Enumeration of Semigroups of Order 9

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Outline

Background

This was an open mathematical problem for 14 years.

There is a large number of solutions, and the problem has many symmetries.

The solution involved

- the development of an enumeration formula for most of the solutions;
- computer search for the remaining solutions.

Basic definitions

Definition (Semigroup)

A set S with a binary operation $\circ$ satisfying

- A1 $(x \circ y) \circ z = x \circ (y \circ z)$ associativity

It may also be true that

- A2 $x \circ e = e \circ x = x$ identity element e
- A3 $x \circ x^{-1} = e$ inverse elements

Examples: $(\mathbb{N}, +), (\mathbb{Z}_n, *)$

Definition (Band)

A semigroup in which all elements are idempotents, i. e.

- A4 $x \circ x = x$

History

order	# groups	# semigroups	
1	1	1	
2	1	4	
3	1	18	
4	2	126	[Forsythe ’54]
5	1	1,160	[Motzkin, Selfridge ’55]
6	2	15,973	[Plemmons ’66]
7	1	836,021	[Jürgensen, Wick ’76]
8	5	1,843,120,128	[Satoh, Yama, Tokizawa ’94]

The problem

For a given number $n \in \mathbb{N}$ find all structural types of semigroups of order n.

A bijection $\sigma : S \to T$ is an *anti-isomorphism* if

$$(xy)^\sigma = y^\sigma x^\sigma \text{ for all } x, y \in S.$$

Definition

Two semigroups are *equivalent* if they are isomorphic or anti-isomorphic.

Nilpotent semigroups

Of the 1,843,120,128 semigroups of order 8 . . .

- . . . less than 0.1% are monoids.
- . . . 99.5% are semigroups with a zero element.
- . . . 99.5% are nilpotent semigroups.
- . . . 99.4% are 3-nilpotent semigroups.

Definition

A semigroup S is *nilpotent* if $|S^r| = 1$ for some $r \in \mathbb{N}$.
$(S^r = \{s_1 \circ s_2 \circ \cdots \circ s_r \mid s_i \in S\})$
A nilpotent semigroup is *r-nilpotent* if $r \in \mathbb{N}$ is the smallest number such that $|S^r| = 1$.

Theorem (Kleitman, Rothschild, Spencer '76)

Asymptotically 'almost all' finite semigroups are 3-nilpotent.

3-nilpotent semigroups

Let $n \geq 2$. Denote $[n] = \{1, \ldots, n\}$.

Define a semigroup S on $[n]$ as follows:

- Take a proper subset B of $[n]$ and set $A = [n] \backslash B$.
- Choose an element $z \in B$.
- Choose a function $\psi : A \times A \to B$.
- For $x, y \in A$ define $x \circ y = \psi(x, y)$.
- Define $x \circ y = z$ in any other case.

Example

$$n = 7, B = \{1, 3, 5, 7\} \Rightarrow A = \{2, 4, 6\}, z = 7$$

$\circ$	1	2	3	4	5	6	7
1	7	7	7	7	7	7	7
2	7	3	7	5	7	7	7
3	7	7	7	7	7	7	7
4	7	3	7	7	7	1	7
5	7	7	7	7	7	7	7
6	7	5	7	3	7	1	7
7	7	7	7	7	7	7	7

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3	7	7	7	7	7	7	7
4	7	3	7	7	7	1	7
5	7	7	7	7	7	7	7
6	7	5	7	3	7	1	7
7	7	7	7	7	7	7	7

Example

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$\circ$	1	2	3	4	5	6	7
1	7	7	7	7	7	7	7
2	7	**3**	7	**5**	7	**7**	7
3	7	7	7	7	7	7	7
4	7	**3**	7	**7**	7	**1**	7
5	7	7	7	7	7	7	7
6	7	**5**	7	**3**	7	**1**	7
7	7	7	7	7	7	7	7

The number of 3-nilpotent semigroups

Theorem

The number $P(n,r)$ of isomorphism types of 3-nilpotent semigroups of order n with $|B| = r$ equals

$$\sum_{\substack{(j)\vdash n-r \\ (k)\vdash r-1}} \left(\prod_{i=1}^{n-r} j_i! \, i^{j_i}\right)^{-1} \left(\prod_{i=1}^{r-1} k_i! \, i^{k_i}\right)^{-1} \prod_{i_1,i_2=1}^{n-r} \left(1 + \sum_{\substack{d \mid lcm(i_1,i_2) \\ d \leq r}} d k_d\right)^{j_{i_1} j_{i_2} gcd(i_1,i_2)}$$

Thus the number of non-isomorphic 3-nilpotent semigroups of order n equals

$$\sum_{r=2}^{n-1} P(n,r) - P(n-1,r-1).$$

The search

Find all semigroups of order n that are not 3-nilpotent.

