Enumeration of some classes of words avoiding two generalized patterns of length three

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In the present work we deal with pattern avoidance on words. This topic has first appeared in [R], and has been systematically developed in [B]. Subsequently, several authors have studied this kind of matters, and in particular in [BM] exact formulas and/or generating functions for the number of words avoiding a single generalized pattern of length 3 have been found. Here we use a general method to count words on a totally ordered alphabet avoiding a set of generalized patterns of length 3 of type $(1,2)$ (i.e., having a dash between the first and the second element). Our approach consists of inserting a letter at the end of a given word of length n, thus obtaining a word of length $n+1$. Our construction implies that part of the preceding letters could be renamed. The choice of the letter to be inserted depends on the patterns to be excluded. Obviously, the above described insertion technique, if applied to a word avoiding the listed patterns, produces words in which the only occurrence of a forbidden pattern could involve the newly inserted element. Moreover, the particular type of the patterns to be excluded allows to easily control the correct generation of words of increasing length. The first appearance of this technique goes back to [BFP]; later, in [E] the author used it to count several classes of generalized pattern avoiding permutations.

In this note we consider only the case of words avoiding two patterns of length 3 of type $(1,2)$ having two distinct letters. For each class of pattern avoiding words of this kind we are able to determine a succession rule describing the growth of the class, and then to derive from it either the generating function or the closed formula. Here we give only three examples, postponing to the days of the conference the complete picture.

In what follows, we will denote $[m]^n(\sigma,\tau)$ the set of all words over the totally ordered alphabet $\{1,\dots,m\}$ of length n avoiding the two patterns σ,τ. Moreover, the generating function of $[m]^n(\sigma,\tau)$ will be written $f^{(m)}_{\sigma,\tau}(x)$. Finally, $(a)_b$ is the falling factorial $a(a-1)\cdot\ldots\cdot(a-b+1)$.

$$|[m]^n(1-12,2-21)| = \begin{cases} \sum_{k=0}^{n-1} k\cdot(m)_k + (m)_n \ , & n<m \ , \\ \sum_{k=0}^{m-1} k\cdot(m)_k \ , & n\geq m \ . \end{cases}$$

$$|[m]^n(1-21,2-12)| = \sum_{k=0}^{n}\binom{m}{k} k\cdot(n-1)_{k-1} \ .$$

$$f^{(m)}_{1-11,1-12}(x) = \sum_{k=0}^{m-1}\binom{m}{k+1}\left(1+\sum_{h=0}^{k} x(1+x)^h\right)C_k(x) - x(1+x)^{m-1}C_{m-1}(x) \ ,$$

where

$$C_k(x) = \sum_{i=k}^{\frac{k^2+3k}{2}}\left(\sum_{i_1+\cdots+i_k=i}\binom{k+1}{i_1}\cdot\ldots\cdot\binom{2}{i_k}\right)x^i \ .$$

References

[BFP] A. Bernini, L. Ferrari, R. Pinzani, *Enumerating permutations avoiding three Babson-Steingrímsson patterns*, Ann. Comb. 9 (2005) 137-162.

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[B] A. Burstein, *Enumeration of words with forbidden patterns*, PhD thesis, University of Pennsylvania, 1998.

[BM] A. Burstein, T. Mansour, *Words restricted by 3-letter generalized multipermutation patterns*, Ann. Comb. 7 (2003) 1-14.

[E] S. Elizalde, *Generating trees for permutations avoiding generalized patterns*, local proceedings of Permutation Patterns 2006.

[R] A. Regev, *Asymptotics of the number of k-ary words with an l-descent*, Electron. J. Combin. 5 (1998) #R15.