-}\sigma_i\text{-}k_i\}_{i \in I}$
then

$$Dh(x) = g(x)^2$$

The differential equation

$$Dh(x) = g(x)^2$$

has the unique solution

$$\begin{aligned} h(x) &= 1 + \int_0^x g(t)^2 dt \\ &= 1 + \int_0^x \exp\left(2\int_0^t f(s)ds\right) dt \end{aligned}$$

where $f(x)$ is the EGF for avoiders of σ

Avoidance of the pattern 3-121-3

We can use these equations to find the EGF for avoiders of 3-121-3

$$\mathrm{EGF}_{(1)}(x) = 1$$

$$\mathrm{EGF}_{(121)}(x) = \mathrm{EGF}_{(1\text{-}2\text{-}1)}(x) = \mathrm{EGF}_{(2\text{-}1\text{-}2)}(x)$$
$$= 1 + \int_0^x \exp\left(2\int_0^t 1\, ds\right) dt = \frac{e^{2x}+1}{2}$$

$$\mathrm{EGF}_{(3\text{-}121\text{-}3)}(x) = 1 + \int_0^x \exp\left(2\int_0^t \frac{e^{2x}+1}{2}\, ds\right) dt$$
$$= 1 + \int_0^x \exp\left(\frac{e^t + 2t - 1}{2}\right) dt$$

Avoidance of the pattern 3-121-3

We can use these equations to find the EGF for avoiders of 3-121-3

$$\mathrm{EGF}_{(1)}(x) = 1$$

$$\begin{aligned}\mathrm{EGF}_{(121)}(x) &= \mathrm{EGF}_{(1\text{-}2\text{-}1)}(x) = \mathrm{EGF}_{(2\text{-}1\text{-}2)}(x) \\ &= 1 + \int_0^x \exp\left(2\int_0^t 1\, ds\right) dt = \frac{e^{2x}+1}{2}\end{aligned}$$

$$\begin{aligned}\mathrm{EGF}_{(3\text{-}121\text{-}3)}(x) &= 1 + \int_0^x \exp\left(2\int_0^t \frac{e^{2x}+1}{2}\, ds\right) dt \\ &= 1 + \int_0^x \exp\left(\frac{e^t+2t-1}{2}\right) dt\end{aligned}$$

Avoidance of the pattern 3-121-3

We can use these equations to find the EGF for avoiders of 3-121-3

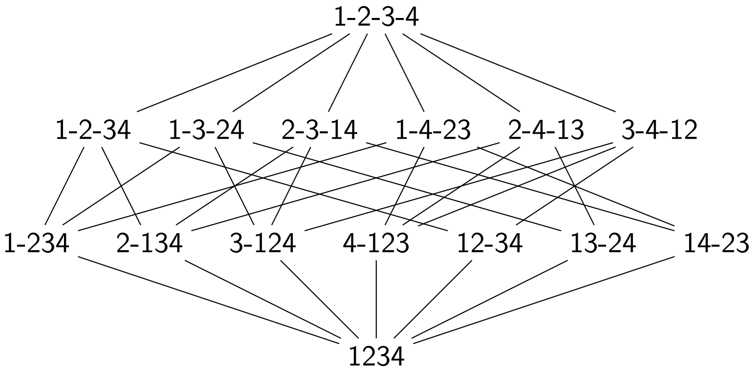
$$\mathrm{EGF}_{(1)}(x) = 1$$

$$\begin{aligned}\mathrm{EGF}_{(121)}(x) &= \mathrm{EGF}_{(1\text{-}2\text{-}1)}(x) = \mathrm{EGF}_{(2\text{-}1\text{-}2)}(x) \\ &= 1 + \int_0^x \exp\left(2\int_0^t 1\,ds\right)dt = \frac{e^{2x}+1}{2}\end{aligned}$$

$$\begin{aligned}\mathrm{EGF}_{(3\text{-}121\text{-}3)}(x) &= 1 + \int_0^x \exp\left(2\int_0^t \frac{e^{2x}+1}{2}\,ds\right)dt \\ &= 1 + \int_0^x \exp\left(\frac{e^t+2t-1}{2}\right)dt\end{aligned}$$

