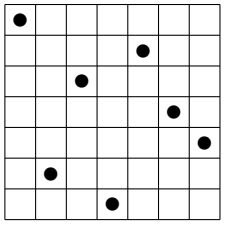
Restricted Symmetric Permutations

Eric S. Egge
Department of Mathematics
Carleton College
Northfield, MN 55057 USA

eggee@member.ams.org

Let S_n denote the set of permutations of $\{1, \dots, n\}$, written in one-line notation. To each $\pi \in S_n$ we associate the *diagram* of π, which is constructed as follows. Begin with an $n \times n$ square, subdivided into n rows of equal height and n columns of equal width, forming n^2 subsquares. Number the rows (resp. columns) of subsquares $1, 2, \dots, n$ from left to right (resp. bottom to top). To complete the diagram, place dots in those squares whose (row,column) coordinates are $(1, \pi(1)), (2, \pi(2)), \dots, (n, \pi(n))$, where $\pi(i)$ is the ith entry of π. For example, the diagram of 7251643 is pictured on the right. The symmetry group of the square acts naturally on these diagrams, and therefore on permutations; we write D_8 to denote this group. In this context it is traditional to single out the reverse map r, which reverses the order of the entries of π, the complement map c, which replaces each entry $\pi(j)$ of $\pi \in S_n$ with $n + 1 - \pi(j)$, and the inverse map i, which takes every permutation to its group-theoretic inverse. On diagrams r is the reflection over a vertical line, c is the reflection over a horizontal line, and i is the reflection over the diagonal from the lower left corner to the upper right corner. Note that r, c, and i together generate D_8, each of them has order two, $rc = cr$, $ri = ic$, and $ci = ir$.



The diagram of 7251643.

Over the past twenty-five years much attention has been paid to pattern-avoiding permutations and pattern-avoiding involutions. From the perspective of the action of D_8, these are the pattern-avoiding permutations which are invariant under the subgroups $\{e\}$ and $\{e, i\}$, respectively. (Here e is the identity.) However, pattern-avoiding permutations which are invariant under other subgroups of D_8 have received little attention. As a first step in changing this, it is not difficult to show that the subgroups of D_8 are the following.

- $H_0 = \{e\}$
- $H_1 = \{e, rc\}$
- $H_2 = \{e, i, rci, rc\}$
- $H_3 = \{e, rc, ri, ci\}$
- $H_{4a} = \{e, i\}$
- $H_{4b} = \{e, rci\}$
- $H_{5a} = \{e, r\}$
- $H_{5b} = \{e, c\}$
- $H_6 = \{e, r, c, rc\}$
- $H_7 = D_8$

Here we use subscripts with the same numerical part for conjugate subgroups. The pattern-avoiding permutations which are invariant under H_0 are the classical pattern-avoiding permutations, and the pattern-avoiding permutations which are invariant under H_{4a} are the pattern-avoiding involutions. No permutation of length 2 or more is invariant under r or c, so no permutation of length 2 or more is invariant under H_{5a}, H_{5b}, H_6, or H_7. In addition, π is invariant under H_{4b} and avoids σ if and only if π^r is invariant under H_{4a} and avoids σ^r. As a result, we are most interested in pattern-avoiding permutations which are invariant under H_1, H_2, or H_3. In this talk I will summarize our knowledge of these permutations. Many of the usual sequences will appear, including powers of 2, Fibonacci numbers, Catalan numbers, and central binomial coefficients.