Where the monotone pattern (mostly) rules

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We consider pattern containment and avoidance with a very tight definition that was used first by Riordan more than 60 years ago.

Let $p = p_1p_2\cdots p_n$ be a permutation, let $k < n$, and let $q = q_1q_2\cdots q_k$ be another permutation. We say that p *very tightly* contains q if there is an index $0 \leq i \leq n-k$ and an integer $0 \leq a \leq n-k$ so that $q_j < q_r$ if and only if $p_{i+j} < p_{i+r}$, and,

$$\{p_{i+1}, p_{i+2}, \cdots, p_{i+k}\} = \{a+1, a+2, \cdots, a+k\}.$$

That is, p very tightly contains q if p tightly contains q and the entries of p that form a copy of q are not just in consecutive positions, but they are also consecutive as entries (in the sense that their set is an interval).

Using this definition, we prove the monotone pattern is easier to avoid than almost any other pattern of the same length. We also show that with this definition, almost all patterns of length k are avoided by the same number of permutations of length n. The corresponding statements are not known to be true for more relaxed definitions of pattern containment.

Let $V_n(q)$ denote the number of n-permutations that very tightly avoid q. While we cannot prove that $V_n(q) \leq V_n(12\cdots k)$ for all patterns q of length k, we will be able to prove that this inequality holds for *most* patterns q of length k. As a byproduct, we will prove that for all k, there exists a set W_k of patterns of length k so that $\lim_{k\rightarrow\infty} \frac{|W_k|}{k!} = 1$, and $V_n(q)$ is identical for all patterns $q \in W_k(q)$. In other words, almost all patterns of length k are equally difficult to very tightly avoid. There are no comparable statements known for the other two discussed notions of pattern avoidance.