

# Aspects of Separability

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PP 2007, St Andrews

What colour is this?

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Oh yes . . .

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Oh yes . . .

I am *not* going to thank the organizers.

# Outline

- Separable permutations
- Subclasses are smaller
- Six degrees of separation?
- An example

# Separable permutations

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A more illuminating definition is that  $\mathcal{S}$  is the closure of the singleton permutation under the operations  $\oplus$  and  $\ominus$ , where:

$$\alpha \oplus \beta = \mathbf{12}[\alpha, \beta] = \frac{\alpha}{\beta}$$

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Or:  $\mathcal{S}$  is the smallest non-trivial substitution closed class.

# The master equations

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$$\begin{aligned}\mathcal{S} &= 1 \cup 12[\mathcal{S}^+, \mathcal{S}] \cup 21[\mathcal{S}^-, \mathcal{S}] \\ \mathcal{S}^+ &= 1 \cup 21[\mathcal{S}^-, \mathcal{S}] \\ \mathcal{S}^- &= 1 \cup 12[\mathcal{S}^+, \mathcal{S}]\end{aligned}$$

Where  $\mathcal{S}^+$  are the *plus indecomposable* elements of  $\mathcal{S}$  and  $\mathcal{S}^-$  the *minus indecomposable* elements of  $\mathcal{S}$ . This translates directly into a system of equations for the generating functions  $s$ ,  $s^+$  and  $s^-$ .

The separable permutations are enumerated by the large Schröder numbers:

$$\begin{aligned}s &= \frac{1 - t - \sqrt{1 - 6t + t^2}}{2t} \\ &= t + 2t^2 + 6t^3 + 22t^4 + 90t^5 + 394t^6 + 1806t^7 + 8558t^8 + \dots\end{aligned}$$

The radius of convergence of  $s$  is  $r = 3 - 2\sqrt{2}$  and hence:

$$\lim_{n \rightarrow \infty} |S_n|^{1/n} = \frac{1}{3 - 2\sqrt{2}} = 3 + 2\sqrt{2}.$$

# Growth rate criticality

A class,  $\mathcal{C}$ , of permutations is said to be *growth rate critical* if, for any proper subclass  $\mathcal{D} \subset \mathcal{C}$ :

$$\limsup_{n \rightarrow \infty} |\mathcal{D}_n|^{1/n} < \limsup_{n \rightarrow \infty} |\mathcal{C}_n|^{1/n}.$$

A few sufficient conditions for growth rate criticality are known, but they do not apply to  $\mathcal{S}$ . So:

## Question

*Is  $\mathcal{S}$  growth rate critical?*

Consider subclasses  $\mathcal{S}_\pi$  defined by a single additional pattern restriction. Argue inductively on  $|\pi|$ . In the inductive step, suppose without loss of generality that  $\pi = \alpha \ominus \beta$  where  $\alpha$  is minus indecomposable. Then define a superset  $\mathcal{Q}_\pi$  of  $\mathcal{S}_\pi$

$$\begin{aligned}\mathcal{Q}_\pi &= 1 \cup 12[\mathcal{Q}_\pi^+, \mathcal{Q}_\pi] \cup 21[\mathcal{S}_\alpha^-, \mathcal{Q}_\pi] \cup 21[\mathcal{Q}_\pi^- \setminus \mathcal{S}_\alpha^-, \mathcal{S}_\beta] \\ \mathcal{Q}_\pi^+ &= 1 \cup 21[\mathcal{S}_\alpha^-, \mathcal{Q}_\pi] \cup 21[\mathcal{Q}_\pi^- \setminus \mathcal{S}_\alpha^-, \mathcal{S}_\beta] \\ \mathcal{Q}_\pi^- &= 1 \cup 12[\mathcal{Q}_\pi^+, \mathcal{Q}_\pi].\end{aligned}$$

Now use the fact that the generating functions  $a$  and  $b$  of  $\mathcal{S}_\alpha^-$  and  $\mathcal{S}_\beta$  have radius of convergence larger than  $r$ , as well as  $a, b < s$  to establish that the solution of this system has its smallest positive singularity at some  $r' > r$ .

# Not many degrees of separation

Every proper subclass  $\mathcal{C}$  of  $\mathcal{S}$  has a generating function which is algebraic over  $\mathbb{Q}(t)$ . A *strong subclass* of a class  $\mathcal{C}$  is one each of whose basis elements is involved in some basis element of  $\mathcal{C}$ .

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## Theorem

*Let  $\mathcal{C}$  be a proper subclass of  $\mathcal{S}$ . If the basis of  $\mathcal{C}$  contains both plus and minus decomposable elements, then its generating function is rational over  $\mathbb{Q}(t)$  and the generating functions of all its strong subclasses. Otherwise, it is of degree one or two over this field.*

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## Corollary

*You can't trisect the angle in  $\mathcal{S}$ .*

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## Corollary

*The degree over  $\mathbb{Q}(t)$  of the generating function of any subclass of  $\mathcal{S}$  is a power of 2.*

## But as many as can be expected

Let  $(\pi_n)_{n \geq 1}$  be the sequence of permutations:

132, 4132, 15243, 615243, 1726354, ...

that is,  $\pi_1 = 132$  and

$$\pi_{n+1} = \begin{cases} 1 \oplus \pi_n & n \text{ even} \\ 1 \ominus \pi_n & n \text{ odd.} \end{cases}$$

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It is *certainly true* that:

### Proposition

*The degree over  $\mathbb{Q}(t)$  of  $c_n$  is precisely  $2^n$ .*

But, is it actually true?

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The result is *obvious* by substituting in the master equation. But ... We must ensure that there's no accidental collapse – i.e. that the quadratic which  $c_n$  satisfies over  $\mathbb{Q}(t, c_{n-1})$  is truly irreducible. It turns out that:

$$c_{n-1} = \frac{c_n - c_n t - t}{c_n^2 + c_n + t}$$

and this gives a recurrence for a polynomial satisfied by  $c_n$  namely:

$$P_n(x) = (x^2 + x + t)^{2^{n-1}} P_{n-1} \left( \frac{x - xt - t}{x^2 + x + t} \right)$$

where  $P_1$  is the irreducible quadratic polynomial satisfied by  $c_1$ .

# Proof

If, when we formally substitute  $t = 1$  in this recurrence we get a sequence of irreducible polynomials over  $\mathbb{Q}$ , we'd be done. And of course we do.

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To see *that* we further reduce to work over  $GF(2)$  rather than  $\mathbb{Q}$  and consider a sequence of elements in some algebraic closure defined by:  $\alpha_3^2 + \alpha_3 + 1 = 0$ , and thereafter

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A bit of algebraic sleight of hand gives an infinite descent if  $\alpha_n \in GF(2)[\alpha_{n-1}]$ .

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How does this relate to the *Vatter dichotomy*?

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Thanks to the organizers of PP2007!