Formulate a *Constraint Satisfaction Problem* (CSP) L:

- Variables: $T(i,j), 1 \leq i,j \leq n$
- Domains: $\{1, 2, \ldots, n\}$
- Constraints: $T(i, T(j,k)) = T(T(i,j),k), 1 \leq i,j,k \leq n$,
  $\exists i,j,k : T(i, T(j,k)) \neq "0"$,
  $T \leq T^g, g \in S_n \times C_2$

We use the Minion CSP solver to obtain solutions for (variations on) L.

Problem:
There are $2n!$ symmetry breaking (SB) constraints.

Case split

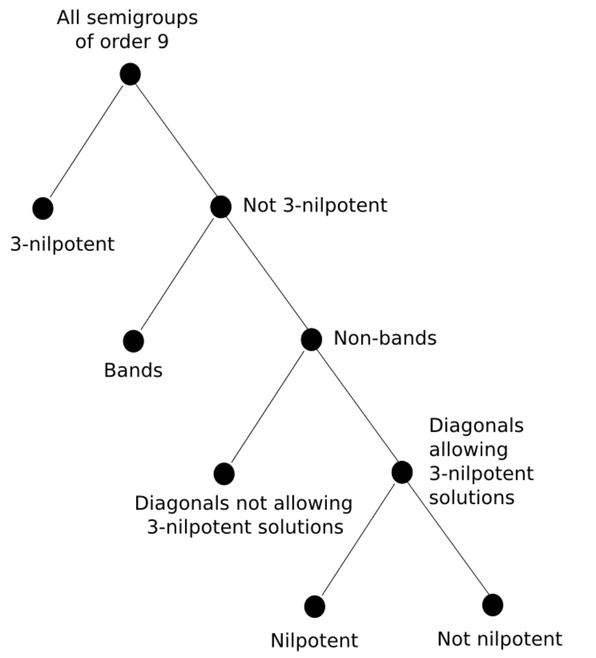
The action on tables induces an action on diagonals.
We construct a set of orbit representatives.

For every diagonal D formulate a CSP based on L.

- Add constraints: $T(i,i) = D(i), 1 \leq i \leq n$,
- Replace SB constraints: $T \leq T^g, g \in \mathrm{Stab}_{S_n}(D) \times C_2$

What we care about in a case split:

- different cases have non-equivalent solutions,
- all cases together have the solutions of L,
- most instances have fewer symmetries,
- the time and space complexities are small enough for us to be able to solve the instances on our compute servers.


All semigroups of order 9
Not 3-nilpotent
3-nilpotent
Non-bands
Bands
Diagonals allowing 3-nilpotent solutions
Diagonals not allowing 3-nilpotent solutions
Nilpotent
Not nilpotent

A case for bands

Theorem

Let B be a band with $\mathcal{D}$-classes $\{E_\alpha : \alpha \in Y\}$. Then each E_α is a rectangular band and Y forms a semilattice (the join being defined by $E_\alpha E_\beta \subseteq E_{\alpha\beta}$).

Formulate CSPs for non-equivalent sets of rectangular bands indexed by semilattices.

Results

order	# semigroups	# 3-nilpotent semigroups
1	1	0
2	4	0
3	18	1
4	126	8
5	1,160	84
6	15,973	2,660
7	836,021	609,797
8	1,843,120,128	1,831,687,022
9	**52,989,400,714,478**	**52,966,239,062,973**

Challenges

It took 30 months to design and implement the case split searches. These will now run in a few days on a single computer to give the non-3-nilpotent semigroups of order 9.

The same approach would require several years search for order 10 semigroups.

The challenges now are to:

- re-model certain CSP instances to utilise knowledge of the possible generators of solution semigroups;
- distribute and/or parallelise the search across a large number of processors;
- deal with the $2 \times 10!$ symmetries effectively and efficiently.