A conjecture for the packing density of 2413

Cathleen Battiste Presutti and Walter Stromquist

Abstract for PP2007
April 15, 2007

We conjecture that the packing density of the pattern $\pi = 2413$ is $\delta(2413) \approx 0.1047241758\ldots$ and show how this value can be realized as the limit of the packing densities $\delta(2413, \sigma_n)$ for a sequence of permutations σ_n. Along the way we define the packing rate of a pattern with respect to a probability measure, and show that maximizing the packing rate of a pattern π over all measures gives the packing density of π.

The packing density $\delta(\pi, \sigma)$ of a pattern $\pi \in S_m$ in a permutation $\sigma \in S_n$ is the fraction of the m-element subsequences of σ that have the same order type as π. We define $\delta(\pi, n) = \max_{|\sigma|=n} \delta(\pi, \sigma)$, and then the packing density of π is $\delta(\pi) = \lim_{n\to\infty} \delta(\pi, n)$.

Now let μ be a probability measure (probability distribution) on the unit square, and let $\pi \in S_m$. If m points are chosen independently from the distribution μ then the *packing rate of* π *with respect to* μ, $\delta'(\pi, \mu)$, is the probability that the points form a configuration with the same order type as the graph of π. If a permutation $\sigma \in S_n$ is drawn from μ in the same way, then the expected value of $\delta(\pi, \sigma)$ is $\delta'(\pi, \mu)$. We show that if μ^* maximizes $\delta'(\pi, \mu)$ over all measures μ, then $\delta'(\pi, \mu^*) = \delta(\pi)$. Finding the packing density of π is thus equivalent to finding an optimal measure.

In the case of $\pi = 2413$ we conjecture that the optimal measure μ^* has the same fourfold symmetry as π itself, and that it is concentrated (roughly) along four segments, one running from $(1/4, 1/4)$ to $(3/4, 0)$ and the others placed symmetrically. Given these assumptions, we calculate exactly the optimal distribution of probability along each segment. In some places, parts of the segments can be expanded to create small square *recursion bubbles* in which the measure μ^* recurs at a smaller scale. We conjecture that in the optimal measure there is an infinite sequence of recursion bubbles at each end of each segment, leaving a middle portion of each segment in which probability is distributed continuously. The resulting measure gives a packing rate of $0.1047241758\ldots$ which we conjecture to be optimal.

Cathleen Presutti, Ohio University - Lancaster, Lancaster, Ohio 43130 USA
battiste@ohio.edu

Walter Stromquist, Swarthmore College, Swarthmore, PA 19312 USA
wstromq1@swarthmore.edu