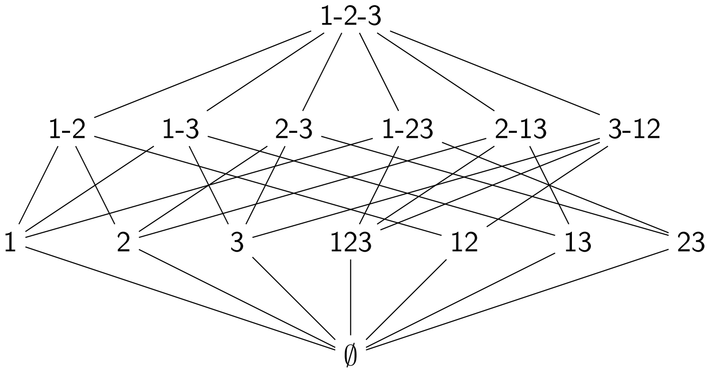
Dowling Numbers

The lattice for set partitions



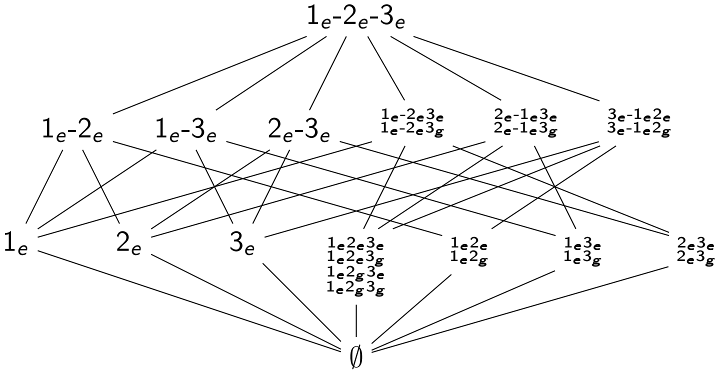
Dowling Numbers

Remove the blocks containing the element 4



Dowling Numbers

Dowling Lattice over the group $G = \{e, g\}$



Back to 3-121-3

3 9 1 5 14 10 7 12 16 15 11 6 13 8 2 4

3 | 9 1 5 | 14 10 7 12   15 | 11 6 13 | 8 | 2 4

$\{3_e\}\{1_e, 5_g, 9_e\}\{7_e, 10_e, 12_g, 14_e\}\{6_e, 11_e, 13_g\}\{2_e, 4_g\}$

Back to 3-121-3

3 9 1 5 14 10 7 12 16 15 11 6 13 8 2 4

3 | 9 1 5 | 14 10 7 12 15 | 11 6 13 | 8 | 2 4

$\{3_e\}\{1_e, 5_g, 9_e\}\{7_e, 10_e, 12_g, 14_e\}\{6_e, 11_e, 13_g\}\{2_e, 4_g\}$

Back to 3-121-3

3 9 1 5 14 10 7 12 16 15 11 6 13 8 2 4

3 | 9 1 5 | 14 10 7 12 15 | 11 6 13 | 8 | 2 4

$\{3_e\}\{1_e, 5_g, 9_e\}\{7_e, 10_e, 12_g, 14_e\}\{6_e, 11_e, 13_g\}\{2_e, 4_g\}$

Back to 3-121-3

3 9 1 5 14 10 7 12 16 15 11 6 13 8 2 4

3 | 9 1 5 | 14 10 7 12 15 | 11 6 13 | 8 | 2 4

$\{3_e\}\{1_e, 5_g, 9_e\}\{7_e, 10_e, 12_g, 14_e\}\{6_e, 11_e, 13_g\}\{2_e, 4_g\}$

Back to 3-121-3

3 9 1 5 14 10 7 12 16 15 11 6 13 8 2 4

3 | 9 1 5 | 14 10 7 12 15 | 11 6 13 | 8 | 2 4

$\{3_e\}\{1_e, 5_g, 9_e\}\{7_e, 10_e, 12_g, 14_e\}\{6_e, 11_e, 13_g\}\{2_e, 4_g\}$

Back to 3-121-3

3 9 1 5 14 10 7 12 16 15 11 6 13 8 2 4

3 | 9 1 5 | 14 10 7 12 15 | 11 6 13 | 8 | 2 4

$\{3_e\}\{1_e, 5_g, 9_e\}\{7_e, 10_e, 12_g, 14_e\}\{6_e, 11_e, 13_g\}\{2_e, 4_g\}$

Back to 3-121-3

3 9 1 5 14 10 7 12 16 15 11 6 13 8 2 4

3 | 9 1 5 | 14 10 7 12 15 | 11 6 13 | 8 | 2 4

$\{3_e\}\{1_e, 5_g, 9_e\}\{7_e, 10_e, 12_g, 14_e\}\{6_e, 11_e, 13_g\}\{2_e, 4_g\}$

Back to 3-121-3

3 9 1 5 14 10 7 12 16 15 11 6 13 8 2 4

3 | 9 1 5 | 14 10 7 12 15 | 11 6 13 | 8 | 2 4

$\{3_e\}\{1_e, 5_g, 9_e\}\{7_e, 10_e, 12_g, 14_e\}\{6_e, 11_e, 13_g\}\{2_e, 4_g\}$

Back to 3-121-3

3 9 1 5 14 10 7 12 16 15 11 6 13 8 2 4

3 | 9 1 5 | 14 10 7 12 15 | 11 6 13 | 8 | 2 4

$$\{3_e\}\{1_e, 5_g, 9_e\}\{7_e, 10_e, 12_g, 14_e\}\{6_e, 11_e, 13_g\}\{2_e, 4_g\}$$

Back to 3-121-3

Permutations avoiding 3-121-3 are in one-to-one correspondence with Dowling partitions

That is, permutations avoiding all of the patterns

5-132-4
5-231-4
4-132-5
4-231-5

are in one-to-one correspondence with Dowling partitions

Back to 3-121-3

Permutations avoiding 3-121-3 are in one-to-one correspondence with Dowling partitions

That is, permutations avoiding all of the patterns

5-132-4
5-231-4
4-132-5
4-231-5

are in one-to-one correspondence with Dowling partitions