Q-counting descent pairs with prescribed tops and bottoms

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Given sets X and Y of positive integers and a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$, an X,Y-*descent* of σ is a descent pair $\sigma_i > \sigma_{i+1}$ whose "top", σ_i, is in X and whose "bottom", σ_{i+1}, is in Y. For a fixed n we define the polynomial

$$P_n^{X,Y}(x) = \sum_{s\geq 0} P_{n,s}^{X,Y} x^s := \sum_{\sigma\in S_n} x^{des_{X,Y}(\sigma)}. \tag{0.1}$$

Thus, the coefficient $P_{n,s}^{X,Y}$ is the number of $\sigma \in S_n$ with exactly s X,Y-descents.

Hall and Remmel [1] gave direct combinatorial proofs of a pair of formulas for $P_{n,s}^{X,Y}$. First of all, for any set $S \subseteq \mathbb{N}$, let

$$\begin{aligned} S_n &= S\cap[n], \text{ and} \\ S_n^c &= (S^c)_n = [n] - S. \end{aligned}$$

Hall and Remmel proved

Theorem 1

$$P_{n,s}^{X,Y} = |X_n^c|! \sum_{r=0}^{s} (-1)^{s-r} \binom{|X_n^c|+r}{r}\binom{n+1}{s-r} \prod_{x\in X_n} (1+r+\alpha_{X,n,x}+\beta_{Y,n,x}), \tag{0.2}$$

and

Theorem 2

$$P_{n,s}^{X,Y} = |X_n^c|! \sum_{r=0}^{|X_n|-s} (-1)^{|X_n|-s-r} \binom{|X_n^c|+r}{r}\binom{n+1}{|X_n|-s-r} \prod_{x\in X_n} (r+\beta_{X,n,x}-\beta_{Y,n,x}), \tag{0.3}$$

where for any set S and any $j, 1\leq j\leq n$, we define

$$\begin{aligned} \alpha_{S,n,j} &= |S^c\cap\{j+1,j+2,\ldots,n\}| = |\{x : j < x \leq n \ \& \ x\notin S\}|, \text{ and} \\ \beta_{S,n,j} &= |S^c\cap\{1,2,\ldots,j-1\}| = |\{x : 1\leq x < j \ \& \ x\notin S\}|. \end{aligned}$$

We prove q-analogues of both Theorem 1 and Theorem 2. For example, define

$$P_n^{X,Y}(q,x) = \sum_{\sigma\in S_n} q^{inv_X(\sigma)+rlmaj_{X,Y}(\sigma)+xycoinv_{X,Y}(\sigma)} x^{des_{X,Y}(\sigma)},$$

where

$$\begin{aligned} inv_X(\sigma) &= \sum_{i=1}^{n} \left(\#j\in X^C \text{ s.t. } j \text{ appears to the left of } i \text{ and } j>i\right) \\ rlmaj_X(\sigma) &= \sum_{\sigma_i\in X_n, \sigma_i>\sigma_{i+1}} (n-i)\,, \text{ and} \\ xycoinv_{X,Y}(\sigma) &= \sum_{x\in X_n} \left(\#z\in Y^C \text{ s.t. } z \text{ appears to the left of } x \text{ and } z<x\right). \end{aligned}$$

Then we can give a direct combinatorial proof of the following.

$$P_{n,s}^{X,Y}(q) = [|X_n^c|]_q! \sum_{r=0}^{s} (-1)^{s-r} q^{\binom{s-r}{2}} \begin{bmatrix} |X_n^c|+r \\ r \end{bmatrix}_q \begin{bmatrix} n+1 \\ s-r \end{bmatrix}_q \prod_{xinX_n} [1+r+\alpha_{X,n,x}+\beta_{Y,n,x}]_q \tag{0.4}$$

This work generalizes many of the results in [2,3,4,5].

References

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